Article

Quadrex Algorithm for Negative Definite Quadratic Programming Models

Abstract: In this paper, a quadrex algorithm for quadratic programming problems is introduced (n = 2) under linear and quadratic constraints. The quadrex algorithm considers on the behavior of the quadratic function near the origin or a translate of the origin, performs a series of translations and orthogonal rotations to obtain the optimal solution of the objective function as well as taking considerations on the constraints of the problem. The method works provided that the eigenvalues of the matrix on quadratic form of the objective function is strictly negative, that is, Q is negative-definite. The quadrex algorithm is a parallel counterpart of the simplex algorithm for linear programming models.

Keywords: simplex, quadratic, quadratic programming, quadrex, NP-hard, negative definite.
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1. Introduction

Linear programming (LP) models is one of the simplest ways to perform optimization in many mathematical applications. Its popularity is rooted from its reputation in computing various problems using the simplex algorithm [3]. In mathematical optimization, the simplex algorithm or also known as simplex method is a popular algorithm in dealing with linear programming. The simplex algorithm is performed by examining each of the corner points o'f the convex polyhedron constituting the set of constraints through algebraic manipulation of the simplex tableau. The simplicity of this algorithm is the reason why most non-linear optimization models are linearly done such as the Frank-Wolfe algorithm by considering a linear approximation of the objective function and moves towards a minimizer of the linear function [4].

It is rare to find a simple algebraic algorithm currently that could take the place of simplex method in linear programming for non-linear programs (NLP). Mostly in these cases, an extensive knowledge is needed such as the Kuhn-Tucker conditions or Lagrange multipliers in finding the optimal solution. In 2012, Lacpao et al. [5] introduced a paper on algebraic algorithm called quadrex algorithm for quadratic programming problems under linear and quadratic constraints. This is a quadratic counterpart of the simplex algorithm for linear programming models. The quadrex algorithm centers on the behavior of the quadratic function at the origin and performs a series of translations and orthogonal rotations to obtain the extremum of the objective function. The method works provided that the eigenvalues of the quadratic form part of the objective function is strictly positive or that Q is strictly positive-definite.

In working with quadratic algorithm, we need to have a deep mathematical knowledge such as the Kuhn-Tucker condition and the Lagrange multiplier in obtaining the optimal solution. This present study will develop an algorithm in solving quadratic functions with complexity which is parallel to a simplex method. The method is less non-deterministic polynomial-time hard compared with the most known algorithms at the present. This would probably attract mathematicians working on optimization problems since this method is more efficient and simpler than the existing methods in solving non-linear algorithms.

Quadratic programming is particularly simple when there are only equality constraints; specifically, the problem is linear. By using Lagrange multipliers [13] and seeking the extremum of the Lagrangian, it may be readily shown that the solution to the equality constrained problem is given by the linear system: "

Q E^T
E 0
#
=
"
x
/
#
=
"

□**c**
d
#
(1)

where *l* is a set of Lagrange multipliers which come out of the solution alongside **x**.

Consider a nonlinear programming problem of the form:

Minimize: f (x) =
1
2
$\mathbf{x}^T Q \mathbf{x} + \mathbf{c}^T \mathbf{x}$

Subject to (one or more constraints of the form): **Ax _ b**(inequality constraint)

Ex = **d**(equality constraints)

The Lagrangian for this problem is

L(x, *l*, *s*) = f (x) □ $l^T b(x)$ □ $s^T c(x)$. (2)

where *l* and *s* are Lagrange multipliers.

At an iterate , a basic sequential quadratic programming algorithm defines an appropriate search direction as a solution to the quadratic programming subproblem [13].

Of course, there are other methods for solving (2) but these methods are equally inaccessible to non-mathematicians. Optimization by vector space methods, for instance, rely on embedding the NLP in an appropriate Hilbert space. The solution is found by an application of the Projection Theorem [12]. It is, however, ironic that majority of the users of NLP's are non-mathematicians who apply this programming model to business, economics, and social science problems.

This paper examines the quadratic programming model from an elementary algebra and analytic geometry perspective by developing an algorithm called quadrex.

2. The Quadrex Programming

We restrict our attention to the case where n = 2. The general quadratic function in two variables:

$f(x, y) = ax^2 + by^2 + cxy + dx + ey + f$; b, c 6= 0 (3)

can be written as

$z = f(x, y) = X^T Q X + A^T X^T + K$ (4)

where

Q =
"
a c2
c2
b
#
, $A^T = (d, e)$, X =
"
x
y
#
(5)

We note in passing that if z = m , a constant, then (3) represents a conic section.

If c = d = e = 0, then we obtain:

$f(x, y) = ax^2 + by^2$. (6)

If a and b are both positive, then the origin (0, 0) is a minimum while if a and b are both negative, then (0, 0) is a maximum.

Lemma 1. Let $z = ax^2 + by^2$, then:

(i) the point (0, 0, 0) is a minimum whenever a, b > 0; and

(ii) the point (0, 0, 0) is a maximum whenever a, b < 0.

Proof: Construct a neighborhood around the origin

$N_e(0, 0, 0)$ = fj(x, y, f (x, y))j$x^2 + y^2 < e^2$g.

Let $(x_0, y_0, f(x_0, y_0))$ 2 $N_e(0, 0, 0)$. Suppose that a, b > 0. Without loss of generality, let a = minfa, bg. Then:

$$0 < ax_0^2 + ay_0^2 < ae^2 < ax_0^2 + by_0^2 \quad (7)$$

It follows that $f(x_0, y_0) > 0 = f(0, 0)$ and the origin is therefore a minimum. The proof in the case of a maximum is similar.

3. General Quadratic Function of the form $f(x, y) = ax^2 + by^2 + dx + ey + f$

The effect of adding a linear term to (6) shifts the origin. That is,

$$f(x, y) = ax^2 + by^2 + dx + ey + f \quad (8)$$

$$= a\left(x^2 + \frac{d}{a}x + \frac{d^2}{4a^2}\right) + b\left(y^2 + \frac{e}{b}y + \frac{e^2}{4b^2}\right) + K$$

$$= a\left(x + \frac{d}{2a}\right)^2 + b\left(y + \frac{e}{2b}\right)^2 + K$$

where

$$K = f - \left(\frac{d}{4a} + \frac{e}{4b}\right) \quad (9)$$

Theorem 1. Let $ax^2 + by^2 + dx + ey + f$. Then

(i) if $a, b > 0$, the point

$\left(-\frac{d}{2a}\right.$

,
□e
2b
, f
□□d
2a , □e
2b
__
is a minimum; and
(ii) if a, b < 0, the point
_ d
2a
,
e
2b
, f
□ d
2a , e
2b
__
is a maximum.

Proof. Apply Lemma 1 to the shifted origin.

If a and b have opposite signs, the behavior of the function is quite interesting.

Lemma 2. Let $f(x, y) = ax^2$ □ by^2 such that a and b have opposite signs. Then the origin is neither a maximum nor a minimum point.

Proof. Let $f(x) = ax^2 + by^2$ such that a and b have opposite signs. Then
$f(x) = ax^2$ □ by^2
= (
p
ax +
p
by)(
p
ax □
p
by)
and so, the origin becomes the saddle point where the lines
y =
p
a
p
b
x
and
y = □
p
a
p
b
x
intersect. This proves the claim.

4. General Quadratic Function of the form $f(x, y) = ax^2 + by^2 + cxy + dx + ey + f$

Consider the equation

$f(x, y) = ax^2 + by^2 + cxy + dx + ey + f$ (10)

represented in matrix form

$$f(x, y) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{N}^T\mathbf{x}^T + f \quad (11)$$

where

$$\mathbf{Q} = \begin{bmatrix} a & c2 \\ c2 & b \end{bmatrix}, \mathbf{N} = \begin{bmatrix} d \\ e \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

Since **Q** is symmetric, we may write

$$\mathbf{Q} = \mathbf{O}\mathbf{M}\mathbf{O}^T \text{ or } \mathbf{M} = \mathbf{O}^T\mathbf{Q}\mathbf{O}$$

where

$$\mathbf{M} = \begin{bmatrix} l_1 & 0 \\ 0 & l_2 \end{bmatrix}$$

is a diagonal matrix whose elements are the eigenvalues of **Q** and

$$\mathbf{O} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}$$

is an orthogonal matrix whose columns are the eigenvectors corresponding to the eigenvalues.
Introduce the new variables x_0 and y_0 for x and y variables and let $\mathbf{X}_0$ be

$$\mathbf{X}_0 = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

such that

$$\mathbf{X} = \mathbf{O}\mathbf{X}_0. \quad (12)$$

Then (11) becomes

$$f(x, y) = \mathbf{O}\mathbf{X}_0{}^T\mathbf{Q}\mathbf{X} + \mathbf{N}^T\mathbf{O}\mathbf{X}_0{}^T + \mathbf{f}$$
$$= l_1 x_0^2 + l_2 y_0^2 + h_1 x_0 + h_2 y_0 + f \quad (13)$$

which is the same form in (8).

Theorem 2. Let

$$f(x, y) = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{N}^T\mathbf{x}^T + f.$$

If **Q** is negative definite and $a, b < 0$, then $f(x, y)$ is a maximum obtained by Theorem (1).

Proof. Use orthogonal rotation and translation of axes and apply Theorem 1 and Lemma 1. This proves the result.

5. Constrained Optimization

Consider a maximization problem:

Maximize: $f(x, y) = ax^2 + by^2$, $a, b < 0$

Subject to:
$cx^2 + dy^2 > K, K > 0$ (14)

The point (0, 0, f (x, y)) is a global maximum and is a feasible solution since the point satisfies the constraint.
It is interesting to note in a case when the constraint does not include the origin as shown in the following maximization problem:
Maximize: $f(x, y) = ax^2 + by^2$, $a, b < 0$
Subject to:
$cx^2 + dy^2$ _ $K, K > 0$ (15)
In this case, the origin is not a feasible solution.
Consider the arc traced by
$cx^2 + dy^2 = K$. (16)
We claim that the optimal solution which is feasible lies along this arc. Observe that the arc is an ellipse.
If c > d, the major axis of the ellipse is along the vertical axis while if c < d, the major axis is along the horizontal axis.
Suppose c > d, then the major axis has coordinates (
q
Kc
, 0) and (□
q
Kc
, 0). The minor axis has
coordinates (0,
q
Kd
) and (0,□
q
Kd
). The maximum occurs along the arc at a point closest to the global
(infeasible) minimum (0, 0), namely, (0,
q
Kd
) and (0,□
q
Kd
).
Theorem 3. The optimal solution to
Maximize: $f(x, y) = ax^2 + by^2$, $a, b < 0$
Subject to:
$cx^2 + dy^2$ _ $K, K > 0$ (17)
occurs at the endpoints of the minor axis of (16).
Proof. Choose any point (x_0, y_0) satisfying cx^2
$_0 + dy^2_0$
= K where c < d. The value of f (x, y) at (0,
q
Kd
)
is
f (x, y)_ =
bK
d
(18)
and the value of f (x, y) at (x_0, y_0) is
$f(x_0, y_0) = ax^2$
$_0 + by^2_0$
< by_0 _ b _

K
d
= f (x, y)_. (19)
Therefore, f (x, y)_ > f (x_0, y_0).
The extension of Theorem 3 to several quadratic constraints is obvious. The optimal solution can be found by finding the intersections of the elliptical constraints
$c_i x^2 + d_i y^2$ _ k_i, $k_i > 0$ (20)
and choosing the intersection closest to the origin.
In cases where the objective function has a linear component such as
f (x, y) = $ax^2 + by^2$ + dx + ey + f , (21)
Theorem 1 can easily be determined and the global maximum occurs at
_□d
2a
,
□e
2b
, f
□□d
2a , □e
2b
__
. We
examine the constraints
$c_i x^2 + d_i y^2$ _ k_i, $k_i > 0$
and determine whether the global maximum is feasible or not. Otherwise, take the intersection point of the constraint closest to the global maximum. The search for the closest point of intersection is

facilitated by the fact that it is the one which matches the signs of the components of the global maximum.

6. Illustrations

The following are some numerical illustrations of the quadrex algorithm:
Illustration 1. Obtain the extrema of
1. f (x, y) = □3x^2 □ 2y^2 + 4
2. f (x, y) = x^2 + 2y^2 □ 6x + 4y + 1
3. f (x, y) = □6x^2 + 4xy □ 9y^2 + 6x □ 15y + 2

Solution

1. Note that a = □3 and b = □2 and a, b < 0. It follows that the minimum is at (0, 0, f (0, 0)) = (0, 0, 4).
2. Rewrite f (x, y) = x^2 + 2y^2 □ 6x + 4y + 1, that is,
f (x, y) = x^2 + 2y^2 □ 6x + 4y + 1
= x^2 □ 6x + 9 + 2(y^2 + 2y + 1) + 1 + 9 + 2
= (x □ 3)^2 + 2(y + 1)^2 + 12
Since a = 2, b = 3 are both positive, the maximum is at
_
3,□1, f (3,□1)
_
=
_
3,□1,□10
_
.
3. Rewrite
f (x, y) = □6x^2 + 4xy □ 9y^2 + 6x □ 15y + 2 (22)
in matrix form.
f (x, y) =

$$\begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} -6 & 2 \\ 2 & -9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 6 & -15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 & 0 \end{bmatrix} = \mathbf{x}^T\mathbf{Q}\mathbf{x} + \mathbf{N}^T\mathbf{x} + K \quad (23)$$

where

$$Q = \begin{bmatrix} -6 & 2 \\ 2 & -9 \end{bmatrix}, \quad N^T = (6, -15), \quad X = \begin{bmatrix} x \\ y \end{bmatrix}, \quad K = \begin{bmatrix} 2 \\ 0 \end{bmatrix}. \quad (24)$$

Since Q is symmetric, it follows that Q can be represented as

$$Q = RDR^T \text{ or } D = R^TQR$$

where

$$D = \begin{bmatrix} -5 & 0 \\ 0 & -10 \end{bmatrix} \quad (25)$$

is a diagonal matrix whose elements are the eigenvalues of Q and

$$R = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}$$

5
#
(26)
is an orthogonal matrix whose columns are the eigenvectors corresponding to the eigenvalues.
Let X_0 =
"
x_0
y_0
#
such that
$X = RX_0$. (27)

Then (23) becomes
$f(x, y) = (\mathbf{RX}_0)^T\mathbf{Q}(\mathbf{RX}_0) + \mathbf{NRX}_0 + K$
$= \mathbf{X}_0^T(\mathbf{R}^T\mathbf{QR})\mathbf{X}_0 + \mathbf{NRX}_0 + K$
$= \mathbf{X}_0^T\mathbf{DX}_0 + \mathbf{NRX}_0 + K$
= □5x_0^2 □ 10y_0^2 □
3
p
5
5
x_0 □
36
p
5
5
y_0 + 2
The equation does not contain any cross product so that it can now be translated and is the same form with (8). Applying Theorem 1, x_0 = □3
p
5
50 and y_0 = □9
p
5
25 . The coordinate of the original function can be found using (27). Since a = □6 and b = □9 are both negative, f (x, y) = 857
100 is a
maximum at x = 6
25 and y = □39
50 .

Illustration 2.

Maximize: f (x, y) = □2x^2 □ 3y^2 □ 2x + 3y □ 5
Subject to: □2x^2 □ 3y^2 _ 2

Solution

Since the global maximum (□12
, 12
) is not feasible, we locate the minor axis of the ellipse □2x^2 □
3y^2 = 2. We found that the endpoints are
_
0,
q
23
_
and
_
0,□

q
23

_
. The closest point to (□12
, 12) is

_
0,
q
23

_
. Therefore, the optimal solution is

_
0,
q
23
, 7 □
p
6

_
.

7. Conclusion

The quadrex algorithm proposed in this paper checks the behavior of the quadratic objective function near the origin or a translate of the origin. The optimal solution is the origin if it is feasible. Otherwise, the point of intersection of the constraint set closest to that origin is the optimal solution. The algorithm works whenever the matrix of the quadratic form in the objective function is strictly negative-definite with no positive eigenvalue.

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