Sensitivity of Selected Method of Estimation Based on the Lee-Carter Model

Abstract

Efficiency and the unbiasedness structure of three method of estimation namely: the Weighted Least Square, Maximum Likelihood Estimation and the Method of Moment were examined in this study. Using three scenarios labelled as Case I, II & III; where Case I is model simulation based on the distributional properties of the Lee Carter model, Case II and III were majorly data contamination by self-inclusion of Outliers to the simulated dataset for High (Case II) and Low (Case III), respectively. The behavioral patterns of the estimators were assessed at different sample sizes. Under Case I, it was noted that the larger the sample size n, the more the distribution tends to Normal but for Case II and III, the data gets inflated. The performance of each method of estimation was tested in accordance to the properties of estimators. MLE was best among the three estimators for the Lee Carter Model. The estimated parameters a_x, b_x and $k_t$were observed to approach the true parameter as the sample size increases.

Introduction

The Lee-Carter model has attracted great attention in literature concerning the projection of population and mortality rates. Thus, for statistical computation purposes, long-term mortality prediction is required. For example, the U.S. Social Security Administration (SSA) normally provides mortality forecasts for a horizon of up to 90 years as a reference for the life insurance industry and retirement system which is not the same case as Nigeria data. Sources of data in Nigeria only provide death count data (not mortality data) and data are available for a short-term period (not up to 100 years) as against the conditions of applying a Lee carter Model. However, the focus of the study is not to examine the limitation of the available data but to justify the unbiasedness of methods (which are not commonly in use) in the estimation of the assumed Lee-carter's parameters compared to the estimated parameter from the US mortality forecast. This is because, over the years, 99% of researchers (such as Simon Reese (2015), Siu Hang et al. (2005), Lee et al. (2002), Chukwu and Oladipupo (2012).) who worked on the lee-Carter Model (and its extension) apply the Single Decomposition Value Method for the estimation of Lee-Carter's parameter with no justification as to why the method is most preferably used. Hence, this study is designed to substantiate the unbiasedness of methods such as the Maximum Likelihood Estimation (MLE) Method, the Method of Moments (MM) and the Weighted Least Square Method (WLS) in estimating the Lee Carter parameter. Hence, this study is focused on examining the asymptotic characteristics of selected estimators in modelling mortality using the Lee-Carter model especially when Mortality data are limited or contaminated.

Methodology

The data are age-specific mortality rates obtained from the application of the Lee Carters Model to the United States of America Data (both sex) within the age interval of 15-84 years between the years 1933 and 1987. Note that the U.S. Age Specific Mortality Estimate is most preferred in this study because of the availability of data over a long period of time. Similarly, it has been affirmed repeatedly by various researchers that the US Mortality Fits perfect for the Lee Carters Model when compared to other countries including Nigeria. The method of Estimation of the previous study was basically the Singular Value Decomposition. It is also important to mention that the observed data is extracted from Understanding the Lee Cater Mortality Forecast Method (Federico. Girosi and Gary King 2007).

The Lee-Carter methodology for forecasting mortality rates is a simple bilinear model in the variables x (age) and t (calendar year). The model is defined as:

$$\ln m_{xt} = \hat{a}_x + \hat{b}_x \hat{k}_t + \varepsilon_{xt}$$

Where;

m_{xt} : is the matrix of the observed age-specific death rate at age *x* during year t. It is obtained from observed deaths divided by population exposed to risk. It is subject to random fluctuation.

$\hat{a}_x$: is the average of $\ln m_{xt}$ over time t. It describes the (average shape of the age profile) general pattern of mortality by age.

$\hat{k}_t$: is the time trend for the general mortality. It captures the main time trend on the logarithmic scale in mortality rates at all ages. $\hat{k}_t$ is also referred to as the mortality index.

$\hat{b}_x$: indicates the relative pace of change in mortality by age as $\hat{k}_t$ varies. It describes the pattern of deviations from the age profile when the parameter $\hat{k}_t$ varies. It modifies the main time trend according to whether change at a particular age is faster or slower than the main trend.

ε_{xt} : is the residual term at age x and time t. It reflects the age specific influences not captured by the model. It is expected to be Gaussian $\varepsilon_{xt} \sim N(0, \sigma^2)$

Method of Moment

Method of moment utilizes the relationship between moments, which is defined as the expectation of the powers of a random variable, and unknown parameters to estimate the values of parameters. If $M_{x,t}$ is set to be a random variablek_t, its moments can be calculated and be set to equal to randomly assigned letters:

$$\ln m_{x,t} = \hat{a}_x + \hat{b}_x \hat{k}_t + \varepsilon_{xt}$$

(1)

$$\widehat{a_x} = In\ \bar{y} - \hat{b}_x \bar{x}$$

Where $\bar{y} = \frac{\sum In(M_{x,t})}{T}$ and $\bar{x} = \frac{\sum(k_t)}{T}$

(2)

$$\hat{b}_x = \frac{\delta_{x,y}}{\sigma^2} = \frac{\sum(x-\bar{x})(y-\bar{y})}{\sigma^2},$$

y = $Inm_{x,t}$ and x= $\hat{k}_t$

$$\ln \hat{m}_{x,t} = \hat{a}_x + \hat{b}_x \hat{k}_t$$

Maximum Likelihood Estimation

The MLE is referred to as the Poisson Log Bilinear Model. It gives optimal solution of the LC model under a Poisson model and avoid assumption of error with constant variance. This was introduced by (Wilmoth, 1993)

$$D_{x,t} \sim P(m_{x,t,} E_{x,t})$$

(3)

Where,

$$m_{x,t,} = exp(a_x + b_x k_t)$$

MLE is given by:

$$l(a,b,k) = \prod x \prod t \; In \; [\frac{D^{D_{x,t}}{}_{x,t} exp(-\overline{D_{x,t})}}{D_{x,t}!}]$$

(4)

$$\sum_x \sum_t D_{x,t} In\widehat{D_{x,t}} - D_{x,t} In\widehat{D_{x,t}}!$$

$$\sum_x \sum_t (D_{x,t}(a_x + b_x \hat{k}_t) - E_{x,t} exp(a_x + b_x \hat{k}_t) constant$$

By differentiating both sides of the equation, we can immediately see that the observed and the fitted number of death overtime are equal when the algorithm converges.

The Weighted Least Square Method

$$\sum w_i e_i{}^2 = \sum w_i (In \; m_{x,t} - a_x - b_x k_t)^2 \quad (5)$$

Let $x = k_t$

For easy computation, let L represent the sum of square of the residuals, so that

$$L = \sum w_i e_i^2$$

$$L = \sum w_{x,t}\left(\left(In\, m_{x,t} - a_x - b_x k_t\right)^2 \qquad (6)$$

We need to minimize L with respect to a and b to obtain the estimated parameters of a_x , $b_x and\ k_t$ is obtained by differentiating L and equating the derivative to zero(0)

Where $w_{x,t}$can be taken as the reciprocal of the number of deaths at age x and in time period t.

Result and Discussion

The data are the age-specific mortality rates obtained from the application of the Lee Carters Model to the United State of America Data (both sexes) within the age interval 15-84 years between the year (with unequal space interval) 2000, 2002 and 2009. Note that the U.S. Age Specific Mortality Estimate is most preferred in this study because of the availability of data over a long period of time. Also, it has been affirmed repeatedly by various researchers that the US Mortality Fits for the Lee Carters Model when compared to other country including Nigeria. The method of Estimation of the previous study was basically the Singular Value Decomposition. It is important to mention that the observed data was from (Federico. Girosi and Gary King 2007)

Table 1: Table of estimated Average-Age Specific Mortality (a_x) and Relative Pace of Change in Mortality Rate$(b_x)$from previous study to be set as a baseline for the estimated Mortality (m_x)

AGE INTERVAL	FEMALE (a_x)	MALE (a_x)	FEMALE (b_x)	MALE (b_x)

15-19	-5.64997	-6.42310	-0.06130	0.09090
20-24	-5.12436	-5.51284	0.34879	0.32413
25-29	-4.83250	-5.09400	0.52788	0.65062
30-34	-4.70209	-4.78734	0.49725	0.51733
35-39	-4.59599	-4.53278	0.38556	0.34114
40-44	-4.55521	-4.34265	0.25600	0.20567
45-49	-4.45605	-4.15319	0.20744	0.13879
50-59	-4.21795	-3.92900	0.17364	0.10563
60-64	-3.87750	-3.60555	0.15133	0.05830
65-69	-3.60475	-3.31625	0.12048	0.04365
70-74	-3.16506	-2.96741	0.09718	0.03098
75-79	-2.72629	-2.56271	0.08359	0.01515
80-84	-2.30800	-2.16764	0.06808	0.01019
85-89	-1.90624	-2.16764	0.05308	0.00739

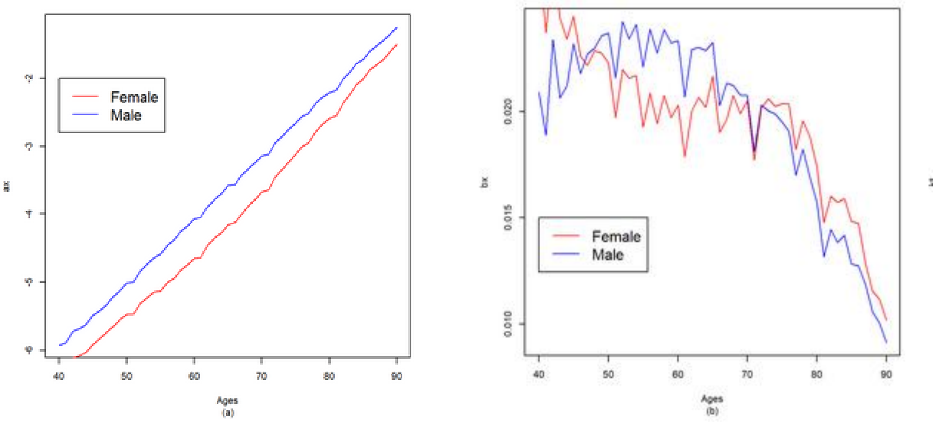


Figure 1: Plots of estimated Average-Specific Mortality Rate(a_x) and Relative Pace Interval(b_x) for the estimated Mortality (m_x)

The figure with non-zero origin above explains the regularities in the observed data. The Average Specific Mortality Rate(a_x) of both male and female increases with increase in age and this implies that as $a_x \longrightarrow \infty$ as x increases. Observing closely the constant irregularities in Figure 2, It obviously describes the pattern of deviations from the age profile. Hence $b_x \longrightarrow 0$ as x increases. The essence of the figures and discussion above is to reveal the nature and structure of data used as a baseline for this study.

Estimation of parameters

Estimated parameters a_x, b_x and k_t for:

1. Case I: In this scenario, the observed data is simulated by generating 50 replicates T with sample sizes 10, 100, 1000. And parameters are estimated using the three Methods of Estimation discussed in Chapter 3
2. Conterminated data
   - **(i)** Case 1I: The simulated data is injected with small values which are imposed as outliers
   - **(ii)** Case III: The simulated data is injected with big values which are imposed as outliers

Descriptive statistics of the simulated data when n=10, 100 and 1000

Case I (when data is not defective)

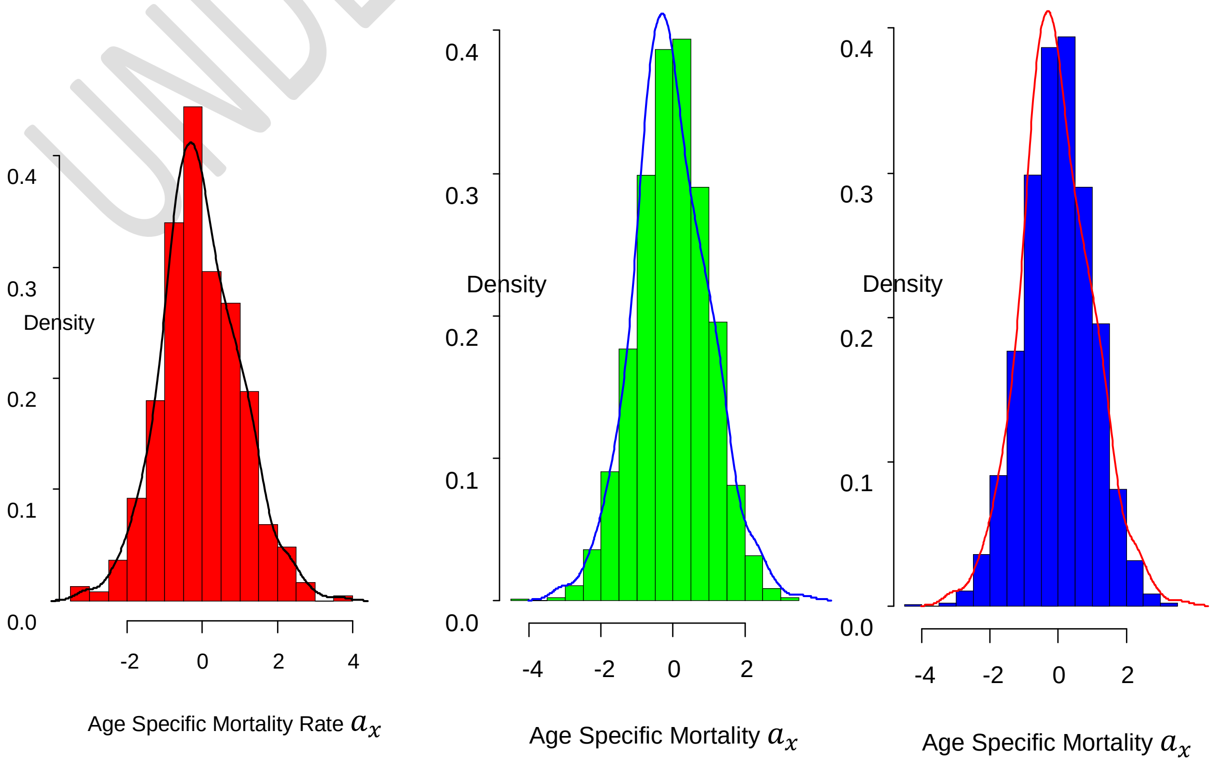


Figure 2: Histogram of the simulated data for n = 50, 100, 1000 for T = 50

The histogram (red) represents simulated data for n = 50 replicated 50 times. The distribution is normal which showed that the highest Age-Specific Mortality Rate is within the interval -0.5 and 0 which implies that the net reduction in Death Rate is above 0.4 and the lowest is between the interval 3.5 to 4 which implies the increase in Death Rate. Hence, there is more reduction in death rate. The green histogram represents simulated data (50 randomly generated number) with sample size 100 which tends to normal. It is also deduced that the highest Age-Specific Mortality Rate is within the interval 0 and 0.5 and the lowest is between the interval 3.5 to 4 which depicts a net increase in Death Rate. Hence, there is an increase in death rate. The blue histogram represents simulated data (50 randomly generated number) with sample size 1000 which behaves better than other sample sizes as it tends to normal. It is also deduced that the highest Age-Specific Mortality Rate is within the interval 0 and 0.5 and the lowest is between the interval 3.5 to 4 which depicts a net increase in Death Rate. Therefore, there is reduction in death rate. We therefore conclude from figure 3 follows the Central Limit theorem which states that the higher the sample size, the more the distribution tends to normal because as the sample sizes increases, the histogram tends to a normal distribution

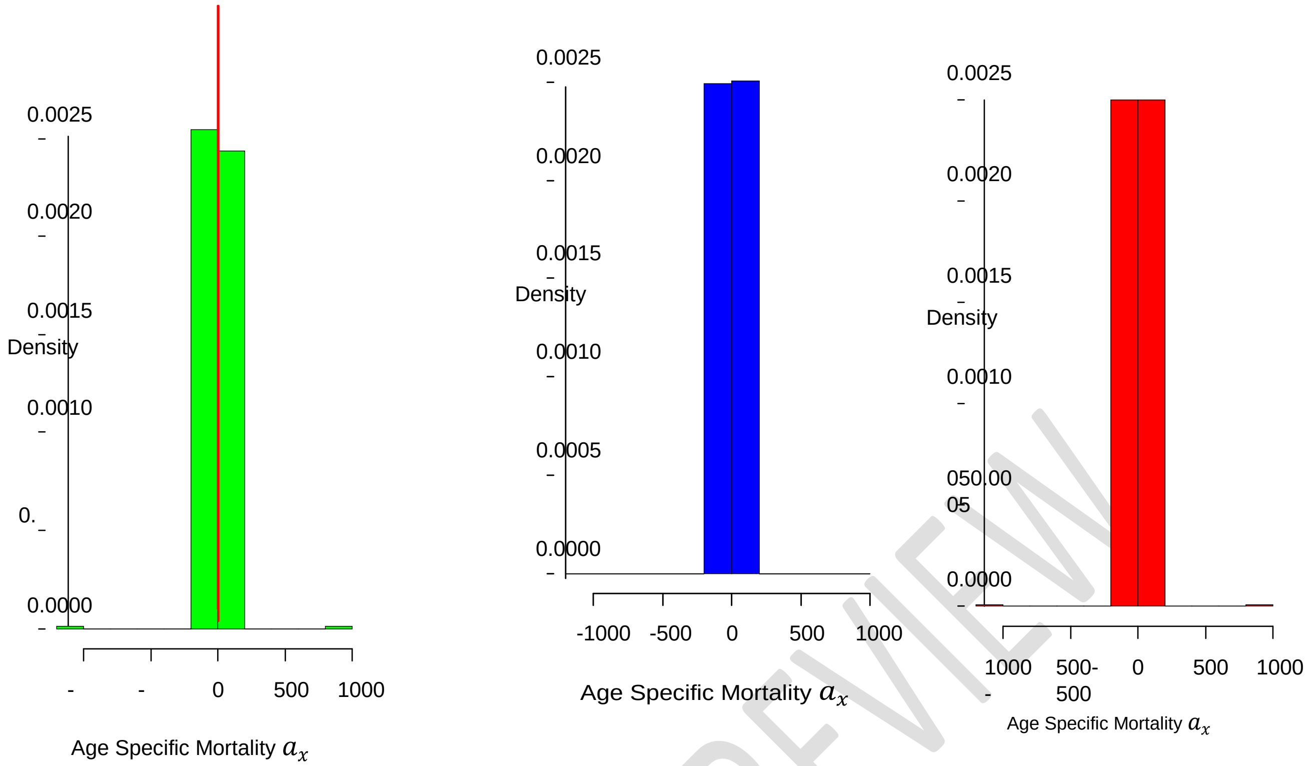


Figure 3: Histogram of the simulated data when n= 10, and T= 50

The figure above (green) represents simulated data with imposed outliers (50 randomly generated number) and sample size 10. It is obvious that the histogram is clustered at the middle. This is a proof that the injected outlier, inflated the Age Specific Mortality Rate. Also, it is observed that the higher density of Age-Specific Mortality Rate is within the interval 0 and -250 and the lowest is within the interval 0 and 250 which depicts that Death Rate increases. The figure (blue) denotes simulated data with imposed outliers (50 randomly generated number) and sample size 100. It is obvious that the histogram is clustered at the middle. This is a proof that the injected outlier, inflated the Age Specific Mortality Rate. Also, it is observed that the high density of Age-Specific Mortality Rate is within the interval 0 and 250 and the low is within the interval 0 and -250 which depicts that Death Rate is increased. The figure (red) denotes simulated data with imposed outliers (50 randomly generated number) and sample size 1000. It is obvious that the histogram is clustered at the middle. This is a proof that the injected outlier, inflated the Age Specific Mortality Rate. Also, it is observed that the high density of Age-Specific Mortality Rate is within the interval 0 and 250 and the low is within the interval 0 and -250 have the same density level of 0.0025. This means that there is neither an increase nor a decrease in Death Rate.

Case I: Normal Situation (no outlier is imposed) For Assumed Parameter $\boldsymbol{a_x} = \boldsymbol{-3.97}$

Table 2: Average parameter estimate of $\boldsymbol{a_x}$ in a normal scenario (when data is not defective)

Estimation Method	$n = 10$	$n = 100$	$n = 1000$
MOM	-5.01000	-4.95000	-4.34000
WLS	-5.22000	-4.85000	-4.11000
MLE	-4.75000	-4.72000	-4.02000

Comparing the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 2, we observe that the estimated parameter $\hat{\boldsymbol{a}}_x = \boldsymbol{-4.02}$ of the Maximum Likelihood Estimation MLE with sample size 1000 is the closest to the assumed parameter $\boldsymbol{a_x} = \boldsymbol{-3.97}$ We could deduce from the above tables that $n \longrightarrow \infty$ as $\widehat{a_x} \longrightarrow a_x$

For Assumed Parameter $\boldsymbol{b_x} = \boldsymbol{0.184}$

Table 3: Average parameter estimate of $\boldsymbol{b_x}$in a normal scenario (when data is not defective)

Estimation Method	$n = 10$	$n = 100$	$n = 1000$
MOM	2.11000	1.94000	0.67000

WLS	1.22000	0.67000	0.34000
MLE	0.85000	0.54000	0.20000

Observing closely the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 3, we observe that the value of the estimated parameter $\hat{\boldsymbol{b}}_{\boldsymbol{x}} = \mathbf{0.2}$ Maximum Likelihood Estimation MLE with sample size 1000 is the closest to the assumed parameter $\boldsymbol{b}_{\boldsymbol{x}} = \mathbf{0.184}$. This implies that $n \longrightarrow \infty$ as $\widehat{b_x} \longrightarrow b_x$

For Assumed Parameter $k_t = 4.52$

Table 4: Average parameter estimate of $\boldsymbol{K_t}$ in a normal scenario (when data is not defective)

Estimation Method	$\boldsymbol{n = 10}$	$\boldsymbol{n = 100}$	$\boldsymbol{n = 1000}$
MOM	5.65000	5.21000	4.76000
WLS	5.23000	4.85000	4.23000
MLE	4.98000	4.55000	4.46000

Studying the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 4, we observe that the value estimated Parameter $\widehat{k_t} = 4.46$ of Maximum Likelihood Estimation MLE with sample size 1000 is the closest to the assumed parameter $k_t = 4.52$. We could deduce from the above tables that $n \longrightarrow \infty$ as $\widehat{k_t} \longrightarrow k_t$. We could deduce from the above tables 2, 3, and 4 that $n \longrightarrow \infty$ as $\widehat{a_x} \longrightarrow a_x$, $\widehat{b_x} \longrightarrow b_x$ and $\widehat{k_t} \longrightarrow k_t$. Hence, we conclude that the larger the sample size the closer the estimated parameter is to the true parameter.

Defective data (outliers are imposed)

Case I1: Situation when outliers (lower values) are imposed on the generated data

Assumed Parameter $\boldsymbol{a_x} = \boldsymbol{-3.9}$

Table 5: Average parameter estimate of $\boldsymbol{a_x}$

Estimation Method	$\boldsymbol{n = 10}$	$\boldsymbol{n = 100}$	$\boldsymbol{n = 1000}$
Mom	-8.22000	-6.95000	-6.34000
WLS	-6.32000	-5.56000	-5.44000
MLE	-6.75000	-5.72000	-5.66000

Studying the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 5, we observe that the value estimated Parameter $\widehat{a_x} = -5.44$ of Weighted Least Square with sample size 1000 is the closest but with -1.5 difference from the assumed parameter $a_x = -3.9$.

Assumed Parameter $\boldsymbol{b_x} = \boldsymbol{0.184}$

Table 6: Average parameter estimate of $\boldsymbol{b_x}$

Estimation Method	$\boldsymbol{n = 10}$	$\boldsymbol{n = 100}$	$\boldsymbol{n = 1000}$
Mom	3.11000	2.94000	1.27000
WLS	3.12000	2.64000	1.64000
MLE	2.85000	2.54000	1.50000

Studying the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 6, we observe that the value estimated Parameter $\widehat{b_x} = 1.50$ of Weighted Least Square with sample size 1000 is the closest but with 0.476 difference from the assumed parameter $b_x = 0.184$

Assumed Parameter $k_t = 4.52$

Table 7: Average parameter estimate of K_t

Estimation Method	$n = 10$	$n = 100$	$n = 1000$
Mom	3.65000	2.14000	1.96000
WLS	2.22000	2.18000	1.92000
MLE	2.10000	2.00000	1.84000

Table 7 above shows that the value of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation, we observe that the value estimated Parameter $\widehat{k_t} = 3.65$ of Weighted Least Square with sample size 1000 is the closest but with 0.87 difference from the assumed parameter $k_t = 4.52$. From table 5, 6, and 7, we could conclude that the imposed outlier ((injecting big values) situation of the estimated parameter is not a perfect situation for estimating the parameters as the values of the estimated parameter gets inflated. Hence, $\widehat{a_x} \nrightarrow a_x$, $\widehat{b_x} \nrightarrow b_x$ and $\widehat{k_t} \nrightarrow k_t$ as $n \longrightarrow \infty$

Case III: Situation when outliers (big values) are imposed on the generated data

Assumed Parameter $\boldsymbol{a_x = -3.97}$

Table 8: Average parameter estimate of a_x

Estimation Method	$n = 10$	$n = 100$	$n = 1000$
Mom	-4.52000	-4.15000	-3.31000
WLS	-4.33000	-4.16000	-3.42000
MLE	-5.85000	-5.44000	-4.01000

Studying the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 8, we observe that the value estimated Parameter $\widehat{a_x} = -4.01$ of Weighted Least Square with sample size 1000 is the closest but with -0.04 difference from the assumed parameter $a_x = -3.97$

Assumed Parameter $\boldsymbol{b_x = 0.184}$

Table 9: Average parameter estimate of b_x

Estimation Method	$n = 10$	$n = 100$	$n = 1000$
Mom	2.11000	1.84000	1.17000
WLS	2.23000	1.54000	0.94000
MLE	1.67000	0.95000	0.30000

Studying the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 9, we observe that the value estimated Parameter $\widehat{b_x} = 0.3$ of Weighted Least Square with sample size 1000 is the closest but with 0.116 difference from the assumed parameter $b_x = 0.184$

Assumed Parameter $k_t = 4.52$

Table 10: Average parameter estimate of K_t

Estimation Method	$n = 10$	$n = 100$	$n = 1000$
Mom	6.65000	5.14000	4.96000
WLS	5.22000	5.18000	4.92000
MLE	5.10000	4.80000	4.67000

Studying the values of the estimated Parameter with respect to sample sizes 10, 100, 1000 and the 3 methods of Estimation in table 10, we observe that the value estimated Parameter $\widehat{k_t} = -4.01$ of Weighted Least Square with sample size 1000 is the closest but with -0.04 difference from the assumed parameter $k_t = 4.52$.

From table 8, 9, and 10, we conclude that the imposed outlier (injecting smaller value) situation of the estimated parameter is not a perfect scenario for estimating the parameters as the values of the estimated parameter gets inflated. Hence, $\widehat{a_x} \nrightarrow a_x$, $\widehat{b_x} \nrightarrow b_x$ and $\widehat{k_t} \nrightarrow k_t$ as $n \longrightarrow \infty$

Replication of the simulated data over equal interval and different sample sizes.

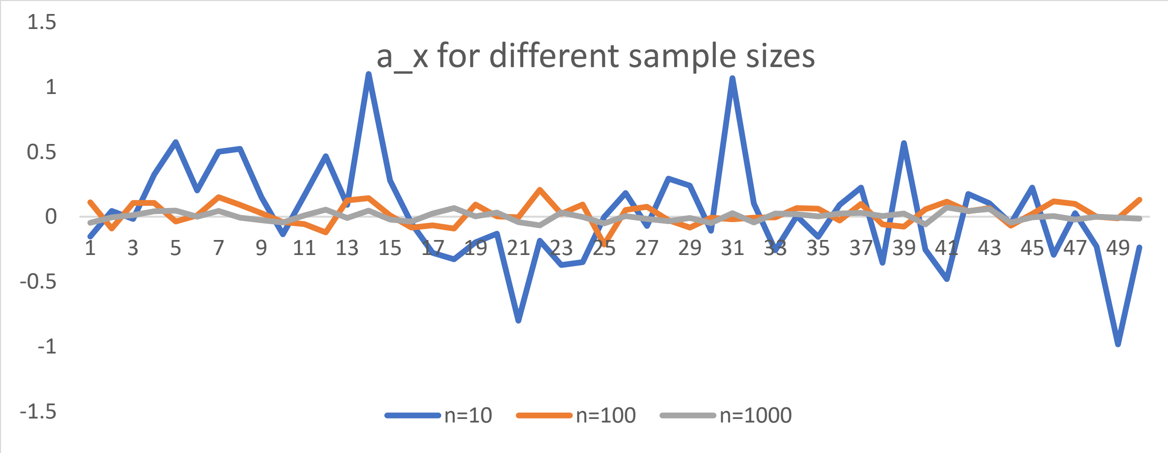


Figure 4: Plot of estimated a_x showing the replication of the simulated data over equal interval and different sample sizes.

Considering figure 4, There is constant irregularities in the trend of the simulated data with equal interval and different sample sizes. From the figure, it is deduced that the sample size 1000 behaves structurally-well when compared to sample size 10 and 100. And sample size 100 behaves better n=10. Hence, we conclude that the behavioral pattern performs better as the sample sizes increases.

Graphical Representation of the three methods of estimation

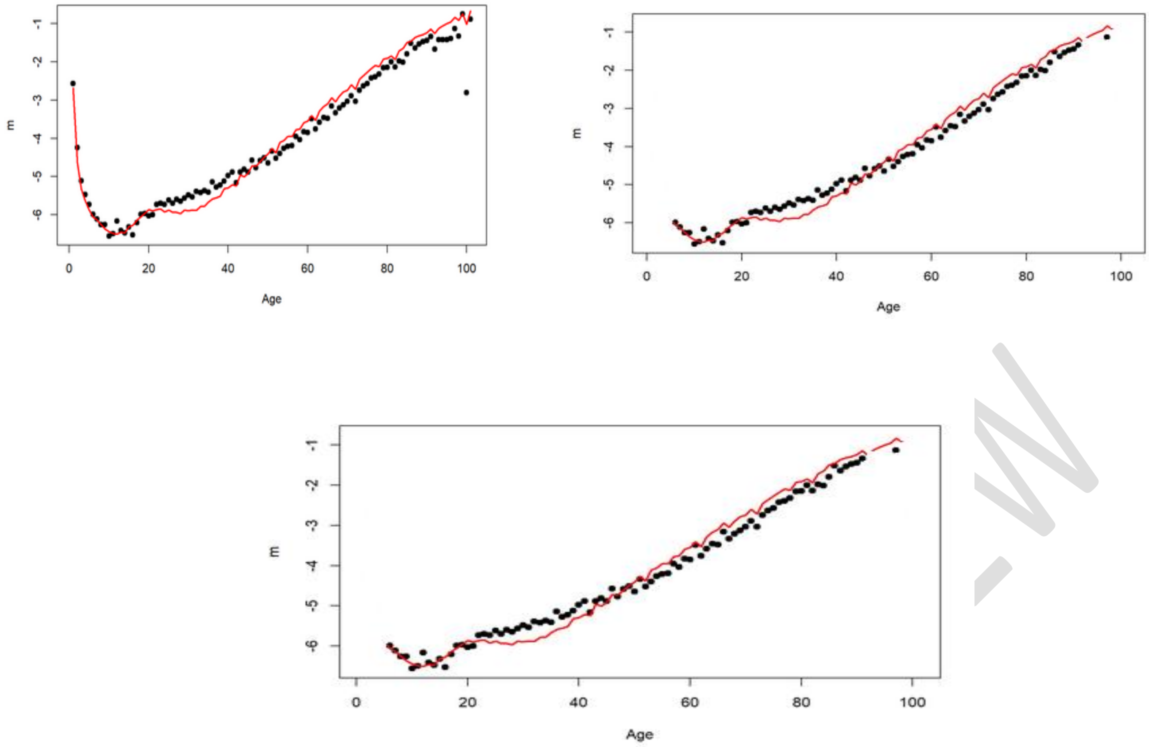


Figure 5: Plot of estimated m_x for the Methods Of Moment, Weighted Least Square and Maximum Likelihood Estimation

Observing Figure 5 very closely, and considering the fact that the plot representing the Method of moments shows that the points are a bit scattered- not clustered around the line. And this depict that the Method of moment, Weighted Least Square are not a perfect method for estimating parameters in the Lee-Carter's Model and considering the fact that the plot representing the Maximum Likelihood Estimation shows that the points are clustered around the line and when compared to the WLS and MOM, we can say that the MLE is the best fit for the Lee Carter Model.

Table 11: Evaluation of the methods of estimation used when $n = 10$

Method	Sex	Mean Square Error	Mean Bias Error
MOM	Male	0.55638	**-0.74591**
	Female	0.57355	**-0.75733**
MLE	Male	0.59427	-0.77089
	Female	0.61200	-0.78231
WLS	Male	1.14696	-1.07096
	Female	1.17155	-1.08238

Table 11 above showed that the values of the estimated parameters with different methods of estimation are not close to zero, hence at sample size n=10 the methods of estimations are all biased except for -0.74591 and -0.75733 with Mean Square Error 0.556382 and 0.573549 under the Method of Moment which though not close to zero, but better than other method of estimation.

Table 12: Evaluation of the methods of estimation used when $n = 100$

Method	Sex	Root Mean Square Error	Mean Bias Error
MOM	Male	0.06205	-0.24910
	Female	0.06416	-0.25330
MLE	Male	0.00015	**-0.01230**

	Female	0.00049	**-0.0221**
WLS	Male	0.18140	-0.42591
	Female	0.18261	-0.42733

Table 12 above explains that the values of the estimated parameters with different methods of estimation are not close to zero, hence at sample size n=100 the methods of estimations are all biased except for -0.0123and -0.0221 with Mean Square Error 0.000151and 0.000488 under the Maximum Likelihood Estimation is very close to zero. And that makes Maximum Likelihood Estimation the best Method of Estimation for sample size n=100

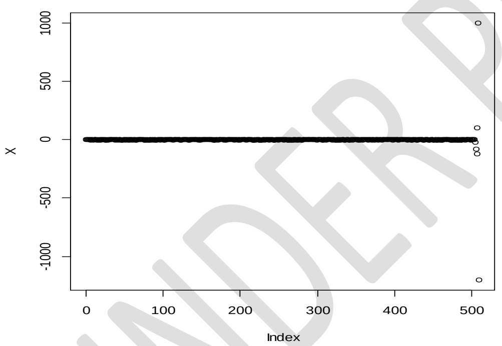
Table 13: Evaluation of the methods of estimation used when $n = 1000$

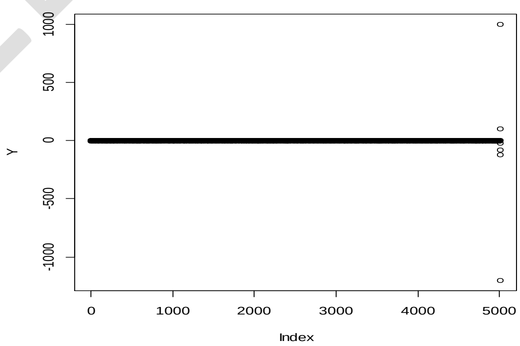
Method	**Sex**	**RMSE**	**Mean Bias Error**
MOM	Male	0.06205	-0.12200
	Female	0.06416	-0.12900
MLE	Male	0.00015	**-0.00120**
	Female	0.00048	**-0.00150**
WLS	Male	0.18139	-0.51100
	Female	0.01488	-0.52400

Table 13 above describes the values of the estimated parameters with different methods of estimation are not close to zero, hence at sample size n=10 the methods of estimations are all biased except for -0.0012and 0.0015 with Mean Square Error 0.000151and 0.000488 under the Method of Moment which is very close to zero. This implies that at n=100, the best Method of Estimation is still Maximum Likelihood Estimation.

Evaluating the 3 methods of estimation when the sample size n=10,100,1000. We infer that the Maximum Likelihood Estimator has a Mean Biased Error value closer to zero as the sample size increases from 10 to 1000. Hence the closer the biased to zero, the more unbiased the method of estimation. We therefore conclude that the MLE is the best method of estimation for Lee Carter Model when compare to Method of Moment and Weighted Least Square Method.

Case scenario II & III: In this scenario, an outlier (small and big values) were deliberately imposed on the generated dataset used and from the simulation observe the performance of the Method of Estimations used.





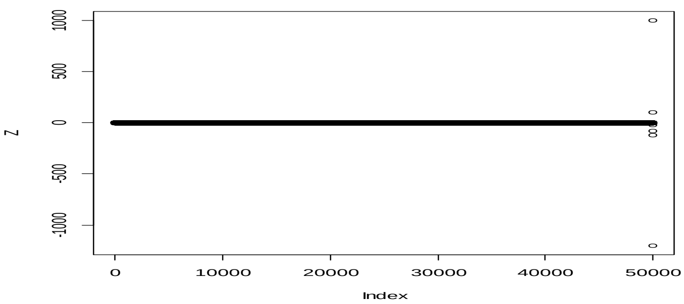


Figure 6: Plots of the replication when injected by an outlier (big and small value) when n=10, 100 and 1000

Table 14: Evaluation of the methods of estimation when $n = 10$ with defective data

Method	**Sex**	**MSE**	**Mean Bias Error**
MOM	Male	1.55002	**-1.24500**
	Female	1.58760	**-1.26000**
MLE	Male	1.65894	-1.28800
	Female	1.63584	-1.27900
WLS	Male	6.27002	-2.50400
	Female	6.52803	-2.5550

Table 14 above describes the values of the estimated parameters with different methods of estimation are not close to zero, hence at sample size n=10 the methods of estimations are all biased except for -1.245 and -1.260 with Mean Square Error 1.550025and 1.5876 under the Method of Moment which is very close to zero. This implies that at n=10

Table 15: Evaluation of the methods of estimation for used when $n = 100$ with defective data

Method	**Sex**	**RMSE**	**Mean Bias Error**
MOM	Male	1.54008	-1.24100
	Female	1.56250	-1.25000

MLE	Male	1.26338	**-1.12400**
	Female	1.25888	**-1.12200**
WLS	Male	2.03063	-1.42500
	Female	2.03633	-1.42700

Table 15 above describes the values of the estimated parameters with different methods of estimation are not close to zero, hence at sample size n=100 the methods of estimations are all biased except for -1.124 and -1.122 with Mean Square Error 1.263376 and 1.258884 under the Method of Moment which is very close to zero. This implies that at n=100, the best Method of Estimation is still Maximum Likelihood Estimation.

Table 16: Evaluation of the methods of estimation for used when $n = 1000$ for defective data

Method	**Sex**	**RMSE**	**Mean Bias Error**
MOM	Male	0.01513	-0.12300
	Female	0.01538	-0.12400
MLE	Male	0.00059	**-0.02420**
	Female	0.00060	**-0.02450**
WLS	Male	1.71872	-1.31100
	Female	1.75298	-1.32400

Table 16 above describes the values of the estimated parameters with different methods of estimation are not close to zero, hence at sample size n=10 the methods of estimations are all biased except for -0.02420 and -0.02450 with Mean Square Error 0.00059 and 0.00060under the Method of Moment which is very close to zero. This implies that at n =100, the best Method of Estimation is still the Maximum Likelihood Estimation. Considering the tables 14, 15 and 16 above, it is obvious that the Root Mean Square Error and Mean Biased Error of the Maximum Likelihood Estimation still serve as the best Method of Estimation even when the data are defective.

Conclusion

The analysis established the fact that the best method of Estimation of Lee carter's Parameter is Maximum Likelihood Estimation for both the data simulated with an outlier and those without the outlier. It is also observed that for all parameters of the Lee-Carter Model, the sample sizes increased with an increase in variable x that is, the age and the estimated parameter approaches the assumed parameter as the sample sizes increase.

COMPETING INTERESTS:

Authors have declared that they have no known competing financial interests OR non-financial interests OR personal relationships that could have appeared to influence the work reported in this paper.

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