*__Original Research Article__*

**The pentagon tells it all. The golden ratio and Fibonacci numbers**


**Abstract**

The golden cut divides a line so that the ratio between the long and the short leg equals the ration between the sum of the two legs and the long leg. Surprisingly, the value for this ratio can be approached by dividing increasing neighboring Fibonacci numbers, numbers where the next in row is the sum of the two previous figures.

Here we show how Fibonacci numbers occur as you construct pentagons where the side length equals the diagonal in the previous pentagon, and how such pentagons can be used to prove that dividing increasing Fibonacci numbers oscillate around the golden ratio.




**Key words:** Fibonacci numbers; golden ratio**;** pentagon

## Introduction

Several attempts have been made to prove the empiric connection between the Fibonacci numbers and the golden ratio (for review, see Livio (2003)), but so far, no simple solution has been presented. Here we show that the proof can be deducted from inspection of a pentagon.

*The golden cut and the golden ratio.* The golden cut and the adjacent golden ratio were first described by Euclid, the father of mathematics, that lived around 300 BC.  (for more details, seeLivio (2003), p.3).  It is defined by the division of a line so that the long leg (a) relates to the short leg (c) as does the whole line ((a)+(c)) to the long leg (a).

The golden cut has had a tremendous impact in almost all areas of human understanding. The ratio is hidden in several of nature's creations, and it has been extensively related to proportions that are considered beautiful in both art and architecture (for review, see Livio (2003). One of its puzzling features is that the ratio is an irrational number meaning that it cannot be defined by dividing any two rational numbers such as 1,2, or any number, it being as large as you can imagine. In contrast, the value can be calculated mathematically from the following steps: (1) (a)/(c)= ((a) + (c))/(a); (2) (a)/(c)=$\varphi$ - insert $\varphi$ in (1); (3) $\varphi$=1 + 1/$\varphi$, which can be rearranged to read: $\varphi^2 - \varphi - 1=0$, a simple quadratic equation with the solution $\varphi$ = 1.618033….

The golden ratio can also be approached by dividing two neighboring Fibonacci numbers with each other. This relation will be further addressed in the following.

*Fibonacci numbers and the golden ratio.* Fibonacci is a famous Italian mathematician, who lived around year 1200. Amongst many other things he eased trade by introducing the Arabic numbers with decimals to replace the rigid Roman system, that only works with whole numbers (integers) ( for more details, see Hauser (2015, ch.2.3). Today Fibonacci's name is still known because of the row of numbers that bears his name. He developed this row by analyzing an imaginary household of rabbits. Fibonacci started with one pair of rabbits and decided that they would give birth to a new pair of rabbits every month. The baby rabbits had to mature for one month, and thus give birth to their first baby pair at the age of two month. If you observe the number of adult and baby rabbit pairs as time pass by you will get the results depicted in Table 1. Note that for both adult and baby rabbit pairs the number increases in a manner so that the next is the sum of the two previous ones. The same goes for the total number of rabbit pairs. Thus, the Fibonacci numbers are defined as a row of numbers where the next in row is the sum of the two previous numbers. Looking at this numbers in relation to the golden ratio surprisingly led to the realization that dividing two neighboring numbers will approach the golden ratio as the absolute figures increase in size ( for more details, see Livio(2003), p 96).

Interestingly, the Fibonacci numbers also occurs if you increase the size of a pentagon in an ordered manner, and this in turn can reveal the value of the golden ratio.

**Table 1.** Fibonacci rabbit pairs and side length in pentagons.

| Rabbits* | | | | Pentagons** | | | |
|---|---|---|---|---|---|---|---|
| Month | Adult pairs | Baby-pairs | Total number | Pentagon no | (a)'s | (c)'s | Side length |
| 1 | 1 | - | 1 | $A_1$ | 1 | - | (a) |
| 2 | 1 | 1 | 2 | $A_2$ | 1 | 1 | ((a)+(c)) |
| 3 | 2 | 1 | 3 | $A_3$ | 2 | 1 | (2(a)+(c)) |
| 4 | 3 | 2 | 5 | $A_4$ | 3 | 2 | (3(a)+(2(c)) |
| 5 | 5 | 3 | 8 | $A_5$ | 5 | 3 | (5(a)+3(c)) |
| 6 | 8 | 5 | 13 | $A_6$ | 8 | 5 | (8(a)+5(c)) |

*At the timepoint one month there is one pair of rabbits. At two month they give birth to a new pair, that mature within a month, and thus can give birth after two months (at month four)– and every month thereafter.

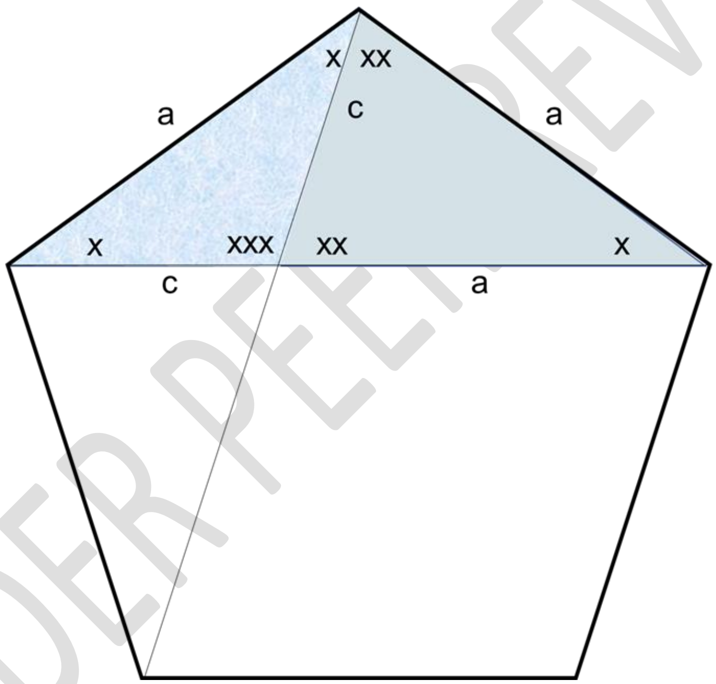
**The pentagons are constructed so that the first have the side length (a) and the diagonal ((a)+(c)). The following have the side length of the diagonal in the previous one. The first pentagon is depicted in Fig 1.

*The pentagon and the golden cut.* As summarized below it is evident that diagonals in a pentagon divides each other by a golden cut.

A pentagon has equal sides (here indicated as (a), Fig 1) and equal angles ($108^o$). If you draw a diagonal, you will see an isosceles triangle. The equal side length is (a)) and the angles are $108^o$, $36^o$ and $36^o$. Drawing a second diagonal will cut the first in two parts and create two triangles on top of the first diagonal. One has the angles of $36^0$ and (108-36) = $72^0$. Thus, the third angle is also $72^0$ (180-36-72) which implies a new isosceles triangle with the side of the pentagon (a). The third side in this triangle is the short part cut of the diagonal, here named (c) (Fig 1). In addition (c) is part of an isosceles triangle with (a) as the third side and with angles of $36^o$ ,$36^o$ and $108^o$, the same angles as present in the triangle created by the first diagonal. The side lengths in two triangles with the same angles relates alike to each other, and thus (a)/(c)=((a)+(c))/(a) which in turn is the equation for the golden ratio. With other words the two diagonals in a pentagon divide each other in the golden cut (fig 1).

**Figure 1.** The pentagon.

The triangle created by drawing a diagonal is shaded. The smaller triangle with the same angles is marked with a structured shading.
X: angle of 36 $^{0}$, XX: angle of 72$^{0}$, XXX: angle of 108$^{0}$.
(a) and (c) indicate the side of the pentagon and the small fragment occurring when two diagonals cut each other.






## Methods and Results

To relate the pentagon to Fibonacci numbers, imagine that you draw a series of pentagons so that the side length of the next pentagon is the diagonal of the previous one.

Your first pentagon($A_1$) has the side length (a), its diagonal is ((a)+(c)) (Fig 1). Now, draw a new pentagon with the side length of the diagonal, that is ($A_2$) has the side length ((a)+(c)), You can deduct that the diagonal in this pentagon consist of a long leg equal to the side of the pentagon ((a)+(c)) and a short leg that is (a). The latter to ensure that the diagonal is indeed divided in the golden cut with a golden ratio of ((a)+(c))/(a). Note that the total length of the diagonal equals the side length in the previous and the present pentagon: ((a)+((a)+(c))) =(2(a)+(c)). This relation is highlighted in Table 1. The table shows the side length as you construct additional pentagons adhering to the rule that the next in row have the side length of the diagonal in the previous one . The integer for (a) and (c) behaves as does the adult and baby rabbit pairs originally used by Fibonacci to develop his row of numbers. Also note that the total side length of the pentagon's follows the same rule ($A_4$) = ($A_2$) + ($A_3$) etc. You need no rabbits. A pentagon will do when you want to create the Fibonacci numbers.

The next step is to show how to reach a value oscillating around the mathematically derived value for the golden ratio. This can be done by constructing a series of pentagons following the same rules as above but this time drawing smaller and smaller pentagons. With other words the side of a given pentagon forms the diagonal in the next in row. The values obtained as the pentagon's grow smaller and smaller are shown in Table 2 You can rewrite the side length in units of $\varphi$ employing (a)/(c)=$\varphi$ (the golden ratio). The values obtained is indicated in the third column of Table 2.

**Table 2.** Side length in regular pentagons. The table shows values where the side length of a given pentagon is the length of the diagonal in the following.

| Pentagon nr | Diagonal length expressed as (a) and (c)* | Side length expressed as (a) and (c)* | Side length expressed as $\varphi$** | Prediction for $\varphi$ | Calculated Value for $\varphi$ |
|---|---|---|---|---|---|
| $A_1$ | ((a) + (c)) | (a) | - | - | - |
| $A_{-1}$ | (a) | (c) | - | - | - |
| $A_{-2}$ | (c) | ((a) - (c)) | $\varphi$ - 1 | >1 | >1 |
| $A_{-3}$ | ((a) - (c)) | (-(a) + 2(c)) | -$\varphi$ + 2 | <2 | <2 |
| $A_{-4}$ | (-(a) + 2(c)) | (2(a) - 3(c)) | 2$\varphi$ - 3 | >3/2 | >1.50000 |
| $A_{-5}$ | (2(a) - 3(c)) | (-3(a) + 5(c)) | -3$\varphi$ + 5 | <5/3 | <1.66667 |
| $A_{-6}$ | (-3(a) + 5(c)) | (5(a) - 8(c)) | 5$\varphi$ - 8 | >8/5 | >1.60000 |
| $A_{-7}$ | (5(a) - 8(c)) | (-8(a) + 13(c)) | -8$\varphi$ + 13 | <13/8 | <1.62500 |
| $A_{-8}$ | (-8(a) + 13(c)) | (13(a) - 21(c)) | 13$\varphi$ - 21 | >21/13 | >1.61538 |
| $A_{-9}$ | (13(a) - 21(c)) | (-21(a) + 34(c)) | -21$\varphi$ + 34 | <34/21 | <1.61904 |
| ….. | | | | | |
| $A_{-14}$ | (-144(a) + 233(c)) | (233(a) - 377(c)) | 233$\varphi$ - 377 | >377/233 | >1.61803 |
| $A_{-15}$ | (233(a) - 377(c)) | (-377(a) + 610(c)) | -377 $\varphi$ + 610 | <610/377 | < 1.61804 |
| $\varphi$*** | | | | | **=1.618033…** |

*The regular pentagon $A_1$ is defined in Fig 1. The side length is (a) and the length of the diagonal is ((a)+(c)).

** The figures in the previous column have been rewritten introducing (a)/(c) = $\varphi$, the golden ratio.

***The mathematically calculated value for $\varphi$.



The side length remains positive since it is the difference between the diagonal and the side in the previous pentagon, but the value obviously approaches zero. Taking these two statements into account you can deduct values for $\varphi$ as indicated in the fourth and the fifth column of Table 2. Note how the values oscillates around the true value for $\varphi$ and how it gets closer and closer as you draw smaller and smaller pentagons.

## Conclusion

Most people interested in the golden ratio and Fibonacci numbers will with great astonishment state that the golden ratio is approached by division of neighboring Fibonacci numbers. The results presented show us that it is the pentagon that defines Fibonacci numbers and proves that the golden ratio can be approached by dividing neighboring Fibonacci numbers.

## References

Hauser FD (2015). *The Golden Ratio: The Facts and the Myths* ,1-50. Francis D Hauser. ISBN-13:978-157518776.

Livio M (2003). *The Golden Ratio: The Story of PHI, the World's Most Astonishing Number*, 1-294. Broadway Books, Inc. 1540 Broadway, New York, NY10036, US. ISBN-13: 978-0767908160