ON MOMENT CONVERGENCE RESULT FOR PROPERLY NORMALIZED DELAYED SUMS

Abstract

Let $\{X_n, n \geq 1\}$ beasequenceofindependentandidenticallydistributedrandom variableswithacommondistributionfunction F.Let (S_n) bethepartialsumsequence. Set

$$T_n = S_{n+a_n} - S_n = \sum_{k=n+1}^{n+a_n} X_k$$. ThesumT$_n$iscalleda (forward)delayedsum.The present work aims toobtainamomentconvergenceresultforthedelayedsumswhentherandom variablesarein thedomainofnormalattractionofastablelawwithan index α, $1 < \alpha < 2$.

This result plays a vital role in studying a local limit theorem.

KeyWordsandPhrases: Domainofnormalattraction, stablelaw,delayed sum, Moment convergence, local limit theorem.

1.Introduction

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identicallydistributed (i.i.d)random variables (r.v.s)withacommondistributionfunction(d.f)F.Let (S_n) bethepartialsum sequence. LetFbelongtothedomainofnormalattraction(DNA)ofastablelawwith an index α, $1 < \alpha < 2$..Let$\{A_n, n \geq 1\}$and$\{B_n, n \geq 1\}$bethesequencesof constants. Set

---

*Corresponding author

$Y_n = \frac{S_n}{B_n} - A_n$ and $T_n = S_{n+a_n} - S_n = \sum_{k=n+1}^{n+a_n} X_k$. The sum T_n is called a (forward) delayed sum and (a_n) be a sequence of real constants with $\limsup_{n \to \infty} \frac{a_n}{n} < \infty$.

WhenFbelongstothe DNAofastablelaw with index α, $1 < \alpha < 2$. Owen[8] obtained a moment convergenceresult for (Y_n). Considering the works of Owen [8], we envisage thatthe moment convergence result holds for properly normalized delayed sumswhentherandomvariablesarein the domainofnormalattractionofastablelawwithan index α, $1 < \alpha < 2$. This result plays a vital role in studying a local limit theorem.We follow the methods and methodology of Owen [8].

In the next section, a needed lemmaand the main theorems were presented. Throughout the paper c, with orwithoutasuffix,standforpositiveconstant.

2. Moment Convergent Result

Lemma

There exists positive constants k and l_0, both independent of n, such that, for $t > l_0$,

$$P\left(\left|\frac{T_n}{n^{\frac{1}{\alpha}}}\right| > t\right) \le \frac{k}{t^{\alpha}} \qquad (1)$$

Proof

Forfixedn and t>1, define $Y_i = \begin{cases} X_i, & \text{if } |X_i| < t\, n^r \\ 0, & \text{otherwise} \end{cases}$, where $r = n^{\frac{1}{\alpha}}$. Then we have,

$$Y_i = \begin{cases} X_i, & \text{if } -t\, n^r < X_i < t\, n^r \\ 0, & \text{if } |X_i| \ge t\, n^r \text{ or } X_i \in \left(-\infty, tn^r\right] \cup \left[tn^r, \infty\right) \end{cases}$$

which implies

$$\sum_{i=n+1}^{n+a_n} Y_i = \begin{cases} \sum_{i=n+1}^{n+a_n} X_i, & \text{if } -(a_n-1)t\,n^r < \sum_{i=n+1}^{n+a_n} X_i < (a_n-1)t\,n^r \\ 0, & \text{if } \sum_{i=n+1}^{n+a_n} X_i \le -(a_n-1)t\,n^r \text{ or } \sum_{i=n+1}^{n+a_n} X_i \ge (a_n-1)t\,n^r \end{cases}$$

Let $V_i = X_i - Y_i$, $T_{1,n} = Y_{n+1}+Y_{n+2}+\dots+Y_{n+a_n}$ and $T_{2,n} = V_{n+1}+V_{n+2}+\dots+V_{n+a_n}$.

Observe that

$$\left\{\left|\frac{T_n}{n^r}\right|>t\right\} \Leftrightarrow \left\{\frac{T_n}{n^r} \in \left((-\infty,t]\cup[t,\infty)\right)\right\} \Leftrightarrow \sum_{i=n+1}^{n+a_n} X_i < -t\,n^r \text{ (or) } \sum_{i=n+1}^{n+a_n} X_i > t\,n^r \quad (2)$$

$$\left\{\left|\frac{T_{1,n}}{n^r}\right|>t\right\} = \left\{\frac{T_{1,n}}{n^r}<-t \text{ (or) } \frac{T_{1,n}}{n^r}>t\right\} = \left\{\sum_{i=n+1}^{n+a_n} Y_i < -t\,n^r \text{ (or) } \sum_{i=n+1}^{n+a_n} Y_i > t\,n^r\right\} \quad (3)$$

and $\{T_{2,n} \ne 0\} = \left\{\sum_{i=n+1}^{n+a_n} V_i \ne 0\right\} = \left\{\sum_{i=n+1}^{n+a_n}(X_i - Y_i) \ne 0\right\} = \left\{\sum_{i=n+1}^{n+a_n} X_i \ne \sum_{i=n+1}^{n+a_n} Y_i\right\} = \left\{\sum_{i=n+1}^{n+a_n} Y_i = 0\right\}$

From the definition of Y_i, we have

$$\left\{\sum_{i=n+1}^{n+a_n} Y_i = 0\right\} \Leftrightarrow \left\{\sum_{i=n+1}^{n+a_n} X_i \le -(a_n-1)tn^r \text{ (or) } \sum_{i=n+1}^{n+a_n} X_i > (a_n-1)tn^r\right\} \quad (4)$$

Bythearguments (2),(3) and (4),we getthat

$$\left\{\left|\frac{T_n}{n^r}\right|>t\right\} \subset \left\{\left(\left|\frac{T_{1,n}}{n^r}\right|>t\right) \cup (T_{2,n} \ne 0)\right\} \text{ and hence } P\left\{\left|\frac{T_n}{n^r}\right|>t\right\} \le P\left(\left|\frac{T_{1,n}}{n^r}\right|>t\right) + P(T_{2,n} \ne 0) \quad (5)$$

Consider $P(T_{2,n} \ne 0) = P\left(\sum_{i=n+1}^{n+a_n} V_i \ne 0\right) = a_n P(V_1 \ne 0)$, where V_i's are i.i.dr.v.s

$= a_n P(X_1 - Y_1 \ne 0) = a_n P(Y_1 \ne X_1) = a_n P(Y_1 = 0) = a_n P(|X_1| > t\,n^r)$, by definition of Y_i.

Since X_n 'sareindomainofnormalattractionofastablelawwith index α ,1<α<2,wehave,forsomeconstantc>0suchthat

$$F(x) \le c|x|^{-\alpha}, \text{ if } x<0 \text{ and } 1-F(x) \le cx^{-\alpha}, \text{ if } x>0 \quad (6)$$

Since $\limsup_{n\to\infty} \frac{a_n}{n} < \infty$ and using (6), one can find some constant $c_1 > 0$, such that

$$P\left(T_{2,n} \neq 0\right) = a_n P\left(\left|X_1\right| > tn^r\right) \square \frac{c}{t^\alpha}\frac{a_n}{n} < \frac{c_1}{t^\alpha}. \tag{7}$$

Notice that $E(X_1) = 0$ implies $0 = E(X_1) = \int_{-\infty}^{\infty} x_1 dF(x) = \int_{|x|>tn^r} x_1 dF(x) + \int_{|x|<tn^r} x_1 dF(x)$

$$\Leftrightarrow \int_{|x|<tn^r} x_1 dF(x) = -\int_{|x|>tn^r} x_1 dF(x) \Leftrightarrow \left|E(Y_1)\right| = \left|-\int_{|x|>tn^r} xdF(x)\right| = \int_{|x|>tn^r} xdF(x).$$

ConsideringEY_1 and integrating by parts, we have,

$$EY_1 = \int_{|x|>tn^r} xdF(x) = x\,F\Big|_{|x|>tn^r} - \int_{|x|>tn^r} Fdx = xF\Big|_{-\infty}^{-tn^r} + xF\Big|_{tn^r}^{\infty} - \int_{-\infty}^{-tn^r} Fdx - \int_{tn^r}^{\infty} Fdx$$

Again using (6), there exists some positive constant $c_2(>c_1)$,such that,

$$EY_1 \le c_2\ x\,|x|^{-\alpha}\Big|_{-\infty}^{-tn^r} + x\left(1 - c_2 x^{-\alpha}\right)\Big|_{tn^r}^{\infty} - \int_{-\infty}^{-tn^r} c_2\,|x|^{-\alpha}dx - \int_{tn^r}^{\infty}(1 - c_2 x^{-\alpha})dx$$

$$\le c_2(-tn^r)(tn^r)^{-\alpha}\Big| + \infty + x\Big|_{tn^r}^{\infty} - c_2 x^{-\alpha+1}\Big|_{tn^r}^{\infty} - c_2\left(\frac{(-x)^{-\alpha+1}}{-\alpha+1}\right)\Bigg|_{-\infty}^{tn^r} - \int_{tn^r}^{\infty} dx + c_2\int_{tn^r}^{\infty} x^{-\alpha}dx$$

$$\le c_2\left(tn^r\right)^{-\alpha+1} + \infty + \infty - tn^r - \infty + c_2\left(tn^r\right)^{-\alpha+1} - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} - \infty - x\Big|_{tn^r}^{\infty} + c_2\frac{x^{-\alpha+1}}{-\alpha+1}\Bigg|_{tn^r}^{\infty}$$

$$\le c_2\left(tn^r\right)^{-\alpha+1} - tn^r + c_2\left(tn^r\right)^{-\alpha+1} - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} - \infty + tn^r + \infty - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1}$$

$$\le c_2\left(tn^r\right)^{-\alpha+1} + c_2\left(tn^r\right)^{-\alpha+1} - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1}$$, whichimplies,

$$\left|EY_1\right| \le 2\,c_2\left(tn^r\right)^{-\alpha+1} + 2\,c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} \le \frac{2c_2}{t^{\alpha-1}n^{1-r}}\left(1 + \frac{1}{\alpha-1}\right) \le \frac{c_3\alpha}{(\alpha-1)t^{\alpha-1}n^{1-r}}$$, where $c_3(>c_2)$ is some

positive constant. Observe that

$$\left|E\left(\frac{T_{1,n}}{n^r}\right)\right| = \left|\frac{1}{n^r}E\left[Y_{n+1} + Y_{n+2} + ... + Y_{n+a_n}\right]\right| \le \frac{1}{n^r}\left[\left|EY_{n+1}\right| + \left|EY_{n+2}\right| + ... + \left|EY_{n+a_n}\right|\right] \le \frac{1}{n^r}\left[\frac{c_3\ n\ \alpha}{(\alpha-1)t^{\alpha-1}n^{1-r}}\right]$$

$$= \frac{c_3\alpha}{(\alpha-1)t^{\alpha-1}}.$$

Since $\alpha>1$ and $t>1$, we must have $\left|E\left(\frac{T_{1,n}}{n^r}\right)\right| \le \frac{c_3\alpha}{\alpha-1}$. Let $B = \frac{c_3\alpha}{\alpha-1}$ and $s=1+B$. Henceforth, we take $t>s$. Then,

$$P\left(\left|\frac{T_{1,n}}{n^r}\right| > t\right) \le P\left(\left|\frac{T_{1,n}}{n^r} - \frac{EY_1}{n^{1-r}}\right| > t - B\right) \tag{8}$$

From Chebyshev's inequality, we get, $P\left(\left|\frac{T_{1,n}}{n^r} - \frac{EY_1}{n^{1-r}}\right| > t - B\right) \le \frac{V\left(\frac{T_{1,n}}{n^r}\right)}{(t-B)^2}$. (9)

Observe that $V\left(\frac{T_{1,n}}{n^r}\right) = \frac{1}{n^{2r}}V(T_{1,n}) = \frac{n}{n^{2r}}V(Y_1) = n^{1-2r}V(Y_1)$.

Since $V(Y_1) \le E(Y_1^2) = \int_{|y|\le tn^r} y^2 dF(y) = \int_{-tn^r}^{tn^r} y^2 dF(y)$. Using integration by parts, we get,

$$V(Y_1) \le y^2F(y)\Big|_{-tn^r}^{tn^r} - \int_{-tn^r}^{tn^r} F(y)dy^2 \le y^2F(y)\Big|_{-tn^r}^{0} + y^2F(y)\Big|_{0}^{tn^r} - \int_{-tn^r}^{0} 2yF(y)dy - \int_{0}^{tn^r} 2yF(y)dy.$$

From the fact (6), we have for some constant $c_4>0$,

$$V(Y_1) \le y^2c_4y^{-\alpha}\Big|_{-tn^r}^{0} + y^2(1 - c_4 y^{-\alpha})\Big|_{0}^{tn^r} - 2\int_{-tn^r}^{0} y\, c_4\, y^{-\alpha}dy - 2\int_{0}^{tn^r} y(1 - c_4y^{-\alpha})dy$$

$$\le 0 - c_4(tn^r)^2(tn^r)^{-\alpha} + (tn^r)^2 - c_4(tn^r)^2(tn^r)^{-\alpha} - 2c_4\left.\frac{y^{-\alpha+2}}{-\alpha+2}\right|_{-tn^r}^{0} - 2\left.\left(\frac{y^2}{2}\right)\right|_{0}^{tn^r} + 2c_4\left.\left(\frac{y^{-\alpha+2}}{-\alpha+2}\right)\right|_{0}^{tn^r}$$

$$\le -2c_4(tn^r)^{2-\alpha} + (tn^r)^2 - \frac{2c_4}{-\alpha+2}\left[0 - (tn^r)^{2-\alpha}\right] - (tn^r)^2 + \frac{2c_4}{-\alpha+2}\left[(tn^r)^{2-\alpha}\right]$$

$$\le -2c_4(tn^r)^{2-\alpha} + \frac{2c_4}{-\alpha+2}(tn^r)^{2-\alpha} + \frac{2c_4}{-\alpha+2}(tn^r)^{2-\alpha} = -2c_4(tn^r)^{2-\alpha} + \frac{4c_4}{-\alpha+2}(tn^r)^{2-\alpha}$$

$$\le -2c_4(tn^r)^{2-\alpha}\left[\frac{2}{2-\alpha}-1\right] = \frac{2\,\alpha\, c_4}{2-\alpha}(tn^r)^{2-\alpha}.$$

There exists a constant $c_5 > 0$, independent of n, such that $V(Y_1) \le c_5(tn^r)^{2-\alpha}$.

$V\left(\frac{T_{1,n}}{n^r}\right) = n^{1-2r}V(Y_1) \le n^{1-2r}c_5(tn^r)^{2-\alpha} \le c_5 t^{2-\alpha}$. From (8) and (9), we get,

$$P\left(\left|\frac{T_{1,n}}{n^r}\right| > t\right) \le c_5(1-B)^{-2}t^{2-\alpha} \le t^{-2}c_5\left(1-\frac{B}{t}\right)^{-2} t^{2-\alpha} \le c_5\left(1-\frac{B}{t}\right)^{-2} t^{-\alpha} \le c_5 (1-B)^{-2} t^{-\alpha} \quad (10)$$

If we let $k=c_5(1-B)^{-2}+c_1$, then (7) and (10) give the result.

Theorem1

For each real number q with $0 \le q < \alpha$, there exists a finite positive real number Q, depending on q but independent of n, such that $E\left(\left|\frac{T_n}{n^r}\right|^q\right) \le Q$

(11)

In particular, there exists a constant M, independent of n, such that $E(|T_n|) \le Mn^r$ (12)

Proof

We can observe that the result is true for q=0. Choose q such that $0<q<\alpha$. Let N_0 and l_0 as in Lemma1. Then $E\left(\left|\frac{T_n}{n^r}\right|^q\right) = \int_0^{l_0} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) + \int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right)$.

Now $\int_0^{l_0} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) \le l_0^q$ and the next integral gets,

$$\int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) = \int_{l_0}^{\infty} x^q d\left(1-P\left(\left|\frac{T_n}{n^r}\right| > x\right)\right) = -\int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| > x\right)$$

$$= -x^q P\left(\left|\frac{T_n}{n^r}\right| > x\right)\Bigg|_{l_0}^{\infty} + q\int_{l_0}^{\infty} x^{q-1}P\left(\left|\frac{T_n}{n^r}\right| > x\right)dx = -\infty + l_0^q P\left(\left|\frac{T_n}{n^r}\right| > l_0\right) + q\int_{l_0}^{\infty} x^{q-1}\frac{c_2}{x^{\alpha}}dx$$, by Lemma1,

$$= -\infty + l_0^q P\left(\left|\frac{T_n}{n^r}\right| > l_0\right) + qc_2\int_{l_0}^{\infty} x^{q-1-\alpha}dx = -\infty + l_0^q P\left(\left|\frac{T_n}{n^r}\right| > l_0\right) + qc_2 \frac{x^{q-\alpha}}{q-\alpha}\Bigg|_{l_0}^{\infty}$$

$$= -\infty + l_0^{\,q}\frac{c_2}{l_0^{\,q}} + \infty + qc_2\frac{l_0^{\,q-\alpha}}{\alpha-q},$$ again,by Lemma1, we have

$$= l_0^{q-\alpha}\left(c_2 + \frac{qc_2}{\alpha-q}\right).$$

Now $\int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) \le l_0^{\,q} + l_0^{q-\alpha}\left(c_2+\frac{qc_2}{\alpha-q}\right)$. Letting $Q = l_0^{\,q} + l_0^{q-\alpha}\left(c_2+\frac{qc_2}{\alpha-q}\right)$ gives(11) and with q=1, equation (12) follows from (11).

Theorem 2

$$\lim_{n\to\infty} E\left(\frac{T_n}{n^r}\right) = E(Y) \tag{13}$$

Moreover, for all q with 0≤q<α, we have, $\lim_{n\to\infty} E\left(\left|\frac{T_n}{n^r}\right|^q\right) = E\left(|Y|^q\right)$ (14)

Proof

Choose p such that 0<q<p<α. Let u= $u = \frac{p}{q}$ and let G(t)=|t|$^\alpha$. Then $\frac{G(t)}{t} \to \infty$, as $t \to \infty$. By the convergence theorem, Loeve[5], page 183, we know that for (3) to hold $\left|\frac{T_n}{n^r}\right|$ should be uniformly integrable. Also, Theorem 22, Meyer, Paul[6], shows that $\left|\frac{T_n}{n^r}\right|^q$ is uniformly integrable, whenever $\sup_n E\left(G\left(\left|\frac{T_n}{n^r}\right|^q\right)\right) < \infty$. But observe that $E\left(G\left(\left|\frac{T_n}{n^r}\right|^q\right)\right) = E\left(\left|\frac{T_n}{n^r}\right|^q\right)$. From Theorem 1, we have $\sup_n E\left(\left|\frac{T_n}{n^r}\right|^q\right) \le Q$. Hence $\left|\frac{T_n}{n^r}\right|^q$ is uniformly integrable and in turn (13) is established.

Inparticular,whenq=1,byobservingthattheuniformintegrabilityof$|T_n|$ impliesthatof$T_n$andagainappealingtotheconvergencetheorem,page183,Loeve [6], (14)getsestablished.

3. Conclusion

In this work, we established amomentconvergenceresultforthedelayedsums whentherandomvariablesarein thedomainofnormalattractionofastablelawwithan index $\alpha, 1 < \alpha < 2.$ This result plays a vital role in studying a local limit theorem. See SujitK. Basu [7] and GootyDivanji and TadewasKoroto[2].

Acknowledgments

The authors sincerely thank the three referees for their positive responses and valuable commentson this work. Also expresses their gratitude to the publishers for their invitation to contribute an article.

Competing interests

We declare that no competing interests exist.

References

[1] Feller W(1996): An Introduction to Probability Theory and ItsApplications. 4thEdn.,NewYork,Wiley.

[2] GootyDivanji and TadewosKoroto(2004):A density version of local limit theorems for weighted sums, stochastic modelling& applications, Vol.7, No.1, 1-9.

[3] Gnedenko,B.V. and Kolmogorov,A.N.(1954):Limit distributionsfor sums of independent random variables. Addison-Wesley, Cam-bridge,Mass.

[4] Lai,T.L(1973):Limittheoremsfordelayedsums.Ann.Probab.2,432-440.

[5] LoeveM(1955),ProbabilityTheory,VonNostrand,Princeton.

[6] MeyerP.A(1966),ProbabilityandPotentials,Blaisdell:Waltham.

[7] SujitK.Basu (1996):On a local limit theorem concerning variables in the the domain of normal attraction of a stable law of index α, 1<α<2, The Annals of Probability, Vol.4, pages 486-489.

[8] Willies L. Owen (1973): An estimate for $E(S_n)$ for variables in thedomain ofnormal attraction of a Stable law of index α, $1 < \alpha < 2$,Ann.Probab.Vol.1,1071-1073.