<u>Short Research Article</u>

Existence and Uniqueness of Almost Non-negative Periodic Solution for a Class of Generalized Sine-Gordon Equation

ABSTRACT

In this paper, we have proved the existence and uniqueness of almost non-negative periodic solution to a class of generalized Sine-Gordon equation. The main method used is the maximum principle of telegraph equation established by Mawhin, Ortega and Robles-Pérez. The main technique used is Banach fixed point theorem in functional analysis. The conclusion is that when the coefficients and nonlinear terms of the equation meet certain conditions, the generalized equation has a unique almost non-negative periodic solution. Generalized the results of Mawhin, Ortega and Robles-Pérez.

Keywords: Sine-Gordon equation, almost periodic solution, maximum principle, Banach fixed point theorem

1. INTRODUCTION

In this paper, we discuss the existence and uniqueness of almost non-negative periodic solution for a class of generalized Sine-Gordon equation by using the maximum principle for bounded weak solutions of telegraph equation established by Mawhin and Ortega. The telegraph equation is an important mathematical and physical equation, and its form is:

$$Lu + \lambda u = u_{tt} - \Delta_x u + cu_t + \lambda u = f(t, x),$$

where $c > 0, \lambda \in \mathbb{R}$. In [1, 2], Mawhin and Ortega studied a nonlinear form of the telegraph equation, namely Sine-Gordon equation:

$$u_{tt} - \Delta_x u + cu_t + \lambda \text{sin} u = f(t, x),$$

which $u \in \mathcal{D}'(\mathbb{R} \times \mathbb{T}^n)$, and its almost periodic solution has been studied. The result is as follows:
If $c > 0, \lambda \in (0, c^2/4], f \in AP(\mathbb{R} \times \mathbb{T}^n), \| f \|_{L^\infty} < \lambda$, then the equation $u_{tt} - \Delta_x u + cu_t + \lambda \text{sin} u = f(t, x)$ has a unique solution $u \in AP(\mathbb{R} \times \mathbb{T}^n)$ that satisfies

$$\| u \|_{L^\infty} < \frac{\pi}{2},$$

which is unique and almost periodic. Almost periodic solution is a generalization of periodic solution. In some practical problems, considering almost periodic solution of differential equation is more practical than considering periodic solution of differential equation. So it is of great significance to discuss almost periodic solutions of differential equations. You can refer to [4-7] for more information on almost periodic functions. In addition, the research on the analytical solutions and exact solutions of Sine-Gordon equation has produced many results, for example, see [8-12]. Inspired by their work in [1, 2, 3], this paper discusses a broader class of generalized Sine-Gordon equation:

$$u_{tt} - \Delta_x u + cu_t + g(\varphi(u)) = f(t, x),$$

where

$$g(x) = \sum_{k=1}^{m} a_k x^{2k-1}, \varphi(u) = \lambda \text{sin} u - \mu \text{cos} u.$$

Compared with Sine-Gordon equation, the nonlinear term of equation above has an additional term $\mu cosu$, and the degree of g is also increased. We will prove the existence and uniqueness of almost non-negative periodic solution to this equation under certain conditions.

2. PREPARATION

There are some basic lemmas about the maximum principle to prepare for proving the main theorem in this paper. The spatial dimension n in this paper refers to $n = 1, 2, 3$.

Definition 1. Let $c > 0$ and $f \in L^\infty(\mathbb{R} \times \mathbb{T}^3)$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Consider the equation

$$Lu + \lambda u = u_{tt} - \Delta_x u + cu_t + \lambda u = f(t, x), \tag{2.1}$$

if the bounded solution $u(t, x)$ of this equation in $\mathbb{R} \times \mathbb{R}^3$ satisfies

$$u(t, x_1, x_2, x_3) = u(t, x_1 + 2\pi, x_2, x_3) = u(t, x_1, x_2 + 2\pi, x_3) = u(t, x_1, x_2, x_3 + 2\pi),$$

we call that u is a solution of equation (2.1) in $L^\infty(\mathbb{R} \times \mathbb{T}^3)$. The definition of bounded solution in $L^\infty(\mathbb{R} \times \mathbb{T}^2)$ and in $L^\infty(\mathbb{R} \times \mathbb{T})$ are similar.

Note: to call u the bounded solution of equation (2.1) in $\mathbb{R} \times \mathbb{R}^n$ means that

$$\int_{\mathbb{R}\times\mathbb{T}^n} (L^*\phi + \lambda\phi)u = \int_{\mathbb{R}\times\mathbb{T}^n} f\phi$$

is true for any $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{T}^n)$, where $L^*\phi = \phi_{tt} - \Delta_x\phi - c\phi_t$.

Definition 2. Given a function $f: \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ and a vector $\alpha = (t_0, x_0) \in \mathbb{R} \times \mathbb{T}^n$, the translate $T_\alpha f$ is defined as

$$(T_\alpha f)(t, x) = f(t + t_0, x + x_0).$$

Definition 3. If a function $f: \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ is continuous and for each sequence $\alpha_n \in \mathbb{R} \times \mathbb{T}^n$ there is a subsequence α_k such that $T_{\alpha_k} f$ has a uniform limit. It means that for any $\epsilon > 0$, there is a $N = N(\epsilon) > 0$ such that $\left|\left(T_{\alpha_k} f\right)(t, x) - g(t, x)\right| < \epsilon$ for any $k > N$ and any $(t, x) \in \mathbb{R} \times \mathbb{T}^n$. Then we say that f is almost periodic, and write

$$f \in AP(\mathbb{R} \times \mathbb{T}^n).$$

It can be proved that $AP(\mathbb{R} \times \mathbb{T}^n)$ becomes Banach space with L^∞ norm, and embedded into $L^\infty(\mathbb{R} \times \mathbb{T}^n) \cap C(\mathbb{R} \times \mathbb{T}^n)$. In order to prove the main result of this paper, we need to use several lemmas about the maximum principle.

Lemma 1 ([1, 2, 5]). For each $\lambda \in (0, c^2/4]$ and $f \in L^\infty(\mathbb{R} \times \mathbb{T}^n)$, equation (2.1) has a unique bounded solution $u(t, x)$. And if $f(t, x) \geq 0$ is true almost everywhere in $\mathbb{R} \times \mathbb{T}^n$, then $u(t, x) \geq 0$ for almost all $(t, x) \in \mathbb{R} \times \mathbb{T}^n$.

Lemma 2 ([1, 2, 5]). Assume that $\lambda \in (0, c^2/4]$ and $f \in L^\infty(\mathbb{R} \times \mathbb{T}^n)$, the bounded solution of equation (2.1) satisfies the following estimate:

$$\| u \|_{L^\infty} \leq \frac{1}{\lambda} \| f \|_{L^\infty}.$$

Lemma 3 ([1, 2]). Assume that $\lambda \in (0, c^2/4]$ and $f \in AP(\mathbb{R} \times \mathbb{T}^n)$, then equation (2.1) has a unique solution $u \in AP(\mathbb{R} \times \mathbb{T}^n)$.

Let's state the main result of this paper.

Theorem. Consider the equation

$$u_{tt} - \Delta_x u + cu_t + g(\varphi(u)) = f(t, x), \tag{2.2}$$

in which

$$g(x) = \sum_{k=1}^{m} a_k x^{2k-1}, a_1 > 0, a_2, \cdots, a_m \geq 0, c > 0,$$

$$\varphi(u) = \lambda \mathrm{sin} u - \mu \mathrm{cos} u, \lambda > 0, \mu \geq 0,$$

$$\sum_{k=1}^{m} (2k-1)a_k(\lambda^2 + \mu^2)^{k-1/2} \leq \frac{c^2}{4}, \sum_{k=2}^{m} (2k-1)^2 a_k(\lambda^2 + \mu^2)^{k-1} \leq a_1,$$

$$\mu \sum_{k=1}^{m} (2k-1)a_k\lambda^{2k-2} < \lambda \sum_{k=1}^{m} (2k-1)a_k\mu^{2k-2},$$

and $f(t, x) \geq 0$ almost everywhere, $f \in AP(\mathbb{R} \times \mathbb{T}^n)$, $\| f \|_{L^\infty} < g(\lambda)$. Then equation (2.2) has a unique non-negative solution in $AP(\mathbb{R} \times \mathbb{T}^n)$ satisfying:

$$\| u \|_{L^\infty} < \frac{\pi}{2}.$$

3. PROOF OF THE THEOREM

Proof. Fix constants A and θ satisfying

$$\| f \|_{L^\infty} \leq A < g(\lambda), \theta \in (0, \pi/2), g(\varphi(\theta)) > A, -g(\varphi(-\theta)) > A,$$

and $\lambda g'(\mu) \geq \varphi'(\theta) g'(\varphi(\theta))$. By continuity, let θ go to $(\pi/2)^-$, we know that such θ exists.
Let $\Omega = \{u \in AP(\mathbb{R} \times \mathbb{T}^n) | \, u \geq 0, \| u \|_{L^\infty} \leq \theta\}$, it is easy to show that Ω is a complete metric space given the metric $d(u_1, u_2) = \| u_1 - u_2 \|_{L^\infty}$. For $u \in \Omega$, we consider the equation

$$v_{tt} - \Delta_x v + c v_t + \frac{c^2}{4} v = \frac{c^2}{4} u - g(\varphi(u)) + f(t, x). \tag{2.3}$$

Let $q(t,x) = \frac{c^2}{4} u - g(\varphi(u)$, from the definition and properties of almost periodic functions, we can conclude that

$$q(t,x) \in AP(\mathbb{R} \times \mathbb{T}^n),$$

so $q + f \in AP(\mathbb{R} \times \mathbb{T}^n)$. According to lemma 3, equation (2.3) has a unique solution $v \in AP(\mathbb{R} \times \mathbb{T}^n)$, and this determines a mapping

$$F: \Omega \to AP(\mathbb{R} \times \mathbb{T}^n), \qquad u \mapsto v.$$

Obviously, the fixed point of F is the almost non-negative periodic solution of equation (2.2) which satisfies $\| u \|_{L^\infty} < \pi/2$.
So let's prove that F is a compressed mapping and maps Ω into Ω.
Since

$$q = \frac{c^2}{4} u - \sum_{k=1}^{m} a_k \varphi(u)^{2k-1},$$

we have

$$q'(u) = \frac{c^2}{4} - \sum_{k=1}^{m} (2k-1) a_k \varphi(u)^{2k-2} \varphi'(u).$$

Because $|\varphi(u)|, |\varphi'(u)| \leq \sqrt{\lambda^2 + \mu^2}$, it can be seen $q'(u) \geq 0$, q is an increasing function of u, then

$$q + f \leq \frac{c^2}{4} \theta - g(\varphi(\theta)) + A < \frac{c^2}{4} \theta.$$

And $\omega \equiv \theta$ is a solution of equation $\omega_{tt} - \Delta_x \omega + c\omega_t + \frac{c^2}{4} \omega = \frac{c^2}{4} \theta$. By comparing this equation with equation (2.3), we know from lemma 1 that $v \leq \theta$. Similarly, since

$$q + f \geq q(0) + 0 = -g(\varphi(0)) = -g(-\mu) \geq 0,$$

we can see that $v \geq 0$ from lemma 1. That is $v \in \Omega$.
Next, we must prove that F is a compressed mapping.
Let $u_1, u_2 \in \Omega$ and $v_1 = F(u_1), v_2 = F(u_2)$, then $d = v_1 - v_2$ is the solution to the following equation:

$$d_{tt} - \Delta_x d + c d_t + \frac{c^2}{4} d = \hat{q}(t,x),$$

where $\hat{q}(t,x) = \frac{c^2}{4}(u_1 - u_2) - g(\varphi(u_1)) + g(\varphi(u_2))$. It is known from the mean value theorem of differential that there exists ξ between u_1 and u_2 such that

$$\hat{q} = (u_1 - u_2) \left[\frac{c^2}{4} - \varphi'(\xi) g'(\varphi(\xi)) \right].$$

Let

$$h(\xi) = \frac{c^2}{4} - \varphi'(\xi) g'(\varphi(\xi)) = \frac{c^2}{4} - \varphi'(\xi) \sum_{k=1}^{m} (2k-1) a_k \varphi(\xi)^{2k-2},$$

then

$$h'(\xi) = \varphi(\xi) \left[a_1 - \sum_{k=2}^{m} (2k-1) a_k \varphi(\xi)^{2k-4} (\varphi(\xi)\varphi''(\xi) + (2k-2)\varphi'(\xi)^2) \right].$$

The equation in square brackets above is non-negative based on $|\varphi''(\xi)| \leq \sqrt{\lambda^2 + \mu^2}$ and the conditions given. It is easy to see that $\varphi(\xi)$ is increasing in $(0, \pi/2)$. Because $\varphi(0) \leq 0, \varphi(\pi/2) > 0,$ we can see that $h(\xi)$ decreases and then increases in $(0, \pi/2)$. By condition obviously there is $h(\xi) \geq 0$, let's estimate delta $\| h(\xi) \|_{L^\infty}$.
Since $0 \leq \xi \leq \theta$, $\lambda g'(\mu) \geq \varphi'(\theta) g'(\varphi(\theta))$, we have

$$\sup h(\xi) = \max\{h(0), h(\theta)\} = h(\theta)$$
$$= \frac{c^2}{4} - \varphi'(\theta) \sum_{k=1}^{m} (2k-1) a_k \varphi(\theta)^{2k-2} < \frac{c^2}{4}.$$

Lemma 2 shows that

$$\| v_1 - v_2 \|_{L^\infty} = \| d \|_{L^\infty} \leq \frac{4}{c^2} \| \hat{q} \|_{L^\infty} \leq \frac{4h(\theta)}{c^2} \| u_1 - u_2 \|_{L^\infty}.$$

Since $\frac{4}{c^2}h(\theta)$ is a constant less than 1, so F is a compressed mapping. Therefore, according to Banach fixed point theorem,

$$F: \Omega \to \Omega$$

has a unique fixed point u, thus u is the almost non-negative periodic solution satisfying equation (2.2). Suppose that equation (2.2) has another almost periodic solution $\tilde{u}$ such that $\| \tilde{u} \|_{L^\infty} < \pi/2$. Adjust θ so that $\| \tilde{u} \|_{L^\infty} < \theta < \pi/2$. It follows that $\tilde{u} \in \Omega$, so $\tilde{u}$ is also the fixed point of F. Further, we know $\tilde{u} = u$ from the uniqueness of fixed points, which means the solution is unique.

4. CONCLUSION

In this paper, the existence result of almost periodic solution of Sine-Gordon equation is generalized, for the reason that equation (2.2) becomes Sine-Gordon equation if $\mu = 0$ and $g(x) = x$. The addition of $\mu \cos u$ to the Sine-Gordon equation looks interesting, and this aspect of work remains to be studied.

REFERENCES

[1] J. Mawhin, R. Ortega, A.M. Robles-Pérez. Maximum principles for bounded solutions of the telegraph equation in space dimensions two and three and applications. J. Differential Equations. 2005;208, 42–63.
[2] J. Mawhin, R. Ortega, A.M. Robles-Pérez. A maximum principle for bounded solutions of the telegraph equations and applications to nonlinear forcings. J. Math. Anal. Appl. 2000;251, 695–709.
[3] R. Ortega, A.M. Robles-Pérez. A maximum principle for periodic solutions of the telegraph equation. J. Math. Anal. Appl. 1998;221, 625–651.
[4] V.Y. Slyusarchuk. Almost periodic solutions of differential equations. J Math Sci. 2021;254, 287–304.
[5] A.M. Fink. Almost periodic differential equations. Lecture Notes in Mathematics, Vol. 377, Springer, Berlin, 1974.
[6] B.M. Levitan. Almost periodic functions [in Russian]. Gostekhizdat, Moscow, 1953.
[7] A. Pankov. Almost periodic functions, Bohr compactification, and differential equations. Seminario Mat. e. Fis. di Milano. 1996;66, 149–158.
[8] A.T. Deresse, Y.O. Mussa, A.K. Gizaw. Analytical solution of two-dimensional Sine-Gordon equation. Adv. Math. Phys. 2021;1–15.
[9] S.P. Joseph. Exact Traveling Wave Doubly Periodic Solutions for Generalized Double Sine-Gordon Equation. Int. J. Appl. Comput. Math 8. 2022;42.
[10] R. Shi, Z. Song, T. Feng, G. Wang, X. Wang. Analytical soliton solutions of the (2+1)-dimensional Sine-Gordon equation. Nonlinear Dyn. 2017;88(1), 255–262.
[11] S. Johnson, P. Suarez, A. Biswas. New exact solutions for the Sine-Gordon equation in 2+1 dimensions. Comput. Math. Math. Phys. 2012;52(1), 98–104.
[12] A.-M. Wazwaz. The tanh method: exact solutions of the Sine-Gordon and the Sinh-Gordon equations. Appl. Math. Comput. 2005;167(2), 1196–1210.