Original Research Article

GLOBAL WEAK SOLUTIONS FOR THE WEAKLY DISSIPATIVE DULLIN-GOTTWALD-HOLM EQUATION

ABSTRACT. In this thesis, we are concerned with the existence and uniqueness of global weak solutions for the weakly dissipative Dullin-Gottwald-Holm equation describing the unidirectional propagation of surface waves in shallow water regime:

$$u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} + \lambda(u - \alpha^2 u_{xx}) = \alpha^2(2u_x u_{xx} + uu_{xxx}).$$

Our main conclusion is that on $c_0 = -\frac{\gamma}{\alpha^2}$ and $\lambda \geq 0$, if the initial data satisfies certain sign conditions, then we show that the equation has corresponding strong solution which exists globally in time, finally we demonstrate the existence and uniqueness of global weak solutions to the equation.

Keywords: Global weak solutions; Existence and uniqueness; Dissipative Dullin-Gottwald-Holm equation

AMS Subject Classification (2010): 35G25, 35L05

1. INTRODUCTION AND MAIN RESULT

Recently, Novruzov [24] studied the finite-time blowup criteria on the Cauchy problem for the weakly dissipative Dullin-Gottwald-Holm equation:

$$\begin{cases} u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} + \lambda(u - \alpha^2 u_{xx}) \\ \qquad = \alpha^2(2u_x u_{xx} + uu_{xxx}), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the constants α^2 $(\alpha > 0)$ and $\frac{\gamma}{c_0}$ are squares of length scales, $c_0 \geq 0$ is the linear wave speed for undisturbed water resting at spatial infinity and $u(t,x)$ stands for the fluid velocity. On account of the weakly dissipative term $\lambda(u - \alpha^2 u_{xx})$, the Eq.(1.1) is called the weakly dissipative Dullin-Gottwald-Holm equation.

When $\alpha = 1,\ \gamma = 0$ and $\lambda = 0$, Eq.(1.1) becomes the celebrated Camassa-Holm (CH) equation

$$u_t - u_{xxt} + c_0 u_x + 3uu_x = 2u_x u_{xx} + uu_{xxx},\ \ t > 0,\ x \in \mathbb{R}, \tag{1.2}$$

where c_0 is a dispersive coefficient related to the critical shallow water speed. Eq.(1.2) was originally proposed as a model for the unidirectional propagation of shallow water waves over a flat bottom and $u(t,x)$ is the the fluid velocity at time t in the spatial x direction [2, 15]. A rigorous justification of the derivation of the Camassa-Holm equation as an approach to the governing equations for water waves was provided by Constantin and Lannes [11]. Eq.(1.2) was also found independently as a model for nonlinear waves in cylindrical hyperelastic rods [12]. It was claimed in [18] that the equation might be relevant to the modeling of tsunami, also see the discussion in [10]. In the last 30 years, the CH equation and its various generalizations were intensively studied in the PDE community due to its many very interesting and remarkable properties: complete integrability in the sense of an infinite-dimensional Hamiltonian systems [2, 14], existence of peakon and multi-peakon [2, 3], geometric formulations [4, 17, 21], existence of both permanent waves and breaking waves [6, 7, 8, 9].

Series interesting results on the blowup issues for the CH type equations have been obtained by Brandolese and his collaborators. To the best of our knowledge, these blowup criterion are sharp (some details can be found in [1]). Their results highlights how local structure of the solution affects the blowups. A key observation in his argument is that the convolution terms are quadratic and positively definite, therefore the nonlocal integration can be estimated below by local terms.

When $\lambda = 0$, Eq.(1.1) becomes the Dullin-Gottwald-Holm(DGH) [13] equation providing a model to describe the unidirectional propagation of surface waves in a shallow water regime. The equation was derived by the method of asymptotic analysis and a near-identity normal form transformation from water wave theory. Recently, several global existence for strong solutions to the DGH equation on the line was studied by Z. Y. Yin [26] and on the circle was presented in [27]. The DGH equation has global solutions [19, 26]. On the other hand, it has low regularity solutions [22] and global weak solutions [28].

Our aim, in this paper, is to prove the existence and uniqueness of global weak solutions to Eq.(1.1) provided the initial data satisfies certain conditions.

Observe that, Setting

$$F(u) := \left(\frac{u^2}{2} - \frac{\gamma}{\alpha^2}u\right) + p * \left(u^2 + \frac{\alpha^2}{2}u_x^2 + (c_0 + \frac{\gamma}{\alpha^2})u\right),$$

where $p(x) := \frac{1}{2\alpha}e^{-|\frac{x}{\alpha}|}$. Then, the weak solution of Eq.(1.1) is defined by

Definition 1.1. Let $u_0 \in H^1(\mathbb{R})$ and $u \in L^\infty_{loc}\left([0,T); H^1(R)\right)$ satisfies the following identity

$$\int_0^T \int_{\mathbb{R}} (u\psi_t - \lambda u\psi + F(u)\psi_x)\, dxdt + \int_{\mathbb{R}} u_0(x)\psi(0,x)dx = 0,$$

for all $\psi \in C_c^\infty([0,T) \times \mathbb{R})$, where $C_c^\infty([0,T) \times \mathbb{R})$ denotes the space of all functions on $[0,T) \times \mathbb{R}$, which may be obtained as the restriction on $[0,T) \times \mathbb{R}$ of a smooth function on $\mathbb{R}^2$ with compact support contained in $(-T,T) \times \mathbb{R}$, then u is called a weak solution to Eq.(1.1). If u is a weak solution on $[0,T)$ for every $T > 0$, then it is called the global weak solution to Eq.(1.1).

The following result proved in [25] clarifies the relation between strong and weak solutions.

Proposition 1.1. Every strong solution is a weak solution. Furthermore, if u is a weak solution and $u \in C([0,T); H^s(\mathbb{R})) \cap C^1\left([0,T); H^{s-1}(\mathbb{R})\right), s > \frac{3}{2}$, then it is a strong solution.

Next, the result of global weak solutions is

Theorem 1.1. *Let $\gamma = -c_0\alpha^2$ and $\lambda \geq 0$. Assume that $u_0 \in H^1(\mathbb{R})$ and $y_0 := (u_0 - u_{0,xx}) \in \mathcal{M}(\mathbb{R})$. Assume further that there exists $x_0 \in \mathbb{R}$ such that*

$$\text{supp} y_0^- \subset (-\infty, x_0) \;\; \textit{and} \;\; \text{supp} y_0^+ \subset (x_0, \infty).$$

Then Eq.(1.1) has a unique weak solution

$$u \in W^{1,\infty}_{loc}\left(\mathbb{R}_+ \times \mathbb{R}\right) \cap L^{\infty}_{loc}\left(\mathbb{R}_+; H^1(\mathbb{R})\right)$$

with initial data $u(0) = u_0$, *moreover the unique weak solution*

$$u \in C(\mathbb{R}_+; H^1(\mathbb{R})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{R}))$$

and

$$y(t,\cdot) := \left(u(t,\cdot) - \alpha^2 u_{xx}(t,\cdot)\right) \in L^{\infty}_{loc}\left(\mathbb{R}_+; \mathcal{M}(\mathbb{R})\right).$$

Remark 1.1. Let us comment on the proof of Theorem 1.1. With the Lemma 2.4, we know that there exists a unique global solution u to Eq.(1.1) in the space $C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$, when $\gamma = -c_0\alpha^2$, $\lambda \geq 0$ and the initial data u_0 satisfies a certain sign condition. To prove the existence of the global weak solutions to Eq(1.1), we make a suitable approximation of the initial data $u_0 \in H^1(\mathbb{R})$ by smooth functions u_0^n which produces a sequence of global solutions $u^n(t,\cdot)$ of Eq.(1.1) in $H^s(\mathbb{R}),\ s > \frac{3}{2}$, then we prove that there is a subsequence of $\{u^n\}_{n\geq 1}$ which converges pointwise a.e. to a function $u \in H^1_{loc}(\mathbb{R} \times \mathbb{R})$ that satisfies Eq.(1.1) in the sense of distributions. By Gronwall's inequality, the uniqueness of global weak solutions to Eq.(1.1) can be obtained. Note that if $\lambda = 0$, the Theorem 1.1 is reduced to the global weak solutions for Dullin-Gottwald-Holm equation presented in [28].

The remainder of the paper is organized as follows. Some preliminary results on Eq.(1.1), such as the local well-posedness of the Cauchy problem of Eq.(1.1) and global solution to Eq.(1.1), are addressed in Section 2. In the last section, we demonstrate the existence and uniqueness of global weak solutions to Eq.(1.1) provided the initial data satisfies appropriate conditions.

Notation: In the following, for a given Banach space X, we denote its norm by $\|\cdot\|_X$. For $1 \leq p \leq \infty$, the norm in the space $L^p(\mathbb{R})$ will be denoted by $\|\cdot\|_{L^p}$ and $\|\cdot\|_{H^s}$ will stand for the norm in the classical Sobolev spaces $H^s(\mathbb{R})$ for $s \geq 0$. We denote by $*$ the spatial convolution. We use $(\cdot|\cdot)$ to represent the standard inner product in $L^2(\mathbb{R})$. The duality paring between $H^1(\mathbb{R})$ and $H^{-1}(\mathbb{R})$ is denoted by $\langle\cdot,\cdot\rangle$. Let $\mathcal{M}(\mathbb{R})$ be the space of Radon measures on $\mathbb{R}$ with bounded total variation and $\mathcal{M}^+(\mathbb{R})$ be the subset of positive measures. Finally, we write $BV(\mathbb{R})$ for the space of functions with bounded total variation and $\mathbb{V}(f)$ for the total variation of $f \in BV(\mathbb{R})$. We denote $p(x) := \frac{1}{2\alpha}e^{-|\frac{x}{\alpha}|}$ the fundamental solution of $\left(1 - \alpha^2\partial_x^2\right)^{-1}$ on $\mathbb{R}$ and define the two convolution operators p_+ , p_- as

$$p_+ * f(x) = \frac{e^{-\frac{x}{\alpha}}}{2\alpha}\int_{-\infty}^{x} e^{\frac{y}{\alpha}} f(y)dy,$$

$$p_- * f(x) = \frac{e^{\frac{x}{\alpha}}}{2\alpha}\int_{x}^{+\infty} e^{-\frac{y}{\alpha}} f(y)dy.$$

Then $p = p_+ + p_-$, $p_x = \frac{1}{\alpha}p_- - \frac{1}{\alpha}p_+$.

2. Preliminaries

In this section, we will recall several useful results in order to prove the main results.

With $y := u - \alpha^2 u_{xx}$, $\alpha > 0$, Eq.(1.1) takes the form of a quasilinear evolution equation of hyperbolic type

$$\begin{cases} y_t + (u - \frac{\gamma}{\alpha^2})y_x + (\lambda + 2u_x)y + (c_0 + \frac{\gamma}{\alpha^2})u_x = 0, & t > 0,\ x \in \mathbb{R}, \\ y(0,x) = u_0(x) - \alpha^2 u_{0,xx}(x), & x \in \mathbb{R}. \end{cases} \tag{2.1}$$

Note that if $p(x) := \frac{1}{2\alpha}e^{-|\frac{x}{\alpha}|}$, then $(1-\alpha^2\partial_x^2)^{-1}f = p * f$ for all $f \in L^2(\mathbb{R})$ and $p * y = u$. Using this identity, we can rewrite Eq.(2.1) as the equivalent equation

$$\begin{cases} u_t + (u - \frac{\gamma}{\alpha^2})u_x = -\partial_x p * \Big(u^2 + \frac{\alpha^2}{2}u_x^2 + (c_0 + \frac{\gamma}{\alpha^2})u\Big) - \lambda u, & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}. \end{cases} \tag{2.2}$$

The local well-posedness of Cauchy problem for Eq.(2.2) with initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$, can be obtained by applying Kato's theorem [16]. In fact, we have following well-posedness result.

Lemma 2.1. [24] *Given $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$, there exist a maximal $T = T(u_0, \alpha, c_0, \gamma, \lambda) > 0$ and a unique solution u to Eq.(2.2), such that*

$$u = u(\cdot, u_0(x)) \in C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R})).$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$u_0 \to u(\cdot, u_0(x)) : H^s(\mathbb{R}) \to C([0,T); H^s(\mathbb{R})) \cap C^1([0,T); H^{s-1}(\mathbb{R}))$$

is continuous and the maximal time of existence $T > 0$ can be chosen to be independent of s.

The following lemma gives the precise blowup scenario.

Lemma 2.2. [24] *Given $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$, the solution $u = u(\cdot, u_0(x))$ of Eq.(2.2) blows up in a finite time $T > 0$ if and only if*

$$\liminf_{t \to T} \left(\inf_{x \in \mathbb{R}} [u_x(x,t)] \right) \to -\infty.$$

The next lemma, which was addressed in [24], plays a role of conservation laws in the proof of the main results and can be easily proved by integration by parts.

Lemma 2.3. *Let $u_0 \in H^s(\mathbb{R}), s \geq 3$ and $T > 0$ be the maximal existence time of the corresponding solution u to Eq.(1.1). Then we have*

$$\int_{\mathbb{R}} (u^2 + \alpha^2 u_x^2)dx = e^{-2\lambda t} \int_{\mathbb{R}} (u_0^2 + \alpha^2 u_{0,x}^2)dx,\ \forall\, t \in [0,T). \tag{2.3}$$

Proof. In view of Eq.(1.1), it is easy to see that

$$\begin{aligned} \frac{d}{dt}\int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2\right) dx &= \int_{\mathbb{R}} \left(2uu_t + 2\alpha^2 u_x u_{xt}\right) dx = \int_{\mathbb{R}} 2u(u_t - \alpha^2 u_{xxt})dx \\ &= 2\int_{\mathbb{R}} u\left[\alpha^2(2u_x u_{xx} + uu_{xxx}) - c_0 u_x - 3uu_x - \gamma u_{xxx} - \lambda(u - \alpha^2 u_{xx})\right] dx \\ &= -2\lambda \int_{\mathbb{R}} u[u - \alpha^2 u_{xx}]dx = -2\lambda \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2\right) dx, \end{aligned}$$

this completes the proof of Lemma 2.3. □

Next, we consider the following differential equation:

$$\begin{cases} q_t(t,x) = u(t,q(t,x)) - \frac{\gamma}{\alpha^2}, & t \in [0,T), \\ q(0,x) = x, & x \in \mathbb{R}. \end{cases} \tag{2.4}$$

A direct calculation shows that

$$\begin{cases} \dfrac{dq_x(t,x)}{dt} = u_x(t,q(t,x))q_x, & t \in [0,T), \\ q_x(0,x) = 1, & x \in \mathbb{R}. \end{cases} \tag{2.5}$$

Applying classical results in the theory of ordinary differential equations, one can show that the map $q(t,\cdot)$ is an increasing diffeomorphism of $\mathbb{R}$ with

$$q_x(t,x) = \exp\left(\int_0^t u_x(s,q(s,x))ds\right) > 0, \ \forall (t,x) \in [0,T) \times \mathbb{R},$$

where $u(t,x)$ is the corresponding strong solution to Eq.(2.4). From Eq.(2.5), we define $k = c_0 + \frac{\gamma}{\alpha^2}$, then

$$\frac{d}{dt}\left(y(t,q(t,x))q_x^2(t,x)\right) = y_t(q)q_x^2 + y_x q_t q_x^2 + 2y(q)q_x q_{xt} = -\lambda y q_x^2 - k u_x q_x^2.$$

If $k = 0$, then $y(t,q(t,x))q_x^2(t,x) = e^{-\lambda t}y_0(x)q_x^2(0,x) = e^{-\lambda t}y_0(x)$, we can obtain the identity connecting the same sign of potential $y = u - \alpha^2 u_{xx}$ at time t with the same sign of y_0.

Next, we will show the proof of the global solution u to Eq.(2.2) in $C([0,\infty);H^s(\mathbb{R})) \cap C^1([0,\infty);H^{s-1}(\mathbb{R}))$.

Lemma 2.4. *Let $\gamma = -c_0\alpha^2$ and $\lambda \geq 0$. Assume $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$ is such that $y_0(x) = u_0 - \alpha^2 u_{0,xx}$ satisfies $y_0(x) \leq 0$ for $x \in (-\infty, x_0]$, $y_0(x) \geq 0$ for $x \in [x_0,\infty)$ for some point $x_0 \in \mathbb{R}$ and y_0 changes the sign. Then there exists a unique global solution u of Eq.(2.2) in $C([0,\infty);H^s(\mathbb{R})) \cap C^1([0,\infty);H^{s-1}(\mathbb{R}))$. Moreover,*

$$u_x(t,x) \geq -\frac{1}{\alpha}|u(t,x)|$$

and

$$u_x(t,x) \geq -\frac{1}{\alpha}\|u(t,x)\|_{L^\infty} \geq -\frac{1}{\alpha}\frac{\max(1,\alpha)}{\min(1,\alpha)}\|u_0\|_{H^1}, \ (t,x) \in [0,\infty) \times \mathbb{R}.$$

Proof. Since $q(t,x)$ is an increasing diffeomorphism of $\mathbb{R}$ for $t \in [0,T)$, it is easy to see that

$$\begin{cases} y(t,x) \leq 0, & \text{if } x \leq q(t,x_0), \\ y(t,x) \geq 0, & \text{if } x \geq q(t,x_0) \end{cases} \tag{2.6}$$

and $y(t,q(t,x_0)) = 0$, $t \in [0,T)$. With $p(x) := \frac{1}{2\alpha}e^{-|\frac{x}{\alpha}|}$ the fundamental solution of $\left(1-\alpha^2\partial_x^2\right)^{-1}$ on $\mathbb{R}$, the two convolution operators p_+ , p_- are defined as

$$p_+ * f(x) = \frac{e^{-\frac{x}{\alpha}}}{2\alpha}\int_{-\infty}^{x} e^{\frac{y}{\alpha}} f(y)dy,$$

$$p_- * f(x) = \frac{e^{\frac{x}{\alpha}}}{2\alpha}\int_{x}^{+\infty} e^{-\frac{y}{\alpha}} f(y)dy,$$

then $p = p_+ + p_-$, $p_x = \frac{1}{\alpha}p_- - \frac{1}{\alpha}p_+ = \frac{1}{\alpha}p - \frac{2}{\alpha}p_+ = \frac{2}{\alpha}p_- - \frac{1}{\alpha}p$. Note that $y := u - \alpha^2 u_{xx}$ and $u(t,x) = p * y(t,x)$, $x \in \mathbb{R}$, it follows that

$$\begin{aligned} u_x(t,x) &= \frac{1}{\alpha}u(t,x) - \frac{2}{\alpha}p_+ * y(t,x) \\ &= \frac{2}{\alpha}p_- * y(t,x) - \frac{1}{\alpha}u(t,x). \end{aligned}$$

As a consequence, it can be deduced from the above equations and (2.6) that for $x \geq q(t,x_0)$,

$$\begin{aligned} u_x(t,x) &= \frac{2}{\alpha}p_- * y(t,x) - \frac{1}{\alpha}u(t,x) \\ &= -\frac{1}{\alpha}u(t,x) + \frac{1}{\alpha^2}e^{\frac{x}{\alpha}}\int_{x}^{+\infty} e^{-\frac{z}{\alpha}} y(t,z)dz \geq -\frac{1}{\alpha}u(t,x), \end{aligned}$$

while, for $x \leq q(t,x_0)$,

$$\begin{aligned} u_x(t,x) &= \frac{1}{\alpha}u(t,x) - \frac{2}{\alpha}p_+ * y(t,x) \\ &= \frac{1}{\alpha}u(t,x) - \frac{1}{\alpha^2}e^{-\frac{x}{\alpha}}\int_{-\infty}^{x} e^{\frac{z}{\alpha}} y(t,z)dz \geq \frac{1}{\alpha}u(t,x). \end{aligned}$$

By (2.3), the above two inequalities and the Sobolev embedding theorem, we obtain

$$u_x(t,x) \geq -\frac{1}{\alpha}\|u(t,\cdot)\|_{L^\infty} \geq -\frac{1}{\alpha}\frac{\max(1,\alpha)}{\min(1,\alpha)}\|u_0\|_{H^1}, \ (t,x) \in [0,\infty) \times \mathbb{R}. \tag{2.7}$$

In view of Lemma 2.2, this shows the existence time $T = \infty$ and the proof of Lemma 2.4 is completed. □

Let us now recall a partial integration result for spaces (below $\langle \cdot, \cdot \rangle$ is the $H^{-1}(\mathbb{R})$ duality bracket).

Lemma 2.5. [20] *Let $T > 0$. If*

$$f, g \in L^2((0,T); H^1(\mathbb{R})) \quad and \quad \frac{df}{dt}, \frac{dg}{dt} \in L^2((0,T); H^{-1}(\mathbb{R})),$$

then f, g are a.e. equal to a function continuous from $[0,T]$ into $L^2(\mathbb{R})$ and

$$\langle f(t), g(t) \rangle - \langle f(s), g(s) \rangle = \int_s^t \left\langle \frac{df(\tau)}{d\tau}, g(\tau) \right\rangle d\tau + \int_s^t \left\langle \frac{dg(\tau)}{d\tau}, f(\tau) \right\rangle d\tau,$$

for all $t, s \in [0,T]$.

Throughout this paper, we will employ the mollifiers which are denoted by $\{\rho_n\}_{n\geq 1}$,

$$\rho_n(x) := \left(\int_{\mathbb{R}} \rho(\xi)d\xi\right)^{-1} n\rho(nx),\ x\in\mathbb{R},\ n\geq 1,$$

where $\rho \in C_c^\infty(\mathbb{R})$ is defined by

$$\rho(x) := \begin{cases} e^{\frac{1}{x^2-1}}, & \text{for } |x|<1,\\ 0, & \text{for } |x|\geq 1.\end{cases}$$

Then we have the following auxiliary result which plays a key role in the proof of unique global weak solutions.

Lemma 2.6. [5]
(a). Let $f:\mathbb{R}\to\mathbb{R}$ be uniformly continuous and bounded.
(i) If $\mu\in\mathcal{M}(\mathbb{R})$, then

$$\lim_{n\to\infty}[\rho_n * (f\mu) - (\rho_n * f)(\rho_n * \mu)] = 0 \qquad in\ L^1(\mathbb{R}).$$

(ii) If $g\in L^\infty(\mathbb{R})$, then

$$\lim_{n\to\infty}[\rho_n * (fg) - (\rho_n * f)(\rho_n * g)] = 0 \qquad in\ L^\infty(\mathbb{R}).$$

(b). Assume that $u(t,\cdot)\in W^{1,1}(\mathbb{R})$ is uniformly bounded in $W^{1,1}(\mathbb{R})$ for all $t\in\mathbb{R}_+$. Then for a.e. $t\in\mathbb{R}_+$

$$\frac{d}{dt}\int_{\mathbb{R}}|\rho_n * u|dx = \int_{\mathbb{R}}|\rho_n * u_t|sgn(\rho_n * u)dx$$

and

$$\frac{d}{dt}\int_{\mathbb{R}}|\rho_n * u_x|dx = \int_{\mathbb{R}}|\rho_n * u_{xt}|sgn(\rho_n * u_x)dx.$$

3. Global Weak Solutions

The last section will show that there exists unique global weak solutions to Eq.(2.2).

3.1. **Proof of Theorem 1.1.** Before proceeding with the existence of global weak solutions, it still requires to put the following lemma in the first place, which will be effectively applied in the following analysis.

Lemma 3.1. *Let $c_0 = -\frac{\gamma}{\alpha^2},\ \lambda\geq 0$. Assume $u_0\in H^s(\mathbb{R}),\ s\geq 3$ and that there exists $x_0\in\mathbb{R}$ such that*

$$\begin{cases} y_0(x)\leq 0, & \text{if } x\leq x_0,\\ y_0(x)\geq 0, & \text{if } x\geq x_0.\end{cases} \tag{3.1}$$

Then the corresponding strong solution u to Eq.(2.2) satisfies

$(i)\ \|u(t,\cdot)\|_{L^\infty} \leq \dfrac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1},$
$(ii)\ \|u_x(t,\cdot)\|_{L^\infty} \leq \dfrac{1}{2\alpha^2}\|y(t,\cdot)\|_{L^1},$
$(iii)\ \|u(t,\cdot)\|_{L^1} \leq \|y(t,\cdot)\|_{L^1},$
$(iv)\ \|u_x(t,\cdot)\|_{L^1} \leq \dfrac{1}{\alpha}\|y(t,\cdot)\|_{L^1}.$

Moreover, if $y_0 \in L^1(\mathbb{R})$, *then* $y \in C^1\left(\mathbb{R}\,;\ L^1(\mathbb{R})\right)$ *and*

$$\|y(t,\cdot)\|_{L^1} \leq e^{\left[\frac{1}{\alpha}\frac{\max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t}\|y_0\|_{L^1}.$$

Proof. By Lemma 2.3 and Sobolev's embedding theorem, we have

$$\|u(t,\cdot)\|_{L^\infty} \leq \|u(t,\cdot)\|_{H^1} \leq \frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}.$$

Since $y(t,x) := u(t,x) - \alpha^2 u_{xx}(t,x)$, it follows that $u = p * y$ and $u_x = p_x * y$. Note that $\|p_x\|_{L^\infty} = \frac{1}{2\alpha^2}$, $\|p\|_{L^1} = 1$ and $\|p_x\|_{L^1} = \frac{1}{\alpha}$. With Young's inequality, (ii)-(iv) are clear.

With the hypothesis of Lemma 3.1, (2.1) clarifies that $y_t := -(u-\frac{\gamma}{\alpha^2})y_x-(\lambda+2u_x)y$. Since $u \in C\left(\mathbb{R}_+; H^s(\mathbb{R})\right)\cap C^1\left(\mathbb{R}_+; H^{s-1}(\mathbb{R})\right)$, $s > 3$ and $y(t,x) := u(t,x)-\alpha^2 u_{xx}(t,x)$, it follows that $y_t \in C\left(\mathbb{R}_+; L^1(R)\right)$. Note that $y_0 \in L^1(R)$ and $y \in C^1\left(\mathbb{R}_+; L^2(R)\right)$. Then it is easy to deduce that $y \in C^1\left(\mathbb{R}_+\,;\ L^1(\mathbb{R})\right)$.

Since $y_0(x_0) = 0$, it follows from $y(t,q(t,x))q_x^2(t,x) = e^{-\lambda t}y_0(x)$ that $y\left(t,q(t,x_0)\right) = 0$. Based on this relation and $y_t := -(u-\frac{\gamma}{\alpha^2})y_x - (\lambda+2u_x)y$, in view of Lemma 2.4, it follows that

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}} y^+dx = \frac{d}{dt}\int_{q(t,x_0)}^{\infty} ydx &= \int_{q(t,x_0)}^{\infty} -(u-\frac{\gamma}{\alpha^2})y_x - 2yu_x - \lambda ydx\\ &= \int_{q(t,x_0)}^{\infty} -yu_x - \lambda ydx\\ &\leq \left(\sup_{x\in\mathbb{R}}(-u_x)+\lambda\right)\int_{\mathbb{R}} y^+dx.\end{aligned}$$

In view of Lemma 2.3, (2.7) and Sobolev embedding, then

$$\sup_{x\in\mathbb{R}}(-u_x) \leq \frac{1}{\alpha}\|u(t,\cdot)\|_{L^\infty} \leq \frac{1}{\alpha}\|u(t,\cdot)\|_{H^1} \leq \frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}.$$

Thus, it follows that

$$\frac{d}{dt}\int_{\mathbb{R}} y^+dx \leq \left(\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right)\int_{\mathbb{R}} y^+dx.$$

By Gornwall's inequality, we can obtain

$$\int_{\mathbb{R}} y^+dx \leq e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t}\int_{\mathbb{R}} y_0^+dx.$$

Repeating the above proof, the same estimate for y^- is available. The proof of Lemma 3.1 is completed. □

Next, we will show the main result-Theorem 1.1.

Proof of Theorem 1.1. We split the proof of the theorem in two parts:

Part I: Existence proof. To show the existence of global weak solution we proceed in several steps:

Step (i). We make a suitable approximation of the initial data $u_0 \in H^1(\mathbb{R})$ by smooth functions u_0^n which produces a sequence of global solutions $u^n(t,\cdot)$ of Eq.(2.2) in

$H^s(\mathbb{R}),\ s > \frac{3}{2}$, then we prove that there is a subsequence of $\{u^n\}_{n\geq 1}$ which converges pointwise a.e. to a function $u \in H^1_{loc}(\mathbb{R}\times\mathbb{R})$ that satisfies Eq.(2.2) in the sense of distributions.

Assume that and that $y_0 := (u_0 - u_{0,xx}) \in \mathcal{M}(\mathbb{R})$. Then the relation $u_0 = p * y_0$ holds true. By Fubini's theorem and Young's inequality, then

$$\begin{aligned}
\|u_0\|_{L^1} = \|p * y_0\|_{L^1} &= \sup_{\substack{\varphi\in L^\infty(\mathbb{R})\\ \|\varphi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \varphi(x)\,(p * y_0)(x)dx \\
&= \sup_{\substack{\varphi\in L^\infty(\mathbb{R})\\ \|\varphi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} \varphi(x) \int_{\mathbb{R}} p(x-\xi)dy_0(x)dx \\
&= \sup_{\substack{\varphi\in L^\infty(\mathbb{R})\\ \|\varphi\|_{L^\infty(\mathbb{R})}\leq 1}} \int_{\mathbb{R}} (p * \varphi)(\xi)dy_0(\xi) \\
&\leq \sup_{\substack{\varphi\in L^\infty(\mathbb{R})\\ \|\varphi\|_{L^\infty(\mathbb{R})}\leq 1}} \|p*\varphi\|_{L^\infty}\|y_0\|_{\mathcal{M}} \\
&\leq \sup_{\substack{\varphi\in L^\infty(\mathbb{R})\\ \|\varphi\|_{L^\infty(\mathbb{R})}\leq 1}} \|p\|_{L^1}\|\varphi\|_{L^\infty}\|y_0\|_{\mathcal{M}} \\
&= \|y_0\|_{\mathcal{M}}.
\end{aligned} \tag{3.2}$$

Observe that, setting

$$y_0^n = l\left(-\frac{1}{n}\right)(\rho_n * y_0^+) - l\left(\frac{1}{n}\right)(\rho_n * y_0^-),$$

where $l(r)$ denotes the right translations by $r\in\mathbb{R}$, i.e., $l(r)f(x) = f(x+r)$. Due to the definition of ρ_n and the assumptions of the theorem, we get $\text{supp}(\rho_n * y_0^-) \subset (-\infty, x_0 + \frac{1}{n}]$ and $\text{supp}(\rho_n * y_0^+) \subset [x_0 - \frac{1}{n}, \infty)$. Therefore, it follows that

$$\begin{cases} y_0^n(x) \leq 0, & \text{if } x \leq x_0, \\ y_0^n(x) \geq 0, & \text{if } x \geq x_0. \end{cases} \tag{3.3}$$

Let us define $u_0^n := p * y_0^n \in H^\infty(\mathbb{R})$ for $n \geq 1$. By Lemma 2.2, Lemma 2.4 and (3.3), we obtain global strong solutions

$$u^n = u^n(\cdot, u_0^n) \in C([0,\infty); H^s(\mathbb{R})) \cap C^1([0,\infty); H^{s-1}(\mathbb{R}))$$

for each $s > \frac{3}{2}$ and all $(t,x)\in\mathbb{R}_+\times\mathbb{R}$. Note that $p * y_0^{n\pm} \in H^1(\mathbb{R})$ and $\|l(\mp\frac{1}{n})\rho_n\|_{L^1} = 1$. Since $\text{supp}(l(\mp\frac{1}{n})\rho_n) \to \{0\}$ as $n\to\infty$, it can be found that

$$\begin{aligned}
u_0^n = p * y_0^n &= p * \left[l\left(-\frac{1}{n}\right)(\rho_n * y_0^+) - l\left(\frac{1}{n}\right)(\rho_n * y_0^-)\right] \\
&= l\left(-\frac{1}{n}\right)(p * (\rho_n * y_0^+)) - l\left(\frac{1}{n}\right)(p * (\rho_n * y_0^-)) \\
&\to p * y_0^+ - p * y_0^- = u_0 \ \ in\ H^1(\mathbb{R})\ as\ n\to\infty.
\end{aligned} \tag{3.4}$$

With Young's inequality, in view of (3.2), we obtain for all $n \geq 1$

$$
\begin{aligned}
\|u_0^n\|_{L^1} &= \left\| l\left(-\frac{1}{n}\right)\left(p*(\rho_n*y_0^+)\right) - l\left(\frac{1}{n}\right)\left(p*(\rho_n*y_0^-)\right)\right\|_{L^1} \\
&\leq \left\| l\left(-\frac{1}{n}\right)\left(\rho_n*(p*y_0^+)\right)\right\|_{L^1} + \left\| l\left(\frac{1}{n}\right)\left(\rho_n*(p*y_0^-)\right)\right\|_{L^1} \\
&\leq \left\| l\left(-\frac{1}{n}\right)\rho_n\right\|_{L^1}\|p*y_0^+\|_{L^1} + \left\| l\left(\frac{1}{n}\right)\rho_n\right\|_{L^1}\|p*y_0^-\|_{L^1} \\
&\leq \|p\|_{L^1}\|y_0^+\|_{\mathcal{M}} + \|p\|_{L^1}\|y_0^-\|_{\mathcal{M}} = \|y_0\|_{\mathcal{M}}.
\end{aligned} \tag{3.5}
$$

Similarly, we can obtain the following estimates

$$
\begin{aligned}
&\|u_0^n\|_{L^2} \leq \|p\|_{L^2}\|y_0^+\|_{\mathcal{M}} + \|p\|_{L^2}\|y_0^-\|_{\mathcal{M}} = \|p\|_{L^2}\|y_0\|_{\mathcal{M}}, \\
&\|u_0^n\|_{L^\infty} \leq \|p\|_{L^\infty}\|y_0^+\|_{\mathcal{M}} + \|p\|_{L^\infty}\|y_0^-\|_{\mathcal{M}} = \frac{1}{2\alpha}\|y_0\|_{\mathcal{M}}, \\
&\|u_0^n\|_{H^1} \leq \|p\|_{H^1}\|y_0^+\|_{\mathcal{M}} + \|p\|_{H^1}\|y_0^-\|_{\mathcal{M}} = \|p\|_{H^1}\|y_0\|_{\mathcal{M}}, \\
&\|y_0^n\|_{L^1} \leq \left\| l\left(-\frac{1}{n}\right)\rho_n\right\|_{L^\infty}\|y_0^+\|_{\mathcal{M}} + \left\| l\left(\frac{1}{n}\right)\rho_n\right\|_{L^\infty}\|y_0^-\|_{\mathcal{M}} = \|y_0\|_{\mathcal{M}}.
\end{aligned} \tag{3.6}
$$

By virtue of Lemma 2.3, Lemma 2.4 and (3.6), then

$$
\begin{aligned}
\|u^n(t,\cdot)\|_{L^\infty} \leq \|u^n(t,\cdot)\|_{H^1} &\leq \frac{max(1,\alpha)}{min(1,\alpha)}\|u_0^n\|_{H^1} \\
&\leq \frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}},
\end{aligned} \tag{3.7}
$$

$$
\begin{aligned}
\|u^n(t,\cdot)\|_{L^2} \leq \|u^n(t,\cdot)\|_{H^1} &\leq \frac{max(1,\alpha)}{min(1,\alpha)}\|u_0^n\|_{H^1} \\
&\leq \frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}},
\end{aligned} \tag{3.8}
$$

$$
\begin{aligned}
\|u_x^n(t,\cdot)\|_{L^2} \leq \|u^n(t,\cdot)\|_{H^1} &\leq \frac{max(1,\alpha)}{min(1,\alpha)}\|u_0^n\|_{H^1} \\
&\leq \frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}}.
\end{aligned} \tag{3.9}
$$

The above inequalities imply that

$$
\begin{aligned}
&\left\| u_x^n(t)\left(u^n(t) - \frac{\gamma}{\alpha^2}\right)\right\|_{L^2} \\
&\quad\leq \|u^n(t)u_x^n(t)\|_{L^2} + \|\frac{\gamma}{\alpha^2}u_x^n(t)\|_{L^2} \\
&\quad\leq \left(\|u^n(t)\|_{L^\infty} + \frac{\gamma}{\alpha^2}\right)\|u_x^n(t)\|_{L^2} \\
&\quad\leq \frac{\gamma}{\alpha^2}\frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}} + \left(\frac{max(1,\alpha)}{min(1,\alpha)}\right)^2\|p\|_{H^1}^2\|y_0\|_{\mathcal{M}}^2,
\end{aligned} \tag{3.10}
$$

for all $t \geq 0$ and $n \geq 1$. With Young's inequality and Lemma 2.3, then

$$\begin{aligned}
&\left\|\partial_x p * \left([u^n(t)]^2 + \frac{\alpha^2}{2}[u_x^n(t)]^2\right)\right\|_{L^2} \\
&\quad \le \|p_x\|_{L^2}\left\|[u^n(t)]^2 + \frac{\alpha^2}{2}[u_x^n(t)]^2\right\|_{L^1} \le max(1,\frac{\alpha^2}{2})\|p_x\|_{L^2}\|u^n(t)\|_{H^1}^2 \\
&\quad \le max(1,\frac{\alpha^2}{2})\left[\frac{max(1,\alpha)}{min(1,\alpha)}\right]^2\|p_x\|_{L^2}\|p\|_{H^1}^2\|y_0\|_{\mathcal{M}}^2.
\end{aligned} \tag{3.11}$$

By (3.8), (3.10), (3.11) and Eq.(2.2), we find that

$$\left\|\frac{d}{dt}u^n(t,\cdot)\right\|_{L^2} = \|u_t^n(t,\cdot)\|_{L^2} \le Q_1 + Q_2 + Q_3, \tag{3.12}$$

for all $t \ge 0$ and $n \ge 1$. Here

$$\begin{aligned}
Q_1 &= \lambda\frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}}, \\
Q_2 &= \left(\frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}} + \frac{\gamma}{\alpha^2}\right)\frac{max(1,\alpha)}{min(1,\alpha)}\|p\|_{H^1}\|y_0\|_{\mathcal{M}}, \\
Q_3 &= max(1,\frac{\alpha^2}{2})\left[\frac{max(1,\alpha)}{min(1,\alpha)}\right]^2\|p_x\|_{L^2}\|p\|_{H^1}^2\|y_0\|_{\mathcal{M}}^2.
\end{aligned}$$

For fixed $T > 0$, with (3.8),(3.9) and (3.12), we have

$$\int_0^T\int_{\mathbb{R}}\left([u^n(t,\cdot)]^2 + [u_x^n(t,\cdot)]^2 + [u_t^n(t,\cdot)]^2\right)dxdt \le K, \tag{3.13}$$

where K is a positive constant depending only on α, γ, λ, $\|p\|_{H^1}$, $\|p_x\|_{L^2}$, $\|y_0\|_{\mathcal{M}}$ and T. Therefore the sequence $\{u^n\}_{n\ge1}$ is uniformly bounded in the space $H^1((0,T)\times\mathbb{R})$. Thus, we can extract a subsequence such that

$$u^{n_k} \rightharpoonup u \text{ weakly in } H^1((0,T)\times\mathbb{R}) \text{ for } n_k \to \infty \tag{3.14}$$

and

$$u^{n_k} \to u \text{ a.e. on } ((0,T)\times\mathbb{R}) \text{ for } n_k \to \infty, \tag{3.15}$$

for some $u \in H^1((0,T)\times\mathbb{R})$. Given $t \in (0,T)$, it follows Lemma 3.1 and (3.5) that $u_x^{n_k}(t,\cdot) \in BV(\mathbb{R})$ with

$$\begin{aligned}
\mathbb{V}[u_x^n(t,\cdot)] = \|u_{xx}^{n_k}(t,\cdot)\|_{L^1} &\le \alpha^{-2}\left(\|u^{n_k}(t,\cdot)\|_{L^1} + \|y^{n_k}(t,\cdot)\|_{L^1}\right) \\
&\le 2\alpha^{-2}\|y^{n_k}(t,\cdot)\|_{L^1} \\
&\le 2\alpha^{-2}e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t}\|y_0\|_{L^1} \\
&\le 2\alpha^{-2}e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t}\|y_0\|_{\mathcal{M}}.
\end{aligned} \tag{3.16}$$

Using Young's inequality and relation $u_x^{n_k}(t,\cdot) = p_x * y^{n_k}(t,\cdot)$, we can obtain

$$\begin{aligned}\|u_x^{n_k}(t,\cdot)\|_{L^\infty} &= \|p_x * y^{n_k}(t,\cdot)\|_{L^\infty} \\ &\le \|p_x\|_{L^\infty}\|y^{n_k}(t,\cdot)\|_{L^1} \\ &\le \frac{1}{2\alpha^2}e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t}\|y_0\|_{\mathcal{M}}.\end{aligned} \tag{3.17}$$

By Helly's theorem [23], there exists a subsequence,again denoted by $\{u_x^{n_k}(t,\cdot)\}_{n_k\ge 1}$, which converges at every point to some function $v(t,\cdot)$ of finite variation with

$$\mathbb{V}[v(t,\cdot)] \le 2\alpha^{-2}e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]T}\|y_0\|_{\mathcal{M}}.$$

In view of (3.15), we have that for almost all $t\in(0,T)$, $u_x^{n_k}(t,\cdot)\to u_x(t,\cdot)$ as $n_k\to\infty$, in $\mathcal{D}'(\mathbb{R})$. This enables us to identify $v(t,\cdot)$ with $u_x(t,\cdot)$ for a.e. $t\in(0,T)$.Therefore,

$$u_x^{n_k}\to u_x \text{ a.e. on } (0,T)\times\mathbb{R} \text{ for } n_k\to\infty \tag{3.18}$$

and for a.e. $t\in(0,T)$,

$$\mathbb{V}[u_x(t,\cdot)] = \|u_{xx}(t,\cdot)\|_{\mathcal{M}} \le 2\alpha^{-2}e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]T}\|y_0\|_{\mathcal{M}}. \tag{3.19}$$

With Lemma 3.1, (3.7)-(3.9) and (3.17), we get

$$\begin{aligned}&\left\|\,[u^n(t)]^2+\frac{\alpha^2}{2}\,[u_x^n(t)]^2\,\right\|_{L^2} \\ &\quad\le \|u^n(t)\|_{L^\infty}\|u^n(t)\|_{L^2}+\frac{\alpha^2}{2}\|u_x^n(t)\|_{L^\infty}\|u_x^n(t)\|_{L^2} \\ &\quad\le \left(\frac{max(1,\alpha)}{min(1,\alpha)}\right)^2\|p\|_{H^1}^2\|y_0\|_{\mathcal{M}}^2 \\ &\qquad+\frac{1}{4}\frac{max(1,\alpha)}{min(1,\alpha)}e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]T}\|p\|_{H^1}\|y_0\|_{\mathcal{M}}^2.\end{aligned} \tag{3.20}$$

For $t\in(0,T)$, the sequence $\{[u^n(t)]^2+\frac{\alpha^2}{2}\,[u_x^n(t)]^2\}_{n\ge 1}$ is uniformly bounded in $L^2(\mathbb{R})$. Therefore, it has a sequence, denoted again $\{[u^n(t)]^2+\frac{\alpha^2}{2}\,[u_x^n(t)]^2\}_{n\ge 1}$, converging weakly in $L^2(\mathbb{R})$. For a.e. $t\in(0,T)$, we denote from (3.15) and (3.18) that the weak $L^2(\mathbb{R})-\text{limit}$ is $\left(\,[u(t,\cdot)]^2+\frac{\alpha^2}{2}\,[u_x(t,\cdot)]^2\,\right)$. As $p_x\in L^2(\mathbb{R})$, a.e. on $(0,T)\times\mathbb{R}$, then

$$\partial_x p * \left([u^{n_k}(t)]^2+\frac{\alpha^2}{2}\,[u_x^{n_k}(t)]^2\right) \to \partial_x p * \left(u^2+\frac{\alpha^2}{2}u_x^2\right) \text{ as } n_k\to\infty. \tag{3.21}$$

From (3.15), (3.18) and (3.21), we obtain that u satisfies Eq.(2.2) in $\mathcal{D}'\left((0,T)\times\mathbb{R}\right)$.

Step (ii). Showing that $u\in W^{1,\infty}_{loc}(\mathbb{R}_+\times\mathbb{R})\cap L^\infty_{loc}\left(\mathbb{R}_+;H^1(\mathbb{R})\right)$, then we prove $u\in C(\mathbb{R}_+;H^1(\mathbb{R}))\cap C^1(\mathbb{R}_+;L^2(\mathbb{R}))$.

From(3.12) and (3.20), we infer that the sequence $\{u_t^{n_k}(t,\cdot)\}_{n_k\ge 1}$ and $\{u^{n_k}(t,\cdot)\}_{n_k\ge 1}$ are uniformly bounded in $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$, respectly, for all $t\in\mathbb{R}_+$ and $k\in\mathbb{N}$. Hence the family $t\mapsto u^{n_k}\in H^1(\mathbb{R})$ is weakly equicontinuous on $[0,T]$ for any $T>0$. Applying Arela-Ascoli theorem that $\{u^{n_k}\}_{n_k\ge 1}$ contains a subsequence, we denote again by $\{u^{n_k}\}_{n_k\ge 1}$, which converges weakly in $H^1(\mathbb{R})$, uniformly in $t\in[0,T)$. The limit function is u and is locally weakly continuous from $\mathbb{R}_+$ into $H^1(\mathbb{R})$, i.e.,

$$u \in C^{\omega}_{loc}\left(\mathbb{R}_+; H^1(\mathbb{R})\right).$$

Since for a.e. $t \in \mathbb{R}_+$, $u^{n_k}(t,\cdot) \rightharpoonup u(t,\cdot)$ weakly in $H^1(\mathbb{R})$. From (3.8), (3.9), we have

$$\begin{aligned}\|u(t,\cdot)\|_{H^1} &\le \liminf_{n_k\to\infty} \|u^{n_k}(t,\cdot)\|_{H^1} \le \liminf_{n_k\to\infty} \frac{max(1,\alpha)}{min(1,\alpha)} \|p\|_{H^1}\|y_0\|_{\mathcal{M}} \\ &= \frac{max(1,\alpha)}{min(1,\alpha)} \|p\|_{H^1}\|y_0\|_{\mathcal{M}},\end{aligned}$$

for a.e. $t \in \mathbb{R}_+$. The previous relation implies that $u \in L^{\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R})$. By (3.17) and (3.18), for all $t \in \mathbb{R}_+$, we have

$$\|u_x(t,\cdot)\|_{L^\infty} \le \frac{1}{2\alpha^2} e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t} \|y_0\|_{\mathcal{M}}.$$

According to the above inequality, we deduce that $u_x \in L^{\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R})$. Then one can easily deduce that $u \in W^{1,\infty}_{loc}(\mathbb{R}_+ \times \mathbb{R}) \cap L^{\infty}_{loc}\left(\mathbb{R}_+; H^1(\mathbb{R})\right)$. Note that

$$min(1,\alpha^2)\|u\|^2_{H^1} \le \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2\right) dx \le max(1,\alpha^2)\|u\|^2_{H^1},$$

we define $\|u\|^2_{1,\alpha} := \int_{\mathbb{R}} \left(u^2 + \alpha^2 u_x^2\right) dx$ and make $\|u\|^2_{1,\alpha}$ as an equivalent norm of $H^1(\mathbb{R})$. Observe that

$$\|u(t)-u(s)\|^2_{H^1} \le \frac{max(1,\alpha^2)}{min(1,\alpha^2)} \|u(t)-u(s)\|^2_{1,\alpha}, \quad \text{for } t,\ s \in \mathbb{R}_+$$

and

$$\|u(t)-u(s)\|^2_{1,\alpha} = \|u(t)\|^2_{1,\alpha} - 2\left(u(t),u(s)\right)_{1,\alpha} + \|u(s)\|^2_{1,\alpha}, \quad \text{for } t,\ s \in \mathbb{R}_+.$$

By Lemma 2.3, we have $\int_{\mathbb{R}}(u^2+\alpha^2 u_x^2)dx = e^{-2\lambda t}\int_{\mathbb{R}}(u_0^2+\alpha^2 u_{0,x}^2)dx,\ \forall t \in [0,T)$, that is

$$\|u(t)-u(s)\|^2_{H^1} \le \frac{max(1,\alpha^2)}{min(1,\alpha^2)} \left(2e^{-2\lambda t}\|u_0\|^2_{1,\alpha} - 2\left(u(t),u(s)\right)_{1,\alpha}\right), \quad \text{for } t,\ s \in \mathbb{R}_+,$$

the scalar product in the last line converges to

$$\|u(t)\|^2_{1,\alpha} = e^{-2\lambda t}\|u_0\|^2_{1,\alpha}, \quad \text{as } s \to t,$$

we conclude that $u \in C(\mathbb{R}_+; H^1(\mathbb{R}))$. From Eq.(2.2) and Young's inequality, we get $u \in C^1(\mathbb{R}_+; L^2(\mathbb{R}))$.

Step (iii). Showing that $y \in L^{\infty}_{loc}\left(\mathbb{R}_+; \mathcal{M}(\mathbb{R})\right)$.

Note that

$$L^1(\mathbb{R}) \subset (L^\infty(\mathbb{R}))^* \subset (C_0(\mathbb{R}))^* = \mathcal{M}(\mathbb{R}).$$

By Lemma 3.1 and (3.19), for a.e. $t \in \mathbb{R}_+$, then

$$\begin{aligned}\left\|u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right\|_{M(\mathbb{R})} &\leq \|u(t,\cdot)\|_{L^1}+\alpha^2\|u_{xx}(t,\cdot)\|_{M(\mathbb{R})}\\ &\leq \|y(t,\cdot)\|_{L^1}+\alpha^2\|u_{xx}(t,\cdot)\|_{M(\mathbb{R})}\\ &\leq 3e^{\left[\frac{1}{\alpha}\frac{max(1,\alpha)}{min(1,\alpha)}\|u_0\|_{H^1}+\lambda\right]t}\|y_0\|_{\mathcal{M}},\end{aligned}$$

the above inequality implies that

$$\left(u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right)\in L^{\infty}_{loc}\left(\mathbb{R}_+;\mathcal{M}(\mathbb{R})\right).$$

Part II: uniqueness proof. Let $u,\ v\in C(\mathbb{R}_+;H^1(\mathbb{R}))\cap C^1(\mathbb{R}_+;L^2(\mathbb{R}))$ be two global weak solutions of Eq.(2.2) with the same initial data u_0. such that

$$\left(u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right)\in L^{\infty}_{loc}\left(\mathbb{R}_+;\mathcal{M}(\mathbb{R})\right)$$

and

$$\left(v(t,\cdot)-\alpha^2v_{xx}(t,\cdot)\right)\in L^{\infty}_{loc}\left(\mathbb{R}_+;\mathcal{M}(\mathbb{R})\right).$$

Fix $T>0$, then we define

$$M(T):=\sup_{t\in[0,T]}\left(\|u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\|_{\mathcal{M}}+\|v(t,\cdot)-\alpha^2v_{xx}(t,\cdot)\|_{\mathcal{M}}\right).$$

Under the assumption on the theorem, we get the $M(T)<\infty$. With $\|p\|_{L^\infty}=\frac{1}{2\alpha}$, $\|p_x\|_{L^\infty}=\frac{1}{2\alpha^2}$, $\|p\|_{L^1}=1$ and $\|p_x\|_{L^1}=\frac{1}{\alpha}$, it follows that

$$\begin{aligned}|u(t,x)| &= \left|p*\left[u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right](x)\right|\\ &\leq \|p\|_{L^\infty}\|u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\|_{\mathcal{M}}\leq\frac{M(T)}{2\alpha}\end{aligned}\tag{3.22}$$

and

$$\begin{aligned}|u_x(t,x)| &= \left|p_x*\left[u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right](x)\right|\\ &\leq \|p_x\|_{L^\infty}\|u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\|_{\mathcal{M}}\leq\frac{M(T)}{2\alpha^2},\end{aligned}\tag{3.23}$$

for all $(t,x)\in[0,T]\times\mathbb{R}$. Similarly,

$$|v(t,x)|\leq\frac{M(T)}{2\alpha},\quad |v_x(t,x)|\leq\frac{M(T)}{2\alpha^2},\quad (t,x)\in[0,T]\times\mathbb{R}.\tag{3.24}$$

Note that

$$\|u(t,\cdot)\|_{L^1}=\|p*\left[u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right]\|_{\mathcal{M}}\leq M(T),\tag{3.25}$$

$$\|u_x(t,\cdot)\|_{L^1}=\|p_x*\left[u(t,\cdot)-\alpha^2u_{xx}(t,\cdot)\right]\|_{\mathcal{M}}\leq\frac{M(T)}{\alpha},\tag{3.26}$$

for all $(t,x)\in[0,T]\times\mathbb{R}$. Similarly,

$$\|v(t,\cdot)\|_{L^1}\leq M(T),\quad \|v_x(t,\cdot)\|_{L^1}\leq\frac{M(T)}{\alpha},\quad (t,x)\in[0,T]\times\mathbb{R}.\tag{3.27}$$

As one can see comparing with (3.2). Let us indicate

$$w(t,x) := u(t,x) - v(t,x),\ (t,x) \in [0,T] \times \mathbb{R}.$$

Convoluting Eq.(2.2) for u and v with ρ_n and using Lemma 2.5, Lemma 2.6, we get for a.e. $t \in [0,T]$ and all $n \geq 1$ that

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w| dx &= \int_{\mathbb{R}} (\rho_n * w_t) sgn(\rho_n * w) dx \\
&= -\lambda \int_{\mathbb{R}} (\rho_n * w) sgn(\rho_n * w) dx - \int_{\mathbb{R}} [\rho_n * (w u_x)] sgn(\rho_n * w) dx \\
&\quad - \int_{\mathbb{R}} [\rho_n * (v w_x)] sgn(\rho_n * w) dx - \frac{\gamma}{\alpha^2} \int_{\mathbb{R}} [\rho_n * (w_x)] sgn(\rho_n * w) dx \\
&\quad - \int_{\mathbb{R}} (\rho_n * p_x * [w(u+v)])\, sgn(\rho_n * w) dx \\
&\quad - \frac{\alpha^2}{2} \int_{\mathbb{R}} (\rho_n * p_x * [w_x(u_x + v_x)])\, sgn(\rho_n * w) dx.
\end{aligned}$$

Using above inequalities (3.22)-(3.27), Young's inequality and Lemma 2.6. Following the procedure described in [5, pp.56-57], we deduce that

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w| dx = C(T) \int_{\mathbb{R}} |\rho_n * w| dx + C(T) \int_{\mathbb{R}} |\rho_n * w_x| dx + R_n(t), \tag{3.28}$$

for a.e. $t \in [0,T]$ and all $n \geq 1$, where $C(T)$ is a generic constant depending on $M(T),\ \gamma,\ \alpha,$ and λ. Moreover, $R_n(t)$ satisfies

$$\begin{cases} R_n(t) \to 0, & n \to \infty, \\ |R_n(t)| \leq G(T), & n \geq 1, t \in [0,T], \end{cases} \tag{3.29}$$

here $G(T)$ is a positive constant depending on $M(T),\ \gamma,\ \alpha,\ \lambda$, and the $H^1(\mathbb{R})-$norm of $u(0)$ and $v(0)$. Similarly, convoluting Eq.(2.2) for u and v with $\rho_{n,x}$ and using Lemma 2.6, for a.e. $t \in [0,T]$ and all $n \geq 1$, then, we can derive out

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w_x| dx &= \int_{\mathbb{R}} (\rho_n * w_{xt}) sgn(\rho_n * w_x) dx \\
&= -\lambda \int_{\mathbb{R}} (\rho_n * w_x) sgn(\rho_n * w_x) dx - \int_{\mathbb{R}} [\rho_n * (w u_{xx})] sgn(\rho_{n,x} * w) dx \\
&\quad - \int_{\mathbb{R}} [\rho_n * (w_x(u_x + v_x))] sgn(\rho_{n,x} * w) dx \\
&\quad - \int_{\mathbb{R}} [\rho_n * (u w_{xx})] sgn(\rho_{n,x} * w) dx - \frac{\gamma}{\alpha^2} \int_{\mathbb{R}} [\rho_n * (w_{xx})] sgn(\rho_{n,x} * w) dx \\
&\quad - \int_{\mathbb{R}} \left(\rho_n * p_{xx} * \left[u^2 - v^2 + \frac{\alpha^2}{2}(u_x^2 + v_x^2) \right] \right) sgn(\rho_n * w) dx.
\end{aligned}$$

According to (3.22)-(3.27), Young's inequality, Lemma 2.6, the identity $\alpha^2 \partial_x^2 (p * f) = p * f - f$ and following the arguments given in [5, pp.56-57], then

$$\frac{d}{dt}\int_{\mathbb{R}} |\rho_n * w_x| dx = C(T)\int_{\mathbb{R}} |\rho_n * w| dx + C(T)\int_{\mathbb{R}} |\rho_n * w_x| dx + R_n(t), \tag{3.30}$$

for a.e. $t \in [0,T]$ and all $n \geq 1$, where $C(T)$ is a generic constant and $R_n(t)$ satisfies (3.29). Summing (3.28) and (3.30), an application of Gronwall's inequality yields that

$$\begin{aligned}\int_{\mathbb{R}} \left(|\rho_n * w| + |\rho_n * w_x|\right)(t,x) dx \leq \int_0^t e^{2C(T)(t-s)} R_n(s) ds \\ + e^{2C(T)t}\int_{\mathbb{R}} \left(|\rho_n * w| + |\rho_n * w_x|\right)(0,x) dx,\end{aligned}$$

for all $t \in [0,T]$ and all $n \geq 1$. Note that $w = u - v \in W^{1,1}(\mathbb{R})$, In view of (3.29) and Lebesgue's dominated convergence theorem, we obtain for all $t \in [0,T]$ that

$$\int_{\mathbb{R}} \left(|w| + |w_x|\right)(t,x) dx \leq e^{2C(T)t}\int_{\mathbb{R}} \left(|w| + |w_x|\right)(0,x) dx.$$

Note that $w(0) = w_x(0) = 0$, we deduce that $u(t,x) = v(t,x)$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}$. Recalling that T was chosen arbitrary, uniqueness is now plain. $\square$

3.2. **The Periodic Case.** Considering the periodic case, we identify all spaces of periodic functions with function spaces over the unit circle $\mathbb{S}$ in $\mathbb{R}^2$ and the Eq.(1.1) can be rewritten as

$$\begin{cases} u_t - \alpha^2 u_{xxt} + c_0 u_x + 3uu_x + \gamma u_{xxx} + \lambda(u - \alpha^2 u_{xx}) & \\ \qquad = \alpha^2(2u_x u_{xx} + uu_{xxx}), & t > 0,\ x \in \mathbb{R}, \\ u(0,x) = u_0(x), & x \in \mathbb{R}, \\ u(t,x+1) = u(t,x), & t > 0, x \in \mathbb{R}. \end{cases} \tag{3.31}$$

As a corollary of Theorem 1.1, we get

Theorem 3.1. *Let $\gamma = -c_0\alpha^2$ and $\lambda \geq 0$. Assume that $u_0 \in H^1(\mathbb{S})$, and $y_0 := (u_0 - u_{0,xx}) \in \mathcal{M}(\mathbb{S})$. Assume further that exists $x_0 \in (0,1)$ such that*

$$\text{supp} y_0^- \subset (0, x_0) \ \ and \ \ \text{supp} y_0^+ \subset (x_0, 1).$$

Then Eq.(3.31) has a unique weak solution

$$u \in C(\mathbb{R}_+; H^1(\mathbb{S})) \cap C^1(\mathbb{R}_+; L^2(\mathbb{S}))$$

and

$$y(t,\cdot) := \big(u(t,\cdot) - \alpha^2 u_{xx}(t,\cdot)\big) \in L^\infty_{loc}\left(\mathbb{R}_+; \mathcal{M}(\mathbb{S})\right).$$

REFERENCES

[1] L. BRANDOLESE, Local-in-space criteria for blowup in shallow water and dispersive rod equations, *Comm. Math. Phys.*, **330**(2014), 401–414.

[2] R. CAMASSA AND D. D. HOLM, An integrable shallow water equation with peaked solitons, *Phys. Rev. Lett.*, **71**(1993), 1661–1664.

[3] C. S. CAO, D. D. HOLM AND E. S. TITI, Traveling wave solutions for a class of one-dimensional nonlinear shallow water wave models, *J. Dynam. Differential Equations,* **16**(2004), 167–178.

[4] K. S. CHOU AND C. Z. QU, Integrable equations arising from motion of plane curves I, *Physica D*, **162**(2002), 9–33.

[5] A. CONSTANTIN AND L. MOLINET, Global weak solutions for a shallow water equation, *Comm. Math. Phys.*, **211**(2000), 45–61.

[6] A. CONSTANTIN, Existence of permanent and breaking waves for a shallow water equation: a geometric approach, *Ann. Inst. Fourier (Grenoble)*, **50**(2000), 321–362.

[7] A. CONSTANTIN AND J. ESCHER, Wave breaking for nonlinear nonlocal shallow water equations, *Acta Math.*, **181**(1998), 229–243.

[8] A. CONSTANTIN AND J. ESCHER, Global existence and blow-up for a shallow water equation, *Ann. Scuola Norm. Sup. Pisa.*, **26**(1998), 303–328.

[9] A. CONSTANTIN AND J. ESCHER, On the blow-up rate and the blow-up set of breaking waves for a shallow water equation, *Math. Z.*, **233**(2000), 75–91.

[10] A. CONSTANTIN AND R. S. JOHNSON, Propagation of very long water waves, with vorticity, over variable depth, with applications to tsunamis, *Fluid Dyn. Res.*, **40**(2008), 175–211.

[11] A. CONSTANTIN AND D. LANNES, The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations, *Arch. Ration. Mech. Anal.*, **192**(2009), 165–186.

[12] H. H. DAI, Model equations for nonlinear dispersive waves in a compressible Mooney-Rivlin rod, *Acta Mech.*, **127**(1998), 193–207.

[13] H. R. DULLIN, G. A. GOTTWALD AND D. D. HOLM, An integrable shallow water equation with linear and nonlinear dispersion, *Phys. Rev. Lett.*, **87**(2001), 1945–1948.

[14] A. FOKAS AND B. FUCHSSTEINER, Symplectic structures, their Backlund transformation and hereditary symmetries, *Phys. D.*, **4**(1981/82), 47–6.

[15] R. S. JOHNSON, Camassa-Holm, Korteweg-de Vries and related models for water waves, *J. Fluid Mech.*, **457**(2002), 63 – 82.

[16] T. KATO, Quasi-linear equations of evolution with applications to partial differential equations, Springer, Berlin, 1975, pp. 25–70.

[17] S. KOURANBAEVA, the Camassa-Holm equation as a geodesic flow on the diffeomorphism group, *J. Math. Phys.*, **40**(1999), 857–868.

[18] M. LAKSHMANAN, Integrable nonlinear wave equations and possible connections to tsunami dynamics, in Tsunami and nonlinear waves, Springer, Berlin, 2007, pp. 31–49.

[19] Y. LIU, Global existence and blow-up solutions for a nonlinear shallow water equation, *Math. Ann.*, **335**(2006), 717–735.

[20] J. MALEK, J. NECAS AND M. RUZICKA, Weak and Measure-Valued Solutions to Evolutionary PDEs, Chapman Hall, London, 1996.

[21] G. MISIOLEK, A shallow water equation as a geodesic flow on the Bott-Virasoro group, *J. Geom. Phys.*, **24**(1998), 203–208.

[22] O. G. MUSTAFA, Existence and uniqueness of low regularity solutions for the Dullin-Gottwald-Holm equation, *Comm. Math. Phys.*, **265**(2006), 189–200.

[23] I. P. NATANSON, Theory of Functions of a Real Variable, *New York: F. Ungar Publ. Co.*, 1964.

[24] E. NOVRUZOV, Blow-up of solutions for the dissipative Dullin-Gottwald-Holm equation with arbitrary coefficents, *J. Differential Equations*, **261**(2016), 1115–1127.

[25] Z. Y. YIN, On the Cauchy problem for an integrable equation with peakon solutions, *Illinois J. Math.*, **4**(2003), 649–666.
[26] Z. Y. YIN, Well-poseness, blowup, and global existence for an integrable shallow water equation, *Discrete Contin. Dyn. Syst.*, **11**(2004), 393–411.
[27] Z. Y. YIN, Global existence and blow-up for periodic integrable shallow water equation with linear and nonlinear dispersion, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, **12**(2005), 87–101.
[28] S. H. ZHANG AND Z. Y. YIN, Global weak solutions for the Dullin-Gottwald-Holm equation, *Nonlinear Anal.*, **72**(2010), 1690–1700.

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