
Mixed-mode oscillations in Filippov system

ABSTRACT:

The mechanism of the mixed mode oscillations of a class of non-smooth Filippov systems under multistable coexistence was studied in this paper. Based on a Lorenz-type chaotic model with multi-attractor coexistence, the Filippov system was established by introducing non-smooth terms as well as an external excitation. With multiple stable attractors in the discontinuous vector field, the parameter changes have led to complex transition patterns between the attractors and the non-smooth interface, or between the attractors. When an order gap exists between the exciting frequency and the natural frequency, implying the mixed-mode oscillations. Here we have taken several excitation amplitudes to cover different coexistence regions, a set of mixed mode oscillation patterns were obtained. Besides, the bifurcation set of two generalized autonomous subsystems and the coexistence region of attractors were discussed. Combined with the transformed phase diagram method, the bifurcation mechanism of bursting oscillation and the sliding dynamical behaviors of the system at the discontinuous interface has revealed with slow varying parameters access in different regions of multistable attractors coexistence. The alternations between quiescent and spiking states become more frequent and complex, leading to the change of the structure of the bursting oscillation modes. Moreover, the non-smooth partition interface of the system yields multiple non-smooth bifurcations, which will also affect the oscillation modes of the generalized autonomous system.

Keywords: bursting oscillation, multiscale coexistence, multiscale coupling, non-smooth system

1. INTRODUCTION

Multistable coexistence is a unique property of dynamical systems. It has been widely found in various fields of nature and engineering, such as circuit models¹, biological population model² and the food chain model³ etc. When the initial state changes, the multistable coexistence system may be influenced by different attractors, which may appear completely different dynamical behaviors, leading to the dynamical behaviors of the system becoming unpredictable⁴. Therefore, studying the dynamical behaviors and attractive domains of systems with the multistable coexistence is meaningful.

Up to now, the research about multistable systems has focused on the solution of attractors or characterizing the attractive domain⁵. However, in practical engineering, systems often exist multiscale coupling or non-smooth factors⁶ etc., leading to the more complex behaviors. Meanwhile, multi-timescale coupling phenomena is common in the field of living organisms⁷, physics⁸, chemistry⁹ etc. In a period of dynamic behavior, there will be bursting oscillation behaviors that alternate between large oscillation (spiking state) and small oscillation (quiescent state)¹⁰. Due to a lack of theoretical foundation, early scholars mostly used the approximate solution and numerical simulation¹¹ methods to investigate the bursting oscillation mechanism. Until 2000, Rinzel proposed the fast and slow analysis method¹², which is not only provided theoretical support for the bursting oscillation mechanism but also given the classification of the bursting oscillation form, after that more scholars began to study it. When there exists an order gap between the exciting frequency and the natural frequency,

implying the coupling of two scales in the frequency domain exists, slow-fast behaviors may also appear¹³. Such as Han Xiuqing¹⁴ et al found that bursting oscillations of Duffing oscillator can be controlled by forcing frequency and forcing amplitude.

It is worth noting that when the limit cycles coexist with the equilibrium points, two-dimensional torus in the vector field, implying the two scale system exhibits both mixed mode oscillations and quasi-periodic oscillations¹⁵. It is shown that multistable coexistence in the multiscale coupled system may directly affect the fast and slow behavior.

Moreover, when the non-smooth term exists, the discontinuity of the vector field makes the system controlled by different subsystems. Some dynamic behaviors such as sliding bifurcation on the non-smooth interface may appear, making the switching mechanism between the stable attractors more complicated¹⁶. Nowadays, most researches are mainly aimed at the systems containing single stable attractor, which mainly show the transition modes of quiescent states and spiking states with single type in Filippov system under two-scale coupling¹⁷. However, for common dynamical systems, such as chaotic attractors of Lorentz type, stable equilibrium points often coexist with limit cycles and other forms of stable attractors, which undoubtedly leads to more complex mixed oscillation modes.

Here, we introduce a Lorentz-type chaotic system containing multistable coexistence attractors¹⁸. Based on the Lorentz-type chaotic system, a novel three dimensional non-smooth dynamical model is developed by introducing the non-smooth factor and the periodic exciting term:

$$\begin{aligned} \dot{x} &= ax - yz, \\ \dot{y} &= -by + xz, \\ \dot{z} &= -cz + xyz + d + g(x) + A \sin(\Omega t), \end{aligned} \quad (1)$$

The novel model is a Filippov system with discontinuous vector fields. Where $w = A \sin(\Omega t)$ represents the periodic parametric excitation with the excitation amplitude A and frequency Ω . The non-smooth term is expressed in the form $g(x) = -3 \operatorname{sgn} x$, implying the system can be expressed into two subsystems. Because of the existence of the slowly varying excitation term w , when the exciting frequency Ω is far less than the natural frequency of the system, denoted by ω_n , the coupling of two scales in frequency domain exists. Furthermore, the non-smooth boundary, defined as $\Sigma \equiv \{(x, y, z), x = 0\}$, divides the phase space into two regions, described by $D_1 \equiv \{(x, y, z), x > 0\}$ and $D_2 \equiv \{(x, y, z), x < 0\}$, in which the trajectory is governed by different subsystem, respectively. Because of the trajectories of non-smooth systems are represented by two different subsystems, more complicated dynamical behaviors can be obtained. With the variation of the parameters, it is not difficult to find that the number of equilibrium points may change. When the system takes the representative parameters, we explore the dynamical behaviors are able to find its corresponding multistable coexistence region and further analyze its dynamical behaviors.

Based on the slow-fast analysis method, the whole exciting term $w = A \sin(\Omega t)$ can be regarded as a generalized state variable, which forms the slow subsystem, leading to the so-called fast subsystem in generalized autonomous form. Based on the above, we select specific parameters with the excitation frequency at $\Omega = 0.01$. With the overlap of the transformed phase diagram and the bifurcation diagram, we deeply study the dynamic mechanism of the Filippov system in this paper.

2. Mixed-mode oscillations with different excitation amplitudes

Two typical situations are examined below.

Case 1: $A=3.2$

The system presents a quasi-periodic movement on the right side of the interface ($x > 0$), and the radius of movement **varies** with the amplitude of the external excitation. As shown in Fig.1, the D_2 region ($x < 0$) occurs two spiking states, represent by SP_2 and SP_3 respectively. Further observation shows that SP_3 belongs to the quasi-periodic type, its frequency different with SP_2 . The inconsistent frequency indicates that the system trajectory from the spiking state(SP_2) to a short silence and then tends to the limit cycle quickly. Furthermore, more smooth bifurcations appear with the **increase** of external excitation amplitude, leading the SP_3 turn to the QS_1 .

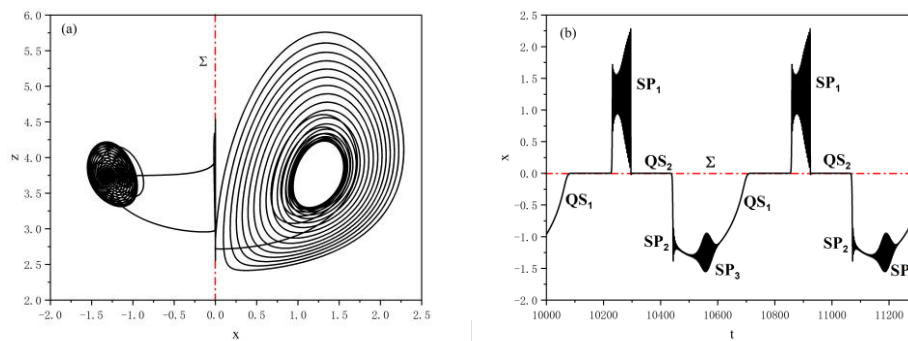


Fig.1 $A=3.2$ (a)Phase portrait on the (x, z) plane ;(b) Time histories of x .

Case 2: $A=3.9$

When the external excitation amplitude increases to $A=3.9$, the periodic oscillation in the D_1 region ($x > 0$) disappears and turns **into** two spiking states, as shown in Fig.2. Among that the SP_2 consistent with case $A=3.2$, the trajectory tends to a smooth equilibrium point, **described by SP_2** . The two spiking states are connected in a quiescent state QS_2 , indicating that more local bifurcations occur in the system.

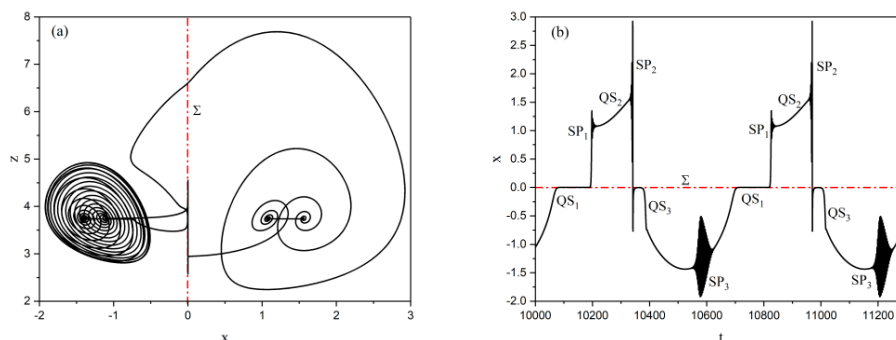


Fig.2 $A=3.9$ (a)Phase portrait on the (x, z) plane ;(b) Time histories of x .

The trajectory in the D_2 region ($x < 0$) has also changed. Compared with the case 1 for $A=3.2$, the number of spiking states decreases. The original oscillations toward the stable equilibrium point disappear, indicating that the trajectory may move to the limit cycle via an additional stable attractor. In order to reveal the mechanism of the dynamics of the full system, now we turn to the attractors and their bifurcations of the fast subsystem with the variation of the

slow-varying parameter w .

3. Equilibrium branches and bifurcations

To simplify the analysis, here we fix the parameters at $a=2$, $b=7$, $c=3$, and $d=8$, and we give the equilibrium cycles and bifurcations of the subsystems, as shown in Fig 3. Where, all black solid lines indicate stable equilibrium branches, and black dashed lines indicate unstable equilibrium branches. Furthermore, all pseudo-equilibrium points are marked with red solid and dashed lines. Due to the non-smooth interface, trajectory actually move within a subsystem of $x > 0$ in D_1 region, likewise, trajectory actually move within a subsystem of $x < 0$ in D_1 region.

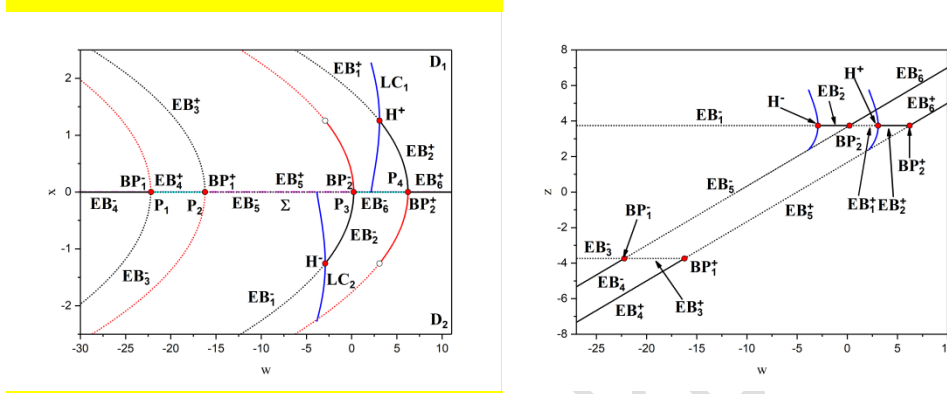


Fig.3 Equilibrium branches and bifurcations.

When $x > 0$, the trajectory is governed by the subsystem D_1 , which the corresponding equilibrium points are represented as the EB^- . Similarly, the equilibrium points of subsystem D_2 can be represented as EB^+ .

For the subsystem in D_1 , the stable equilibrium branch EB_2^+ evolves to a stable limit cycle LC_1 via Hopf bifurcation at H^+ . Sub-critical pitchfork bifurcation occurs at the point P_2 and super-critical pitchfork bifurcation occurs at the point P_4 . For the subsystem in D_2 , similarly, three bifurcation points H^- , BP_1^- and BP_2^- can be observed, which correspond to pitchfork bifurcation and Hopf bifurcation respectively. With the decrease of slow-varying parameter w , the stable equilibrium branch EB_2^- evolves to another stable limit cycle, denoted by LC_2 , see Fig.3.

Since the equilibrium branches of the two subsystems on the $x=0$ interface are complex, for better understanding, the range of the equilibrium branches of different subsystems on the interface is given in the Table 1.

subsystem	D_2	D_2	D_2	D_1	D_1	D_1
equilibrium branches	EB_4^-	EB_5^-	EB_6^-	EB_4^+	EB_5^+	EB_6^+
range	$w < P_1$	$P_1 \sim P_3$	$w > P_3$	$w < P_2$	$P_2 \sim P_4$	$w > P_4$
stability	stable	unstable	stable	stable	unstable	stable

Table.1 Range of equilibrium branches on $x=0$ interface of different subsystems

If we take a certain range, the system will have multiple stable coexisted attractors, in order to show the system in the phase space clearly, we omit all the pseudo-equilibrium points in the (w, z) diagram. From the equilibrium branches as well as the bifurcations on the (w, x) plane and the (w, z) plane plotted in Fig.3, one may find there exist the coexisted attractors, including the stable equilibrium branch EB_2^+ coexists with the stable equilibrium branch EB_6^- , the

coexistence of equilibrium points and the stable limit cycle corresponding to different intervals of w .

Summarize the above, two Hopf bifurcations and four pitchfork bifurcations occur in the system with a single slow parametric excitation. In the next section, we explore the dynamical mechanisms of the mixed-mode oscillations.

4. Analysis of the mixed-mode oscillation mechanism

We overlap the transformed phase portrait with the bifurcation diagram, from which we can obtain the transition mechanism of the trajectory. Moreover, due to the overlap of the non-smooth interface with the equilibrium branches in the x -plane, we introduce the overlap of the transformed phase portrait and equilibrium branches on the (w, x) plane and the (w, z) plane. Combine the two diagrams, we can distinguish the sliding phenomenon at the non-smooth interface, and the movement along the equilibrium point.

Case 1: $A=3.2$

When the amplitude A increases to 3.2, as shown in Fig 4. Assuming when the trajectory starts from the minimum value of w , because of the Hopf bifurcation at the point H^- , it turns to oscillate according to the limit cycle LC_2 , yielding spiking oscillations SP_3 . The trajectory may finally settle down to the stable equilibrium branch EB_2^- and turns to move almost strictly along the branch until the pitchfork bifurcation occurs. Then, the trajectory moves along the stable equilibrium branch EB_6^- until the point M_1 , as shown in Fig5(b). When the trajectory moves to the point M_2 , it enters the subsystem in D_2 region. The trajectory moves along the unstable equilibrium branch EB_5^+ in the (w, z) plane, corresponding to the (w, x) plane it slides along the boundary. When the trajectory oscillates to the point M_3 with the maximum w , affected by the non-smooth sliding bifurcation, it jumps to the equilibrium point. Before the trajectory fully converging to the stable equilibrium branch EB_2^+ , the Hopf bifurcation occurs at the point H^+ , then the trajectory oscillates according to the stable limit cycle LC_1 , resulting in repetitive spiking oscillations SP_1 .

As the w continues to decrease, the homoclinic bifurcation of the limit cycle appears, leading to the trajectory ends the oscillation along the limit cycle. It turns to move along EB_6^- until the point P_3 , where the pitchfork bifurcation occurs, leading to the trajectory moves along the EB_5^- , appearing in quiescent state QS_2 . Then the non-smooth sliding bifurcation occurs, the trajectory jumps to the focal point in the stable equilibrium EB_2^- , yielding spiking oscillations SP_2 . With the w continues to decrease, the trajectory arrives at the starting point, which finishes one period of the bursting oscillations.

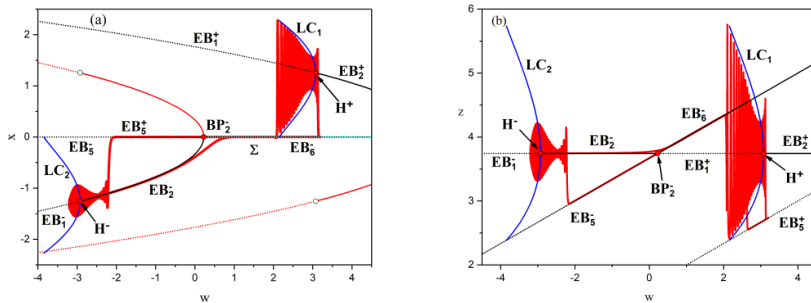


Fig.4 $A=3.2$ Overlap of the transformed phase portrait and equilibrium branches on (w, x) plane (a) and on (w, z) plane (b).

branches on (w, x) plane (a) and on (w, z) plane (b).

From the above analysis, we can find that when the slow variable parameter w visits different multistable coexistence regions, leading to more spiking states and quiescent states appear, which can also affect the structure of mixed mode oscillations. Moreover, due to the existence of the non-smooth partition interface, the trajectory will also be affected by multiple non-smooth bifurcations during the alternations between the spiking states and quiescent states, which makes the mechanism of the mixed-mode oscillation more complicated.

5. Conclusion

For a class of multistable coexisting Lorenz chaotic systems, a periodic external excitation and non-smooth term are introduced to establish a novel non-smooth Filippov system. We studied the bursting oscillation mechanism of non-smooth Filippov systems with two-scale coupling. The system trajectory shows more alternations between quiet states and spiking states in multistable coexistence regions, due to the system trajectory switches between the subsystems in the boundary. In order to analyze the multistable coexistence region and subsystems, here we made the overlap of the transformed phase portrait and equilibrium branches as well as the bifurcations in the (w, x) and (w, z) plane. Based on numerical analysis, trajectory is controlled in turn by the subsystems with sliding movement. We explore the dynamic behaviors of the system on the non-smooth interface, in order to provide a theoretical basis for the study of non-smooth phenomena in practical engineering applications. In addition, the slow passage effect also plays an important role in the bursting oscillations, leading to a new quiescent state trajectory that connecting two spiking states being appear, implying the system trajectory will present more abundant dynamical behaviors. It proves that not only bifurcations can cause the alternations between the spiking states and quiescent states, but also the slow passage effect.

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