A New Generalized Gamma Function and Its Properties

Abstract

In this work, we introduce a new generalized Gamma function, which is named as p-v-Gamma function and provide some properties generalizing those satisfied by the classical Gamma function. We also give some convexity and monotonicity properties. Furthermore, we establish some inequalities related to this new function.

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1 Introduction

The classical Euler's Gamma function $\Gamma(x)$ is defined for $x > 0$ as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt = \lim_{p\to\infty} \frac{p!p^x}{x(x+1)\dots(x+p)}.$$

This function plays central roles in the theory of special functions and have lots of generalizations. There seems to be so much study of literature. Although it is not possible to list all of these papers, we can give some of them to the readers, [2], [4], [6], [7], [8], [9], [10], [11], [13], [15], [17], [18] and the references therein.

Euler, gave an equivalent definition for the function Γ (known as p-analogue of the Gamma function) as

$$\Gamma_p(x) = \frac{p!p^x}{x(x+1)\dots(x+p)}, \quad x > 0$$

where p is a positive integer and

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x),$$

[1]. It satisfies the identities:

$$\Gamma_p(x+1) = \frac{px}{x+p+1}\Gamma_p(x),$$

$$\Gamma_p(1) = \frac{p}{p+1}.$$

In [12], the authors define the p-analogue of the psi function as the logarithmic derivative of the function Γ_p, that is

$$\psi_p(x) = \frac{d}{dx}\ln\Gamma_p(x) = \frac{\Gamma_p'(x)}{\Gamma_p(x)}.$$

and they gave the series representation of the ψ_p with the relation:

$$\psi_p(x) = \ln p - \sum_{n=0}^{p}\frac{1}{x+n}.$$

Also, the authors in [3] introduced a new one-parameter deformation of the classical Gamma function, called v-analogue (v-deformation or v-generalization) of the Gamma function for $x, v > 0$ as

$$\Gamma_v(x) = \int_0^\infty \left(\frac{t}{v}\right)^{\frac{x}{v}-1} e^{-t}dt. \tag{1.1}$$

Note that when $v = 1$, $\Gamma_v(x) = \Gamma(x)$. They also gived the relation

$$\Gamma_v(x) = \lim_{n\to\infty}\frac{n!\left(\frac{n}{v}\right)^{\frac{x}{v}} v^{n+2}}{x(x+v)(x+2v)\dots(x+nv)}. \tag{1.2}$$

Definition 1. *i.* A function $f : (a, b) \to \mathbb{R}$ is said to be convex if

$$f(\alpha x + \beta y) \le \alpha f(x) + \beta f(y), \tag{1.3}$$

ii. A function $f : (a, b) \to \mathbb{R}$ is said to be concave if the inequality (1.3) is reversed,
iii. A function $f : (a, b) \to \mathbb{R}^+$ is said to be logarithmically convex if

$$\log f(\alpha x + \beta y) \le \alpha \log f(x) + \beta \log f(y)$$

for all $x, y \in (a, b)$ and $\alpha, \beta > 0$ such that $\alpha + \beta = 1$.
iv. A function f is said to be totally monotone if f is continuous on $[0, \infty)$, infinitely differentiable on $(0, \infty)$ and satisfies the condition that

$$(-1)^n f^{(n)}(x) \ge 0, \ n = 0, 1, 2, \dots, x > 0.$$

Throughout this work, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of real numbers, positive real numbers and natural numbers respectively.

The main purpose of this paper is to introduce a new generalized Gamma function $\Gamma_{p,v}$, called the p-v-Gamma function. Our motivation to introduce this new function comes from a natural question if a similar definition of Γ can be given for the function Γ_v. It contributes to giving some generalized properties. We establish recurrent relations for $\Gamma_{p,v}$ in Lemma 1 and Lemma 2. Also, we give the convexity property by Theorem 2. After defining the p-v-psi and p-v-polygamma functions we continue to give series representations, monotonicity properties, and some inequalities involving these new functions.

2 Main Results

We begin this section by presenting a new generalized Gamma function as follows:

Definition 2. Let $x, v > 0$ and $p \in \mathbb{N}$. Then the p-v-Gamma function (also called p-v-analogue or p-v-generalization of the Gamma function) is defined as

$$\Gamma_{p,v}(x) = \frac{p!\left(\frac{p}{v}\right)^{\frac{x}{v}} v^{p+2}}{x(x+v)(x+2v)\dots(x+pv)}. \tag{2.1}$$

Note that $\Gamma_{p,v}(x) \to \Gamma_v(x)$ as $p \to \infty$.

Lemma 1. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then the function $\Gamma_{p,v}$ satisfies the identities:*

$$i. \quad \Gamma_{p,v}(x+v) = \frac{xp}{v(x+pv+v)}\Gamma_{p,v}(x), \tag{2.2}$$

$$ii. \quad \Gamma_{p,v}(v) = \frac{p}{p+1}. \tag{2.3}$$

Proof. The result follows immediately by the equation (2.1). □

Also, note that $\Gamma_{p,v}$ satisfies the following commutative diagram:

$$\begin{array}{ccc} \Gamma_{p,v} & \xrightarrow{p\to\infty} & \Gamma_v \\ {\scriptstyle v=1}\downarrow & & \downarrow{\scriptstyle v=1} \\ \Gamma_p & \xrightarrow{p\to\infty} & \Gamma \end{array}$$

Now, we give a recurrent relation for $\Gamma_{p,v}$ which is also a generalization of (2.2).

Lemma 2. *Let $x, v > 0$ and $p, n \in \mathbb{N}$. Then the function $\Gamma_{p,v}$ satisfies the relation:*

$$\Gamma_{p,v}(x+nv) = \left(\frac{p}{v}\right)^n \left(\prod_{i=0}^{n-1} \frac{x+iv}{x+(p+i+1)v}\right) \Gamma_{p,v}(x). \tag{2.4}$$

Proof. By (2.1) we have,

$$\begin{aligned} \frac{\Gamma_{p,v}(x+nv)}{\Gamma_{p,v}(x+nv-v)} &= \frac{p}{v}\,\frac{(x+nv-v)(x+nv)(x+nv+v)\dots(x+nv+(p-1)v)}{(x+nv)(x+nv+v)(x+nv+2v)\dots(x+nv+pv)} \\ &= \frac{p}{v}\,\frac{(x+nv-v)}{(x+nv+pv)}. \end{aligned}$$

Then,

$$\Gamma_{p,v}(x+nv) = \frac{p}{v}\,\frac{(x+nv-v)}{(x+nv+pv)}\Gamma_{p,v}(x+nv-v).$$

In a similar way, we have

$$\Gamma_{p,v}(x+nv-v) = \frac{p}{v}\,\frac{(x+nv-2v)}{(x+nv+(p-1)v)}\Gamma_{p,v}(x+nv-2v).$$

Then we have

$$\begin{aligned} \Gamma_{p,v}(x+nv) &= \frac{p}{v}\,\frac{(x+nv-v)}{(x+nv+pv)}\Gamma_{p,v}(x+nv-v) \\ &= \left(\frac{p}{v}\right)^2 \frac{(x+nv-v)}{(x+nv+pv)}\,\frac{(x+nv-2v)}{(x+(n-1)v+pv)}\Gamma_{p,v}(x+nv-2v). \end{aligned}$$

Continuing in this way, we obtain

$$\begin{aligned} \Gamma_{p,v}(x+nv) &= \left(\frac{p}{v}\right)^n \frac{(x+nv-v)(x+nv-2v)\dots(x+nv-nv)}{(x+nv+pv)(x+(n-1)v+pv)\dots(x+(n-(n-1)v)+pv)}\Gamma_{p,v}(x) \\ &= \left(\frac{p}{v}\right)^n \frac{(x+(n-1)v)(x+(n-2)v)\dots x}{(x+(p+n)v)(x+(p+n-1)v)\dots(x+(p+1)v)}\Gamma_{p,v}(x) \\ &= \left(\frac{p}{v}\right)^n \left(\prod_{i=0}^{n-1} \frac{x+iv}{x+(p+i+1)v}\right)\Gamma_{p,v}(x), \end{aligned}$$

and the result follows. □

Remark 1. Lemma 2 generalizes Lemma 2.1 of [12].

Note that, when taking the limit of both sides of the equation (2.4) as $p \to \infty$, we obtain that

$$\lim_{p\to\infty} \Gamma_{p,v}(x+nv) = \lim_{p\to\infty} \left(\frac{p}{v}\right)^{n} \frac{\prod_{i=0}^{n-1}(x+iv)}{\prod_{i=0}^{n}(x+(p+i)v)} \Gamma_{p,v}(x).$$

Hence we get

$$\Gamma_v(x+nv) = \frac{1}{v^{2n}} \prod_{i=0}^{n-1}(x+iv)\Gamma_v(x)$$

or equivalently

$$\Gamma_v(x) = v^{2n} \frac{\Gamma_v(x+nv)}{x(x+v)\ldots(x+(n-1)v)}.$$

Theorem 1. *Let $x, v > 0$, $r > 1$ and $p \in \mathbb{N}$. Then, the inequalitiy*

$$\Gamma_{p,v}(rx) < \left(\frac{p}{v}\right)^{\frac{rx-x}{v}} \Gamma_{p,v}(x) \tag{2.5}$$

is valid.

Proof. Using the definition of $\Gamma_{p,v}$ we get

$$\frac{\Gamma_{p,v}(rx)}{\Gamma_{p,v}(x)} = \frac{p!\left(\frac{p}{v}\right)^{\frac{rx}{v}} v^{p+2}\, x(x+v)\ldots(x+pv)}{p!\left(\frac{p}{v}\right)^{\frac{x}{v}} v^{p+2}\, rx(rx+v)\ldots(rx+pv)} < \left(\frac{p}{v}\right)^{\frac{rx}{v}-\frac{x}{v}},$$

and the result follows. □

Theorem 2. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the function $\Gamma_{p,v}$ is convex.*

Proof. We have to prove that

$$\Gamma_{p,v}(\alpha x + \beta y) \le [\Gamma_{p,v}(x)]^{\alpha} [\Gamma_{p,v}(y)]^{\beta} \tag{2.6}$$

for all $\alpha, \beta > 0$, $\alpha + \beta = 1$ and $x, y > 0$. Using the concavity of the logarithm function we have

$$x^{\alpha} y^{\beta} \le \alpha x + \beta y. \tag{2.7}$$

By this, we obtain

$$\left(v+\frac{x}{k}\right)^{\alpha}\left(v+\frac{y}{k}\right)^{\beta} \le \alpha\left(v+\frac{x}{k}\right) + \beta\left(v+\frac{y}{k}\right) = v + \frac{\alpha x+\beta y}{k}$$

for all $k \in \mathbb{N}$. Then we have

$$\left(v+\frac{x}{1}\right)^{\alpha}\left(v+\frac{x}{2}\right)^{\alpha}\cdots\left(v+\frac{x}{p}\right)^{\alpha}\left(v+\frac{y}{1}\right)^{\beta}\left(v+\frac{y}{2}\right)^{\beta}\cdots\left(v+\frac{y}{p}\right)^{\beta}$$
$$\le \left(v+\frac{\alpha x+\beta y}{1}\right)\left(v+\frac{\alpha x+\beta y}{2}\right)\cdots\left(v+\frac{\alpha x+\beta y}{p}\right).$$

By using the equation (2.1) and inequality (2.7) we can write

$$\Gamma_{p,v}(\alpha x+\beta y) = \frac{\left(\frac{p}{v}\right)^{\frac{\alpha x+\beta y}{v}} v^{p+2}}{(\alpha x+\beta y)\left(\frac{\alpha x+\beta y}{1}+v\right)\cdots\left(\frac{\alpha x+\beta y}{p}+v\right)}$$

$$\leq \left(\frac{p}{v}\right)^{\frac{\alpha x+\beta y}{v}} v^{p+2} \frac{1}{x^{\alpha} y^{\beta}} \frac{1}{(v+x)^{\alpha}\left(v+\frac{x}{2}\right)^{\alpha}\cdots\left(v+\frac{x}{p}\right)^{\alpha}(v+y)^{\beta}\left(v+\frac{y}{2}\right)^{\beta}\cdots\left(v+\frac{y}{p}\right)^{\beta}}$$

$$\leq \left(\frac{p}{v}\right)^{\frac{\alpha x+\beta y}{v}} v^{p+2} \frac{1}{x^{\alpha} y^{\beta}} \frac{1}{(v+x)^{\alpha}\left(v+\frac{x}{2}\right)^{\alpha}\cdots\left(v+\frac{x}{p}\right)^{\alpha}(v+y)^{\beta}\left(v+\frac{y}{2}\right)^{\beta}\cdots\left(v+\frac{y}{p}\right)^{\beta}} \qquad (2.8)$$

On the other hand, we have

$$[\Gamma_{p,v}(x)]^{\alpha}\,[\Gamma_{p,v}(y)]^{\beta} = \frac{\left(\frac{p}{v}\right)^{\frac{\alpha x}{v}} v^{\alpha(p+2)}}{x^{\alpha}(x+v)^{\alpha}\left(\frac{x}{2}+v\right)^{\alpha}\cdots\left(\frac{x}{p}+v\right)^{\alpha}} \cdot \frac{\left(\frac{p}{v}\right)^{\frac{\beta y}{v}} v^{\beta(p+2)}}{y^{\beta}(y+v)^{\beta}\left(\frac{y}{2}+v\right)^{\beta}\cdots\left(\frac{y}{p}+v\right)^{\beta}}$$

$$= \left(\frac{p}{v}\right)^{\frac{\alpha x+\beta y}{v}} v^{p+2} \frac{1}{x^{\alpha}(x+v)^{\alpha}\left(\frac{x}{2}+v\right)^{\alpha}\cdots\left(\frac{x}{p}+v\right)^{\alpha} y^{\beta}(y+v)^{\beta}\left(\frac{y}{2}+v\right)^{\beta}\cdots\left(\frac{y}{p}+v\right)^{\beta}}. \qquad (2.9)$$

Then by the equations (2.8) and (2.9) we get

$$\Gamma_{p,v}(\alpha x+\beta y) \leq [\Gamma_{p,v}(x)]^{\alpha}\,[\Gamma_{p,v}(y)]^{\beta},$$

and since every logarithmically convex function is also convex [16], the result follows.

□

Corollary 1. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the inequalitiy*

$$\Gamma_{p,v}(x+y) \leq \left(\frac{p}{v}\right)^{\frac{x+y}{2v}} \sqrt{\Gamma_{p,v}(x)\Gamma_{p,v}(y)} \qquad (2.10)$$

is valid.

Proof. By letting $\alpha = \beta = \frac{1}{2}$, writing $2x$ instead of x and $2y$ instead of y in the equation (2.6) we have

$$\Gamma_{p,v}(x+y) \leq \sqrt{\Gamma_{p,v}(2x)\Gamma_{p,v}(2y)}.$$

Now, using the inequality (2.5) for $r = 2$ we get

$$\Gamma_{p,v}(x+y) \leq \sqrt{\left(\frac{p}{v}\right)^{\frac{x}{v}} \Gamma_{p,v}(x) \left(\frac{p}{v}\right)^{\frac{y}{v}} \Gamma_{p,v}(y)} = \left(\frac{p}{v}\right)^{\frac{x+y}{2v}} \sqrt{\Gamma_{p,v}(x)\Gamma_{p,v}(y)},$$

and the result follows. □

We define the p-v-psi (p-v-digamma) function as the logarithmic derivative of the function $\Gamma_{p,v}$; that is

$$\psi_{p,v}(x) = \frac{d}{dx}\ln\Gamma_{p,v}(x) = \frac{\Gamma'_{p,v}(x)}{\Gamma_{p,v}(x)}.$$

By using the equation (2.4) we get the following recurrent relation for $\psi_{p,v}$.

Corollary 2. *Let $x, v > 0$ and $p, n \in \mathbb{N}$. Then the function $\psi_{p,v}$ satisfies the relation:*

$$\psi_{p,v}(x+nv) - \psi_{p,v}(x) = \sum_{i=0}^{n-1} \frac{pv+v}{(x+iv)(x+(p+i+1)v)}. \qquad (2.11)$$

Proposition 1. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the function $\psi_{p,v}$ has the series representation:*

$$\psi_{p,v}(x) = \frac{1}{v}\ln\frac{p}{v} - \sum_{n=0}^{p} \frac{1}{x+nv}. \qquad (2.12)$$

Proof. By using the equation (2.1) we can write

$$\Gamma_{p,v}(x) = \frac{\left(\frac{p}{v}\right)^{\frac{x}{v}} v^{p+2}}{x(x+v)\left(\frac{x}{2}+v\right)\cdots\left(\frac{x}{p}+v\right)}. \tag{2.13}$$

Then,

$$\ln \Gamma_{p,v}(x) = \frac{x}{v}\ln\frac{p}{v} + (p+2)\ln v - \left\{\ln x + \ln(x+v) + \ln\left(\frac{x}{2}+v\right) \ldots \ln\left(\frac{x}{p}+v\right)\right\}.$$

Now, by differentiating both sides of the last equation with respect to x we get

$$\begin{aligned}\psi_{p,v}(x) &= \frac{1}{v}\ln\frac{p}{v} - \left(\frac{1}{x} + \frac{1}{x+v} + \frac{\frac{1}{2}}{\frac{x}{2}+v} + \ldots + \frac{\frac{1}{p}}{\frac{x}{p}+v}\right)\\ &= \frac{1}{v}\ln\frac{p}{v} - \sum_{n=0}^{p}\frac{1}{x+nv},\end{aligned}$$

and the result follows. □

The following proposition is given in [3]:

Proposition 2. *Let $x, v > 0$. Then the v-digamma function ψ_v has the series representation:*

$$\Psi_v(x) = -\frac{\ln v + \gamma}{v} - \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{nv} - \frac{1}{x+nv}\right). \tag{2.14}$$

Note that, γ is the Euler-Mascheroni constant in the proposition 2:

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right).$$

Theorem 3. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the function $\psi_{p,v}$ satisfies the limit property:*

$$\lim_{p\to\infty}\psi_{p,v}(x) = \psi_v(x).$$

Proof. By the equation (2.12) and the Proposition 2 we have

$$\begin{aligned}\lim_{p\to\infty}\psi_{p,v}(x) &= \lim_{p\to\infty}\left[\frac{1}{v}\ln\frac{p}{v} - \frac{1}{x} - \sum_{n=1}^{p}\frac{1}{x+nv}\right]\\ &= \lim_{p\to\infty}\left[\frac{1}{v}\left(\ln p - \ln v - \sum_{n=1}^{p}\frac{1}{n}\right) - \frac{1}{x} - \left(\sum_{n=1}^{p}\frac{1}{x+nv} - \sum_{n=1}^{p}\frac{1}{nv}\right)\right]\\ &= \frac{1}{v}\lim_{p\to\infty}\left(\ln p - \sum_{n=1}^{p}\frac{1}{n}\right) - \frac{\ln v}{v} - \frac{1}{x} - \lim_{p\to\infty}\left(\sum_{n=1}^{p}\frac{1}{x+nv} - \sum_{n=1}^{p}\frac{1}{nv}\right)\\ &= -\frac{1}{v}\gamma - \frac{\ln v}{v} - \frac{1}{x} + \lim_{p\to\infty}\sum_{n=1}^{p}\left\{\frac{1}{nv} - \frac{1}{x+nv}\right\}\\ &= \psi_v(x),\end{aligned}$$

and the result follows. □

By using the equation (2.12) we get the following.

Corollary 3. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the following identities hold:*

$$i. \quad \psi_{p,v}(v) = \frac{1}{v}\left[\ln\frac{p}{v} - H_{p+1}\right]$$

where H_p is the p-th harmonic number, i.e. the sum of the reciprocals of the first p natural numbers:

$$H_p = 1 + \frac{1}{2} + \ldots + \frac{1}{p} = \sum_{n=1}^{p}\frac{1}{n},$$

ii. The function $\psi_{p,v}$ is increasing on $(0, \infty)$.
iii. The function $\psi_{p,v}$ is totally monotone on $(0, \infty)$.

Note that, for an alternative proof of convexity of the function $\Gamma_{p,v}$, we can use monotonicity. Since $\psi_{p,v}$ is increasing on $(0, \infty)$, we have $\psi'_{p,v}(x) > 0$ for $x > 0$. Then $(\ln \Gamma_{p,v}(x))'' > 0$, i.e. $\Gamma_{p,v}$ is logarithmically convex. It follows that $\Gamma_{p,v}$ is convex. Also, since every totally monotone function f is logarithmically convex [5, 14], we get that the function $\psi_{p,v}$ is logarithmically convex and so it is convex.

Now, we define the p-v-polygamma function of order m as the $(m+1)$-th derivative of the logarithm of the function $\Gamma_{p,v}$ as

$$\psi^{(m)}_{p,v}(x) = \frac{d^m}{dx^m}\psi_{p,v}(x) = \frac{d^{m+1}}{dx^{m+1}}\ln\Gamma_{p,v}(x).$$

Thus

$$\psi^{(0)}_{p,v}(x) = \psi_{p,v}(x).$$

Proposition 3. *Let $x, v > 0$ and $p, m \in \mathbb{N}$. Then the function $\psi^{(m)}_{p,v}$ has the series representation:*

$$\Psi^{(m)}_{p,v}(x) = \sum_{n=0}^{p}\frac{(-1)^{m+1}m!}{(x+nv)^{m+1}}. \tag{2.15}$$

Proof. By differentiating m times of the equation (2.12) with respect to x we get the result. □

Note that

$$\lim_{p\to\infty}\psi^{(m)}_{p,v}(x) = \psi^{(m)}_{v}(x),$$

where $\psi^{(m)}_v$ is the v-polygamma function of order m and has the series representation,

$$\Psi^{(m)}_v(x) = (-1)^{m+1}m!\sum_{n=0}^{\infty}\frac{1}{(x+nv)^{m+1}}.$$

given in [3]. By using the equation (2.15) we get the following.

Corollary 4. *Let $x, v > 0$ and $p, m \in \mathbb{N}$. Then, the following identities hold:*
i. The function $\psi^{(m)}_{p,v}$ is positive and decreasing on $(0, \infty)$ if m is odd.
ii. The function $\psi^{(m)}_{p,v}$ is negative and increasing on $(0, \infty)$ if m is even.

Finally, as an application to the definition of $\psi_{p,v}$ we give the following theorems.

Theorem 4. *The following inequalities are valid for $x, v > 0$ and $p \in \mathbb{N}$:*

$$\frac{1}{v}\ln\left(\frac{px}{v(x+pv+v)}\right) - \frac{pv+v}{x(x+pv+v)} < \psi_{p,v}(x) < \frac{1}{v}\ln\left(\frac{px}{v(x+pv+v)}\right).$$

Proof. If we apply the mean value theorem to the function

$$f(x) = \ln \Gamma_{p,v}(x)$$

on $(x, x+v)$, then there is a point $x_0 \in (x, x+v)$ such that

$$\ln \Gamma_{p,v}(x+v) - \ln \Gamma_{p,v}(x) = v\psi_{p,v}(x_0).$$

By the monotonicity of the function $\psi_{p,v}$ on $(0, \infty)$ we get

$$\psi_{p,v}(x) < \psi_{p,v}(x_0) < \psi_{p,v}(x+v) \tag{2.16}$$

and also by (2.2) we get $\psi_{p,v}(x_0) = \frac{1}{v}\ln\frac{px}{v(x+pv+v}$. Now using the equation (2.11) for $n = 1$ in the inequalities (2.16) we get

$$\psi_{p,v}(x) < \frac{1}{v}\ln\left(\frac{px}{v(x+pv+v}\right) < \frac{pv+v}{x(x+pv+v)} + \psi_{p,v}(x),$$

and the result follows.

□

Theorem 5. *Let $x, v > 0$ and $p \in \mathbb{N}$. Then, the function*

$$x \to x\psi_{p,v}(x)$$

is convex.

Proof. We have

$$(x\Psi_{p,v}(x))'' = 2\psi'_{p,v}(x) + x\psi''_{p,v}(x).$$

Then by using the equation (2.15) we have

$$(x\Psi_{p,v}(x))'' = 2\sum_{n=0}^{p}\frac{1}{(x+nv)^2} - x\sum_{n=0}^{p}\frac{2}{(x+nv)^3} = \sum_{n=0}^{p}\frac{2nv}{(x+nv)^3} \geq 0,$$

and the proof is completed. □

Theorem 6. *Let $x \in [0, 1]$, $v > 0$, $p, m \in \mathbb{N}$ and a, b ($a \geq b$) be positive real numbers. Also let c, d be positive real numbers such that $0 < cb^{m+1} \leq da^{m+1}$. Then the function*

$$x \to \left(\ln \frac{[\Gamma_{p,v}(a+bx)]^c}{[\Gamma_{p,v}(b+ax)]^d}\right)^{(m)} \quad is$$

i. decreasing if m is odd,
ii. increasing if m is even.

Proof. Let $g(x) = \frac{[\Gamma_{p,v}(a+bx)]^c}{[\Gamma_{p,v}(b+ax)]^d}$ and $h(x) = \ln g(x)$. Then,

$$\begin{aligned} h^{(m+1)}(x) &= [\ln g(x)]^{(m)} = c\,[\ln(\Gamma_{p,v}(a+bx))]^{(m+1)} - d\,[\ln(\Gamma_{p,v}(b+ax))]^{(m+1)} \\ &= cb^{m+1}\psi^{(m)}_{p,v}(a+bx) - da^{m+1}\psi^{(m)}_{p,v}(b+ax). \end{aligned}$$

Since $x \in [0, 1]$ and $a \geq b$ we have $a + bx \geq b + ax$. Now using the Corollary 4, we get the followings: If m is odd then $0 < \psi_{p,v}^{(m)}(a + bx) \leq \psi_{p,v}^{(m)}(b + ax)$. Then since $0 < cb^{m+1} \leq da^{m+1}$ we can write

$$cb^{m+1}\psi_{p,v}^{(m)}(a + bx) \leq da^{m+1}\psi_{p,v}^{(m)}(a + bx) \leq da^{m+1}\psi_{p,v}^{(m)}(b + ax).$$

So, $h^{(m+1)}(x) \leq 0$. It means that the function $h^{(m)}$ is decreasing on $[0, 1]$ if m is odd.
Similarly, if m is even we have

$$cb^{m+1}\psi_{p,v}^{(m)}(a + bx) \geq da^{m+1}\psi_{p,v}^{(m)}(a + bx) \geq da^{m+1}\psi_{p,v}^{(m)}(b + ax),$$

this implies that the function $h^{(m)}$ is increasing on $[0, 1]$ if m is even, and the result follows. □

3 Conclusions

In the first section, we have given the necessary definitions for our main results. In the main section, we have introduced a new generalized Gamma function $\Gamma_{p,v}$, called the p-v-Gamma function. We have proved a recurrent relation convexity property and related results for $\Gamma_{p,v}$. In addition, we have defined $\psi_{p,v}$ and $\psi_{p,v}^{(m)}$ functions, called the p-v-psi(digamma) and p-v-polygamma functions respectively. Also, we have given some series representations, monotonicity properties, and inequalities involving these new functions.

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