Approximation properties of linear positive operators with differences

Abstract. This paper is the study of approximation properties of differences of linear positive operators. Here we have discussed quantitative estimates for the differences of Baskakov with Baskakov-Szasz and Baskakov-Durrmeyer operators. Difference properties of Baskakov-Szasz and Baskakov-Durrmeyer operators also have given. Finally, we obtain the quantitative estimate in terms of the weighted modulus of smoothness for these operators

Key words: Difference of operators, Linear positive operators, Modulus of continuity, Approximation theory, Baskakov-operators.

AMS Mathematics Subject Classification: 41A25, 41A35.

1. Introduction

In the year 1953, Korovkin found the most powerful and easiest criterion in order to decide approximation process with linear positive operators on continuous functions. After that a considerable amount of research on linear positive operators has been done by various mathematicians. e.g. [1], [2], [3], [4], [5], [6], [7] etc.

In the recent years, the study on the difference of linear positive operators is an active area of research. Difference of summation integral type operators was studied in the last few years. Such problem was initiated by A. Lupas [8]. The operators involved are usually on continuous functions defined on real intervals. Aral-Inoan-Rasa [9], Acu et. al. [10], V. Gupta [11], Gupta-Tachev [12] discovered some noteworthy results on difference of such operators.

What we hope to achieve in this paper is the study of approximation properties of difference on generalized Baskakov operator. First, we recall classical Baskakov operators [13], which for $f \in C[0,\infty)$ are defined as

$$B_n(f,x) = \sum_{u=0}^{\infty} \binom{n+u-1}{u} x^u (1+x)^{-n-u} f\left(\frac{u}{n}\right). \qquad \textbf{(1.1)}$$

We consider $F_{n,u}, G_{n,u}: E \to R,$ where E is a subspace of $C[0,\infty)$ having polynomials of degree up to four. Following operators have been defined for the difference of operators

$$M_n(f,x) = \sum_{u=0}^{\infty} p_{n,u} F_{n,u}(f), \;\; L_n(f,x) = \sum_{u=0}^{\infty} p_{n,u} G_{n,u}(f),$$

where $F_{n,u}(e_0) = G_{n,u}(e_0) = 1$.

We can define operators (1.1) as $B_n(f,x) = \sum_{u=0}^{\infty} p_{n,u}(x) F_{n,u}(f) = \sum_{u=0}^{\infty} p_{n,u}(x) f\left(\frac{u}{n}\right),$ **(1.2)** where $p_{n,u}(x)$ is function having Baskakov basis and it is defined as $p_{n,u}(x) = \binom{n+u-1}{u} x^u (1+x)^{-n-u}$.

Notations: Throughout the present papers following notations are used $d^F = F(e_1)$, $\mu_k^F = F(e_1 - d^F e_0)^k, k \in N$.

Operators applied for differences:

Baskakov-Szasz type operators: In [14] The Baskakov-Szasz type operators are defined as

$$S_n(f,x) = n\sum_{u=0}^{\infty} p_{n,u}(x) \int_0^\infty q_{n,u}(t) f(t)dt, \tag{1.3}$$

$p_{n,u}(x)$ is defined in (1.2) and $q_{n,u}(t) = \frac{e^{-nt}(nt)^u}{u!}$.

Operators (1.3) can also be written as $S_n(f,x) = \sum_{u=0}^{\infty} p_{n,u}(x) G_{n,u}(f)$.

Baskakov-Durrmeyer type operators: These operators are defined as, see [15]

$$D_n(f,x) = \sum_{u=0}^{\infty} p_{n,u}(x) H_{n,u}(f),$$

where,

$$H_{n,u}(f) = \frac{1}{B(u,n+1)} \int_0^\infty \frac{t^{u-1}}{(1+t)^{n+u+1}} f(t)dt.$$

2. Basic Results

Here we establish some lemmas and propositions, which are useful for the proof of main theorems.

Proposition1: Denoting $F_{n,u}(f) = f(\frac{u}{n})$ such that $F_{n,u}(e_0) = 1$, $d^{F_{n,u}} := F_{n,u}(e_1)$ and considering $\mu_2^{F_{n,u}} = F_{n,u}(e_1 - d^{F_{n,u}} e_0)^k$, $k \in N$, then we have

$$\mu_2^{F_{n,u}} = F_{n,u}(e_1 - d^{F_{n,u}} e_0)^2 = 0$$

$$\mu_2^{F_{n,u}} = F_{n,u}(e_1 - d^{F_{n,u}} e_0)^4 = 0.$$

Proposition 2: Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2$. Let $x \in [0,\infty)$, then for $n \in N$, we get

$$|(M_n - L_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'',\delta_1)}{2}\big(1+\gamma(x)\big) + 2\omega(f,\delta_2),$$

where, $\gamma(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_2^{F_{n,u}} + \mu_2^{G_{n,u}}\right)$

and $\delta_1^2 = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_4^{F_{n,u}} + \mu_4^{G_{n,u}}\right)$, $\delta_2^2 = \sum_{u=0}^{\infty} p_{n,u}(x)(d^{F_{n,u}} - d^{G_{n,u}})^2$.

$C_B[0,\infty)$ denotes the class of continuous and bounded functions defined for $x \ge 0$,

$$\|\cdot\| = \sup_{x\in[0,\infty)} |f(x)| < \infty.$$

Lemma 1: Some moments of the operators discussed in (1.2) are as follows

(i) $B_n(e_0,x) = 1$,

(ii) $B_n(e_1,x) = x$,

(iii) $B_n(e_2,x) = \frac{x}{n} + \frac{(n+1)x^2}{n}$,

(iv) $\quad B_n(e_3,x)=\frac{x}{n^2}+\frac{3(n+1)x^2}{n^2}+\frac{(n+1)(n+2)x^3}{n^2},$

(v) $\quad B_n(e_4,x)=\frac{x}{n^3}+\frac{7(n+1)x^2}{n^3}+\frac{6(n+1)(n+2)x^3}{n^3}+\frac{(n+1)(n+2)(n+3)x^4}{n^3}.$

Recurrence relation for above moments is given as

$$B_n(e_{m+1},x)=\frac{x(1+x)}{n}B_n'(e_m,x)+xB_n(e_m,x).$$

3. Difference between the operators

In this section, we allocate quantifiable estimations for difference of Baskakov operators with Baskakov-Szasz type operators, Baskakov operators with Baskakov-Durrmeyer operators and with Baskakov-Szasz operators with Baskakov-Durrmeyer operators.

Proposition 3: By simple calculations as $e_k(t)=t^k$, where $k\in N^{\circ}$, we obtain

$$G_{n,u}(e_k)=\int_0^\infty q_{n,u}(t)t^k dt=\frac{(u+k)!}{u!n^k}.$$

Hence $d^{G_{n,u}}=G_{n,u}(e_1)=\frac{u+1}{n}$

and $\mu_2^{G_{n,u}}=G_{n,u}(e_1-d^{G_{n,u}}e_0)^2=G_{n,u}(e_2,x)+\left(\frac{u+1}{n}\right)^2-2G_{n,u}(e_1,x)\left(\frac{u+1}{n}\right)$

$$=\frac{(u+2)(u+1)}{n^2}-\left(\frac{u+1}{n}\right)^2=\frac{(u+1)}{n^2}$$

also $\mu_4^{G_{n,u}}=G_{n,u}(e_1-d^{G_{n,u}}e_0)^4=G_{n,u}(e_4,x)-4G_{n,u}(e_3,x)\left(\frac{u+1}{n}\right)+6G_{n,u}(e_2,x)\left(\frac{u+1}{n}\right)^2$

$$-4G_{n,u}(e_1,x)\left(\frac{u+1}{n}\right)^3+G_{n,u}(e_0,x)\left(\frac{u+1}{n}\right)^4$$

$$=\frac{(u+4)(u+3)(u+2)(u+1)}{n^4}-4\frac{(u+3)(u+2)(u+1)}{n^3}\left(\frac{u+1}{n}\right)+6\frac{(u+2)(u+1)}{n^2}\left(\frac{u+1}{n}\right)^2$$

$$-4\left(\frac{u+1}{n}\right)\left(\frac{u+1}{n}\right)^3+\left(\frac{u+1}{n}\right)^4=\frac{3(u^2+4u+3)}{n^4}.$$

Theorem 1: (Difference between Baskakov-Szasz operators and Baskakov operators)

Let $f^{(r)}\in C_B[0,\infty), r=0,1,2$. Let $0\le x<\infty$, then for any natural number n, we get

$$|(S_n-B_n)f(x)|\le\frac{\|f''\|}{2}\gamma(x)+\frac{\omega(f'',\delta_1)}{2}\big(1+\gamma(x)\big)+2\omega(f,\delta_2(x)),$$

where, $\gamma(x)=\frac{nx+1}{n^2}$, $\delta_1^2(x)=\frac{3x^2n(n+1)+15nx+9}{n^4}$, $\delta_2^2(x)=\frac{1}{n^2}$.

Proof: Following propositions 1, 2 and 3 and also using Lemma 1, we have

$$\gamma(x)=\sum_{u=0}^{\infty}p_{n,u}(x)\left(\mu_2^{F_{n,u}}+\mu_2^{G_{n,u}}\right)=\sum_{u=0}^{\infty}p_{n,u}(x)\frac{(u+1)}{n^2}$$

$$=\frac{1}{n}B_n(e_1,x)+\frac{1}{n^2}=\frac{nx+1}{n^2}.$$

$$\delta_1^2(x)=\sum_{u=0}^{\infty}p_{n,u}(x)\left(\mu_4^{F_{n,u}}+\mu_4^{G_{n,u}}\right)$$

$$=\sum_{u=0}^{\infty}p_{n,u}(x)\mu_4^{G_{n,u}}=\frac{3x^2n(n+1)+15nx+9}{n^4}$$

$$\delta_2^2(x)=\sum_{u=0}^{\infty}p_{n,u}(x)(d^{F_{n,u}}-d^{G_{n,u}})^2$$

$$= \sum_{u=0}^{\infty} p_{n,u}(x)\left[\frac{u}{n} - \frac{u+1}{n}\right]^2 = \frac{1}{n^2}.$$

Collecting above estimates, we get required result.

Proposition 4: By simple calculations with $e_k(t) = t^k, k \in N^{\circ}$, we obtain $H_{n,u}(e_k) = \frac{(u+r-1)!(n-k)!}{(u-1)!n!}$.

Hence $d^{H_{n,u}} = H_{n,u}(e_1) = \frac{u}{n}$.

$$\mu_2^{H_{n,u}} = H_{n,u}(e_1 - d^{H_{n,u}}e_0)^2 = H_{n,u}(e_2) - 2H_{n,u}(e_1)\left(\frac{u}{n}\right) + H_{n,u}(e_0)\left(\frac{u}{n}\right)^2$$

$$= \frac{u(u+1)}{n^2-n} - \left(\frac{u}{n}\right)^2 = \frac{u(u+n)}{n^2(n-1)},$$

and $\mu_4^{H_{n,u}} = H_{n,u}\left(e_1 - d^{H_{n,u}}e_0\right)^4 = H_{n,u}(e_4, x) - 4H_{n,u}(e_3, x)\left(\frac{u}{n}\right) + 6H_{n,u}(e_2, x)\left(\frac{u}{n}\right)^2$

$$-4H_{n,u}(e_1, x)\left(\frac{u}{n}\right)^3 + H_{n,u}(e_0, x)\left(\frac{u}{n}\right)^4$$

$$= \frac{(u+3)(u+2)(u+1)u}{n(n-1)(n-2)(n-3)} - 4\frac{(u+2)(u+1)u}{n(n-1)(n-2)}\left(\frac{u}{n}\right) + 6\frac{(u+1)u}{n(n-1)}\left(\frac{u}{n}\right)^2$$

$$-4\left(\frac{u}{n}\right)\left(\frac{u}{n}\right)^3 + \left(\frac{u}{n}\right)^4 = \frac{3\left((u^4(n+6)+2nu^3(n+6)+n^2u^2(n+8)+2n^3u\right)}{n^4(n-1)(n-2)(n-3)}.$$

Theorem 2: (Difference between Baskakov-Durrmeyer operators and Baskakov operators)

Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2.$ Let $0 \le x < \infty$, then for any natural number n, we get

$$|(D_n - B_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'', \delta_1)}{2}\left(1 + \gamma(x)\right),$$

where, $\gamma(x) = \frac{(n+1)x(1+x)}{n(n-1)}$ and

$$\delta_1^2(x) = \frac{3x(x+1)(n+1)}{(n-3)(n-2)(n-1)n^3}[n^3x(x+1) + n^2(11x^2 + 11x + 3) + n(36x^2 + 36x + 7) + 6(6x^2 + 6x + 1)].$$

Proof: Using Propositions 1, 3 and Lemma 1, we have the following

$$\gamma(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_2^{F_{n,u}} + \mu_2^{H_{n,u}}\right) = \frac{x(n+1)(n+2)}{n(n-1)}.$$

$$\delta_1^2(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_4^{F_{n,u}} + \mu_4^{H_{n,u}}\right) = \sum_{u=0}^{\infty} p_{n,u}(x)\mu_4^{H_{n,u}} = \frac{3x(x+1)(n+1)}{(n-3)(n-2)(n-1)n^3}$$

$[n^3x(x+1) + n^2(11x^2 + 11x + 3) + n(36x^2 + 36x + 7) + 6(6x^2 + 6x + 1)]$.

Subsequent to proposition 2, we get the result.

Theorem 3: (Difference between Baskakov-Szasz operators and Baskakov-Durrmeyer operators)

Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2.$ Let $0 \le x < \infty$, then for any natural number n, we get

$$|(S_n - D_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'', \delta_1)}{2}\left(1 + \gamma(x)\right) + 2\omega(f, \delta_2(x)),$$

where, $\gamma(x) = \frac{nx+1}{n^2} + \frac{x(1+x)(n+1)}{(n-1)n}$,

$\delta_1^2(x) = \frac{3x^2n(n+1)+15nx+9n}{n^4} + \frac{3x(x+1)(n+1)}{(n-3)(n-2)(n-1)n^3}[n^3x(x+1) + n^2(11x^2 + 11x + 3) + n(36x^2 + 36x + 7) + 6(6x^2 + 6x + 1)]$, $\quad \delta_2^2(x) = \frac{1}{n^2}$.

Proof: According to Lemma 1, Propositions 2, 3 and 4, we have

$$\gamma(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_2^{G_{n,u}} + \mu_2^{H_{n,u}}\right) = \frac{nx+1}{n^2} + \frac{x(1+x)(n+1)}{(n-1)n}$$

$$\delta_1^2(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_4^{G_{n,u}} + \mu_4^{H_{n,u}}\right) \quad = \frac{3x^2n(n+1) + 15nx + 9n}{n^4}$$

$$+ \frac{3x(x+1)(n+1)}{(n-3)(n-2)(n-1)n^3}[n^3x(x+1) + n^2(11x^2 + 11x + 3) + n(36x^2 + 36x + 7) + 6(6x^2 + 6x + 1)]$$

$$\delta_2^2(x) = \sum_{u=0}^{\infty} p_{n,u}(x)(d^{G_{n,u}} - d^{H_{n,u}})^2 = \frac{1}{n^2}.$$

Combining the estimates and according to proposition 2, we get the result.

Conclusion: The present paper deals with the approximations of the differences of linear positive operators defined on bounded or unbounded interval in terms of higher order modulus of continuity.

There are several integral and other modifications, variations, and basic extensions of the Baskakov type operators. Here we have chosen Baskakov basis functions, others can choose other basis functions comparably.

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