Review Article

ECO-EPIDEMILOGICAL MODEL ANALYSIS OF PERY-PREDATOR SYSTEM

Abstract

In this paper disease transmission dynamics of predator-prey population is studied. Thus we have taken a prey-predator population of species containing of three compartment is developed. These species are Susceptible Prey, Infected Prey, and predator. The functional response of predation is assumed to follow modified holling type II functional response & a governing equation non-linear ODE is constructed. Existence, Positivity, and boundedness of solutions to the governing equations are checked. Stability analysis of steady state points of the governing equation has been done by taking different restrictions. Local & global stability of disease free steady state point , endemic steady state points have been studied with the concept of variation matrix & Liapunove functions respectively. Basic reproduction number is drived, and appropriate values are pluged to the parameters. Numerical simulations are done with the help of DEDiscover software that clarify the analytical results.

Key words: Model, Ecology, Epidemiology, Stability, Reproduction Number, Simulation

1. Introduction

"Predator-prey population of interaction of species is a remarkable work of Lotka-Volterra in 1920s" [1,3,5,6,16,17], & "SIR model of Compartmentalization of population dynamics is well known fundamental area of research of Kermack and Mckendrick" [1-3,5-10,15]. Anderson and May Joined these two Mathematical modeling systems, while Chattopadhyay and Arino were the first who have been using the term "eco-epidemiology" for the combined models of ecology and epidemiology [3, 5, 7, 16, and 17]. "The disease transmission dynamics of in predator-prey population is now becomes a vital area of research due to the fact that prey-predator population interaction is rich & complex in nature" [4, 6, 7, 11-13]. "A lot of mathematical models have been constructed & studied on predator-prey populations" [1-7, 9-12]. Several studies focused on the study of disease in a prey only[1-5,7,12], Some researchers were motivated in the study of disease within the predator population only[14], and there are also a few studies on diseases in both prey & predators[6,9,11,16,17] In this study, we have proposed and studied infectious disease only in prey population with predation follows modified holling type II functional responses.

2. Model Formulations and Assumptions

We took a predator-prey population with three compartments having Susceptible prey $x(t)$, infected prey $w(t)$,and predator $y(t)$ populations.

1. Suppose that in absence of disease, susceptible prey population grows logistic function $g(x)$ with intrinsic growth rate r , environmental carrying capacity k . Only susceptible prey can reproduce to reach its carrying capacity, and infected prey does not grow, reproduce and recover from the disease once infected.
2. Disease transmission rate from infected prey $w(t)$ to susceptible prey $x(t)$ follow non-linear incidence function $\mathrm{I}(x,\ w)=\frac{\beta xw}{1+w}$, which was constructed by Gumel and Moghadas and used by different scholars where, the parameter β is infection rate, the simple mass action law βxw measures disease force of infection, and $\frac{1}{1+w}$ measures the inhibition effect from the crowding effect of infected population.
3. The functional response of predator towards the Susceptible prey $x(t)$ and infected prey $w(t)$ are assumed to follow a Modified holling type II functional response $p_1 f_1(x,\ w,\ y)=\frac{p_1 xy}{s+x+pw}$ and $\mathrm{p}_2\mathrm{f}_2(x,\ w,\ y)=\frac{\mathrm{p}_2 wy}{s+x+\mathrm{p}w}$,where $p_1,\ p_2$ are predation coefficients of $x(t)$, $w(t)$ due to predator $y(t)$ with predation preference rate p
4. Supposed that the Consumed susceptible prey & consumed infected prey converted into predator at Conversion rate q_1 & q_2 respectively with half saturated constant s
5. Only infected prey suffers from infectious disease with death rate d_1 and the remaining population predator and susceptible prey suffer with natural death rate d_2, d_3 respectively.

Table 1. Variables Notations & Descriptions

Variables	Descriptions

$\mathbf{X(t)}$	The number of susceptible prey population at time t
$\mathbf{W(t)}$	The Number of infected prey population at time t
$\mathbf{Y(t)}$	The number of predator population at time t

Table 2. Parameters Notations & Descriptions

Parameters	**Description of Parameters**
r, k	Intrinsic growth rate, Carrying capacity of susceptible prey respectively.
q_1, q_2	Conversion rate of susceptible prey , infected prey respectively
p_1, p_2	Predation coefficient of susceptible prey , infected prey respectively
p , s	Predation preference rate , Half saturated Constant respectively
d_1 , d_2	Death rate of infected prey, predator respectively.
$g(x)$	Logistic growth function of susceptible prey
$I(x,w)$	Non-linear incidence rate of disease transmission function
$f_1(x, w, y)$	Predation functional response of predator towards the susceptible prey
$f_2(x, w, y)$	Predation functional response of predators towards the infected prey

Based on the above assumptions, description of variables, & parameters, we establish a model diagram shown in Fig. 1

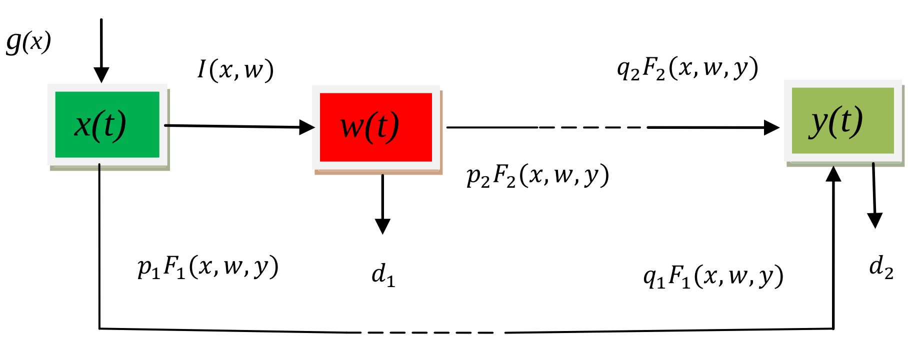

Figure 1. Model Diagram

$$\frac{dx}{dt} = g(x) - I(x, \quad y) - p_1 F_1(x, \quad w, \quad y) \qquad (1)$$

$$\frac{\mathrm{dw}}{\mathrm{dt}} = I(x, \quad y) - p_2 F_2(x, \quad w, \quad y) - \mathrm{d}_1 w \qquad (2)$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = q_1 F_1(x, w, y) + q_2 F_2(x, \quad w, \quad y) - \mathrm{d}_2 y \qquad (3)$$

Based on the model diagram, the governing equation written as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta x w}{1+w} - \frac{p_1 x y}{s+x+pw} \qquad (4)$$

$$\frac{\mathrm{dw}}{\mathrm{dt}} = \frac{\beta x w}{1+w} - \frac{\mathrm{p_2 wy}}{\mathrm{s}+x+pw} - \mathrm{d}_1 w \qquad (5)$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{q}_1 x y}{\mathrm{s}+x+\mathrm{p}w} + \frac{\mathrm{q}_2 w y}{\mathrm{s}+x+\mathrm{p}w} - \mathrm{d}_2 y \qquad (6)$$

with initial conditions $x(0) \geq 0, w(0) \geq 0, y(0) \geq 0$

3. Analysis of the Governing Equations

Let us verify, positivity, boundedness, & existence of solution of the governing equations by establishing the following lemmas and these analyses of the governing equations can be considered as primarly results.

Lemma1. Boundedness All solutions of governing equations (4)-(6) are bounded in feasible region $\mathbb{R}_+^3$

Proof: each solutions $x(t), w(t), y(t)$ of the model is bounded iff total predator-prey population N is bounded. Let us denote the total population of prey-predator $N = x + w + y$

For $\Lambda > 0$ be constant, $\frac{dN}{dt} + \Lambda N = \frac{dx}{dt} + \frac{dw}{dt} + \frac{dy}{dt} + \Lambda N$ (7)

Plug all governing Equations (4)-(6) into (7) and removing all terms with negative coefficient

$\frac{dN}{dt} + \Lambda N \leq rx\left(1 - \frac{x}{K}\right) - (p_1 - q_1)\frac{p_1 xy}{s+x+pw} - (p_2 - q_2)\frac{p_2 wy}{s+x+pw} - d_1 w - d_2 y = \mu$. Then Solving the differential inequality $\frac{dN}{dt} + \Lambda N \leq \mu$ yields $N(t) \leq \frac{\mu}{\Lambda}\left(1 - e^{-\Lambda t}\right) + N(0)e^{-\Lambda t}$ for $t \to \infty, N \to \frac{\mu}{\Lambda}$. It is known that the total prey-predator population is non-negative and hence $0 \leq N(t) \leq \frac{\mu}{\Lambda}$. Thus the invariant feasible region: $\Omega = \left\{(x, \ w, \ y) \in \mathbb{R}_+^3 : 0 \leq N(t) \leq \frac{\mu}{\Lambda}\right\}$ .This proves the Lemma1 and the governing equation is mathematically well posed

Lemma2. Positivity All solutions of governing equations (4)-(6) are positive.

Proof: To show that variables $x(t), w(t), y(t)$ of the Model (4)-(6) are all non-negative $\forall t \geq 0$.

i. **Positivity of x(t):**From the Susceptible prey Model in (4), $\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} - \frac{p_1 xy}{s+x+pw}$ Without loss of generality, After removing all terms with the positive coefficients from the right hand side of the differential equation, we have the following differential inequality; $\frac{dx}{dt} \geq -\left(\frac{rx^2}{k} + \frac{\beta xw}{1+w} + \frac{P_1 xy}{s+x+pw}\right)$ divide both sides by negative yields $-\frac{dx}{dt} \leq \frac{rx^2}{k} + \frac{\beta xw}{1+w} + \frac{p_1 xy}{s+x+pw}$,But It is also clear that the following inequality holds $\frac{rx^2}{k} + \frac{\beta xw}{1+w} + \frac{p_1 xy}{s+x+pw} \leq rx^2 + \beta xw + p_1 xy = x(rx + \beta w + p_1 y)$ Assume that $rx + \beta w + p_1 y = C$,Then the differential inequality reduced to $-\frac{dx}{dt} \leq x(rx + C)$.This inequality can be arranged for integration by partial fraction as $\int \frac{1}{x(rx+C)} dx \geq \int -d\,t$, integrating the integral inequality $\int \left(\frac{1/C}{x} + \frac{-r/C}{rx+C}\right) dx \geq -\int dt$ will give us $\frac{1}{C}\ln|x| - \frac{1}{C}\ln|rx + C| \geq -t + Q$,where Q is integration constant. Using rules of logarithm the inequality can be written as $\ln\left|\frac{x}{rx+C}\right| \geq -Ct + CQ$. Finally solving for x will give as $x(t) \geq \frac{ACe^{-Ct}}{1-rAe^{-Ct}}$, for $A = e^{CQ}$ .Therefore $x(t) > 0$ for $1 - rAe^{-Ct} > 0$. That is $x(t)$ is non-negative for $t > \frac{1}{C}\ln(rA)$

ii. **Positivity of w(t):** From infected prey Model in (5),$\frac{dw}{dt} = \frac{\beta xw}{1+w} - \frac{p_2 wy}{s+x+pw} - d_1 w$,Without loss of original generality, after removing all terms with the positive coefficients $\left(\frac{\beta xw}{1+w}\right)$. we obtain the following differential inequality

$\frac{dw}{dt} \geq -\left(\frac{p_2 wy}{s+x+pw} + d_1 w\right) \Leftrightarrow -\frac{dw}{dt} \leq \left(\frac{p_2 wy}{s+x+pw} + d_1 w\right)$, But it is clear that the inequality $\frac{p_2 wy}{s+x+pw} + d_1 w \leq p_2 wy + d_1 w = (p_2 y + d_1)w$ holds true. Now Assume that $p_2 y + d_1 = C$ .Then we have $-\frac{dw}{dt} \leq Cw$, Now applying integration yield $\ln|w| \geq -Ct + Q$,where Q is integration constant, Then solving for the variable $w(t)$ gives the equation $w(t) \geq e^{-Ct+Q}$ which is exponential function & positive at all time ,Hence $w(t)$ is positive.

iii. **Positivity of y(t):** From the Susceptible predator Model in (6),$\frac{dy}{dt} = \frac{q_1xy}{s+x+pw} + \frac{q_2wy}{s+x+pw} - d_2y$, without loss of original generality, after removing all terms having positive coefficients $\left(\frac{q_1xy}{s+x+pw} + \frac{q_2wy}{s+x+pw}\right)$, we obtain differential equation $\frac{dy}{dt} \geq -(d_2)y$ Then applying integration by separable of variable method results, $ln|y| \geq -(d_2)t + Q$,where Q integration constant. Solving for variable $Y(t)$, we obtain a solution $|y| \geq e^{-(d_2)t+Q}$. Therefore $y(t) \geq e^{-(d_2)t+Q}$ is a positive exponential function. Hence y(t) is positive.

Lemma3.Existence All Solutions of the governing equations (4)-(6) together with the initial conditions $x(0) > 0, w(0) \geq 0, y(0) \geq 0$ exist in $\mathbb{R}^3_+$ i.e., the model variables $x(t)$, $w(t)$, and $y(t)$ exist for all t and remain in $\mathbb{R}^3_+$.

Proof: Let us represent model (1)-(3) given as follows:

$f_1(x, \quad w, \quad y) = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} - \frac{p_1xy}{s+x+pw}$,

$f_2(x, \quad w, \quad y) = \frac{\beta xw}{1+w} - \frac{p_2wy}{s+x+pw} - d_1w$,

$f_3(x, \quad w, \quad y) = \frac{q_1xy}{s+x+pw} - \frac{q_2wy}{s+x+pw} - d_2y$,

Based on Derrick and Groosman theorem, let Ω denote the fesable region

$$\Omega = \left\{(x, \quad w, \quad y) \in \mathbb{R}^3_+; N \leq \frac{\mu}{\Lambda}\right\}.$$

Then governing equation (4)-(6) have a unique solution if $(\partial f_i)/(\partial x_j)$, $i, j = 1,2,3$ are continuous and bounded in Ω. Here, $x_1 = x$, $x_2 = w$, $x_3 = y$, the continuity and the boundedness can be shown as follows:

Table 3. Partial Derivatives of Functions of several variables

For f_1:	For f_3:
$\left\|\frac{\partial f_1}{\partial x}\right\| = \left\|r - \frac{2rx}{k} - \frac{\beta w}{1+w} - \frac{p_1 y(s+pw)}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_1}{\partial w}\right\| = \left\|-\frac{\beta x}{(1+w)^2} + \frac{p_1 pxy}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_1}{\partial y}\right\| = \left\|-\frac{p_1 x}{s+x+pw}\right\| < \infty$	$\left\|\frac{\partial f_3}{\partial x}\right\| = \left\|\frac{q_1 y(s+pw) - q_2 wy}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_3}{\partial w}\right\| = \left\|\frac{(q_2 - q_1)pxy}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_3}{\partial y}\right\| = \left\|\frac{q_1 x + q_2 w}{s+x+pw} - d_2\right\| < \infty$
For f_2: $\left\|\frac{\partial f_2}{\partial x}\right\| = \left\|\frac{\beta w}{1+w}\right\| < \infty$ $\left\|\frac{\partial f_2}{\partial w}\right\| = \left\|\frac{\beta x}{(1+w)^2} - \frac{p_2 y(s+x)}{(s+x+pw)^2} - d_1\right\| < \infty$ $\left\|\frac{\partial f_2}{\partial y}\right\| = \left\|-\frac{p_2 w}{s+x+pw}\right\| < \infty$	

Thus, all the partial derivatives $(\partial f_i)/(\partial x_j)$, $i, j = 1,2,3$ exist, continuous, & bounded in a region Ω for all positive value variables & parameters. Hence, by Derrick Groosman theorem, a solution for the governing equations (4)-(6) exists and unique.

4. Stability Analysis

When there is no predator *(y=0)*, the governing equations (4)-(6) is simplified as:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} = f(x, w) \qquad (8)$$

$$\frac{dw}{dt} = \frac{\beta xw}{1+w} - d_1 w = g(x, w) \qquad (9)$$

This is a subsystem which contains at most three non-negative steady state points, and those steady state points are: trivial steady state point $E_o(0, 0)$, axial steady state point $E_A(k, 0)$, & positive steady state point $E^*(\mathrm{x}, \mathrm{w})$ where

$$x = \frac{2kr + kd_1 - k\sqrt{d_1}\sqrt{4\beta + d_1}}{2r}, \qquad w = \frac{-d_1 + \sqrt{d_1}\sqrt{4\beta + d_1}}{2r}$$

It is clear visible that W is negative. Hence we reject the negative steady state point one and take only the non-negative steady state point, and the Jacobian Matrix of (8) - (9) is given by

$$J(x,w)=\begin{pmatrix} r-\frac{2rx}{k}-\frac{\beta w}{1+w} & -\frac{\beta x}{(1+w)^2} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2}-d_1 \end{pmatrix}$$

Theorem 1. The trivial steady state point $E_o(0,\ 0)$ of sub governing equation (8)- (9) always exists & E_o is a saddle point with locally unstable manifold in x-direction & locally stable manifold in the w-direction .

Proof: Consider the Jacobian matrix of sub governing equation(8)-(9) at trivial steady state point $J(E_o)=\begin{pmatrix} r & 0 \\ 0 & -d_1 \end{pmatrix}$.Thus Eigen values of $J(E_o)$ are $\lambda_1=r>0$ and $\lambda_2=-d_1<0$, Hence E_o is a saddle point with locally unstable manifold in x-direction and local stable manifold in the w-direction

Theorem 2.The axial steady state point $E_A(k,0)$ of sub governing equation (8)-(9) always exists &if if $\beta k-d_1<0$, E_A is locally asymptotically stable point and if $\beta k-d_1>0$, E_o is a saddle point with locally stable manifold in x-direction & locally unstable manifold in w-direction.

Proof: Consider the Jacobian matrix of governing equation (8)-(9) at Axial steady state point, $J(E_A)=\begin{pmatrix} -r & -\beta k \\ 0 & \beta k-d_1 \end{pmatrix}$.Thus Eigen values are $\lambda_1=-r<0$ and $\lambda_2=\beta k-d_1$ Thus if $\beta k-d_1<0$, E_A is locally asymptotically stable point, and if $k-d_1>0$, E_A is a saddle point with locally stable manifold in X-direction and locally unstable manifold in W-direction.

Theorem 3. The positive steady state point $E^*(X^*,W^*)$ of sub models (8) - (9) exists & stable if the inequalities

$$\left\{r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*}\right\}+\left\{\frac{\beta x^*}{(1+w^*)^2}-d_1\right\}>0,\&\ \left\{r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*}\right\}\left\{\frac{\beta x^*}{(1+w^*)^2}-d_1\right\}+\frac{\beta^2x^*w^*}{(1+w^*)^3}>0$$

holds true. Otherwise unstable

Proof: Consider the Jacobian matrix of sub governing equation (8)-(9) at positive steady state point $J^*(x^*,w^*)=\begin{pmatrix} r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*} & -\frac{\beta x^*}{(1+w^*)^2} \\ \frac{\beta w^*}{1+w^*} & \frac{\beta x^*}{(1+w^*)^2}-d_1 \end{pmatrix}$

The characteristic polynomial of the variation matrix is $\det(J(E^*)-\lambda I_2)=0$

That is $\begin{vmatrix} r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*}-\lambda & -\frac{\beta x^*}{(1+w^*)^2} \\ \frac{\beta w^*}{1+w^*} & \frac{\beta x^*}{(1+w^*)^2}-d_1-\lambda \end{vmatrix}=0$

Thus Using Routh Hourwith criterion, the characteristic polynomial:

$\left(r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*}-\lambda\right)\left(\frac{\beta x^*}{(1+w^*)^2}-d_1-\lambda\right)+\frac{\beta^2 x^* w^*}{(1+w^*)^3}=0$ is stable, if and only if

$\left\{r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*}\right\}+\left\{\frac{\beta x^*}{(1+w^*)^2}-d_1\right\}>0$ and $\left\{r-\frac{2rx^*}{k}-\frac{\beta w^*}{1+w^*}\right\}\left\{\frac{\beta x^*}{(1+w^*)^2}-d_1\right\}+\frac{\beta^2 x^* w^*}{(1+w^*)^3}>0$

Otherwise unstable

When there is no infected prey $(w=0)$, the governing equation (4)-(6) simplified as:

$$\frac{dx}{dt}=rx\left(1-\frac{x}{k}\right)-\frac{p_1 xy}{s+x}=f(x,\ y) \quad (10)$$

$$\frac{dy}{dt}=\frac{q_1 xy}{s+x}-d_2 y=h(x,\ y) \quad (11)$$

This represents a subsystem which contains at most three non-negative steady state points. Thus steady state points are: trivial steady state point $E_o(0,\ 0)$, axial steady state point $E_A(k,\ 0)$, and positive steady state point $E^*(x,\ y)$ *where*

$$x=\frac{sd_2}{q_1-d_2},y=\frac{rsq_1(kq_1-kd_2-sd_2)}{(q_1-d_2)^2 p_1 k}$$

The Jacobian matrix of (10)- (11) is obtained as

$$J(x,y)=\begin{pmatrix} r-\frac{2rx}{k}-\frac{sp_1 y}{(s+x)^2} & -\frac{p_1 x}{s+x} \\ \frac{sq_1 y}{(s+x)^2} & \frac{q_1 x}{s+x}-d_2 \end{pmatrix}$$

Theorem 4. The trivial steady state point $E_o(0,0)$ of (10) and (11) always exists & E_o is a saddle point with locally unstable manifold in x-direction & locally stable manifold in the y-direction.

Proof: The Jacobian matrix of sub model (10) and (11) at trivial steady state point E_o is obtained

$$J(E_o)=\begin{pmatrix} r & 0 \\ 0 & -d_2 \end{pmatrix}$$

Thus eigen values of $J(E_o)$ are: $\lambda_1 = r > 0$ and $\lambda_2 = -d_2 < 0$ Hence E_o is a saddle point with locally unstable manifold in x-direction & local stable manifold in the y-direction.

Theorem5. The axial steady state point $E_A(k, 0)$ of sub model (10)-(11) always exists & if $\frac{q_1 k}{s+k} - d_2 < 0$, E_A is locally asymptotically stable point, & if $\frac{q_1 k}{s+k} - d_2 > 0$, E_A is a saddle point with locally stable manifold in x-direction and locally unstable manifold in y-direction.

Proof: The Jacobian matrix of sub governing equation (10) - (11) at axial steady state point E_A is obtained

$$J(E_A) = \begin{pmatrix} -r & \frac{p_1 k}{s+k} \\ 0 & \frac{q_1 k}{s+k} - d_2 \end{pmatrix}$$

Thus eigen values are: $\lambda_1 = -r < 0$ & $\lambda_2 = \frac{q_1 k}{s+k} - d_2$. Thus if $\frac{q_1 k}{s+k} - d_2 < 0$, E_A is locally asymptotically stable point, & if $\frac{q_1 k}{s+k} - d_2 > 0$, E_A is a saddle point with locally stable manifold in x-direction and locally unstable manifold in y-direction.

Theorem6. The positive steady state point $E^*(x^*, y^*)$ of sub governing equation (10)-(11) exists & stable if

$\left\{r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2}\right\} + \left\{\frac{q_1 x^*}{s+x^*} - d_2\right\} > 0$, & $\left\{r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2}\right\}\left\{\frac{q_1 x^*}{s+x^*} - d_2\right\} + \frac{sp_1 q_1 x^* y^*}{(s+x^*)^3} > 0$,holds true. otherwise unstable.

Proof: Consider the Jacobian matrix of sub governing equation (10)-(11) at positive steady state point $J(E^*) = \begin{pmatrix} r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2} & -\frac{p_1 x^*}{s+x^*} \\ \frac{sq_1 y^*}{(s+x^*)^2} & \frac{q_1 x^*}{s+x^*} - d_2 \end{pmatrix}$

The characteristic polynomial of the Jacobian matrix is $\det(J(E^*) - \lambda I_2) = 0$

That is $\begin{vmatrix} r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2} - \lambda & -\frac{p_1 x^*}{s+x^*} \\ \frac{sq_1 y^*}{(s+x^*)^2} & \frac{q_1 x^*}{s+x^*} - d_2 - \lambda \end{vmatrix} = 0$

Using Routh Hourwith criterion, the characteristic polynomial

$\left(r-\frac{2r\,x^*}{k}-\frac{sp_1y^*}{(s+x^*)^2}-\lambda\right)\left(\frac{q_1x^*}{s+x^*}-d_2-\lambda\right)+\frac{sp_1q_1x^*y^*}{(s+x^*)^3}=0$ is stable if

$\left\{r-\frac{2r\,x^*}{k}-\frac{sp_1y^*}{(s+x^*)^2}\right\}+\left\{\frac{q_1x^*}{s+x^*}-d_2\right\}>0,\&\left\{r-\frac{2r\,x^*}{k}-\frac{sp_1y^*}{(s+x^*)^2}\right\}\left\{\frac{q_1x^*}{s+x^*}-d_2\right\}+\frac{sp_1q_1x^*y^*}{(s+x^*)^3}>0$,otherwise unstable.

In this section, I am going to determine the stability analysis of steady state points of governing equation (4)-(6) with no Restriction Imposed on the Prey and Predator population.

$$\frac{dx}{dt}=rx\left(1-\frac{x}{k}\right)-\frac{\beta xw}{1+w}-\frac{p_1xy}{s+x+pw}=f(x,\quad w,\quad y) \qquad (12)$$

$$\frac{dw}{dt}=\frac{\beta xw}{1+w}-\frac{p_2wy}{s+x+pw}-d_1w=g(x,\quad w,\quad y) \qquad (13)$$

$$\frac{dy}{dt}=\frac{q_1xy}{s+x+pw}+\frac{q_2wy}{s+x+pw}-d_2y=h(x,\quad w,\quad y) \qquad (14)$$

This model has at most five non negative steady state points, and the steady points are: i) trivial steady state point $E_o(0,\ 0,\ 0)$, ii)axial steady state point $E_A(k,\ 0,\ 0)$.iii) disease-free steady state point(DFEP) $\bar{E}(\bar{x}\quad 0,\ \bar{y})$ where $\bar{x}=\frac{sd_2}{q_1-d_2}$, $\bar{w}=0$, $\bar{y}=\frac{rsq_1(kq_1-kd_2-sd_2)}{(q_1-d_2)^2p_1k}$ iv) Susceptible prey-free steady state points $E(0,w,y)$ is not applicable where $x=0$, $w=-\frac{sd_2}{pd_2-q_2}$, $y=\frac{sd_1q_2}{(pd_2-q_2)p_2}$ iv) predator-free steady state point $E(x,\ w,\ 0)$, the values of x & y taken from sub governing equation (8) - (9) . v) Positive steady state point$E^*(x^*,\ \ w^*,\ \ y^*)$.

The Jacobian matrix of governing equation (12)-(14) is obtained by

$$J(x,\quad w,\quad y)=\begin{pmatrix} f_x & f_w & f_y \\ g_x & g_w & g_y \\ h_x & h_w & h_y \end{pmatrix}$$

That is the Jacobian matrix of the governing equation (12)-(14) is obtained as

$$J(x,\quad w,\quad y)=\begin{pmatrix} r-\frac{2rx}{k}-\frac{\beta w}{1+w}-\frac{p_1y(s+pw)}{(s+x+pw)^2} & \frac{-\beta x}{(1+w)^2}+\frac{p_1pxy}{(s+x+pw)^2} & \frac{-p_1x}{s+x+pw} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2}-\frac{p_2y(s+x)}{(s+x+pw)^2}-d_1 & \frac{-p_2w}{s+x+pw} \\ \frac{q_1y(s+pw)-q_2wy}{(s+x+pw)^2} & \frac{(q_2-q_1)xyp}{(s+x+pw)^2} & \frac{q_1x+q_2w}{s+x+pw}-d_2 \end{pmatrix}$$

Theorem7. The trivial steady state point $E_o(0, \ 0, \ 0)$ is a saddle point with locally unstable manifold in x-direction & locally stable manifold in wy-plane.

Proof: The Jacobian matrix at E_o is obtained as $J(E_o) = \begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$

Thus the eigen values are:$\lambda_1 = r > 0$, $\lambda_2 = -d_1 < 0$, $\lambda_3 = -d_2 < 0$ which is saddle point with locally unstable manifold in x-direction & locally stable manifold in wy-plane.

Theorem8. The axial steady state point $E_A(k, \ 0, \ 0)$ is stable if $\beta k - d_1 < 0$ & $\frac{q_1 k}{s+k} - d_2 < 0$ and unstable if $\beta k - d_1 < 0$ & $\frac{q_1 k}{s+k} - d_2 < 0$

Proof: The variation matrix at E_A is obtained as

$$J(E_A) = \begin{pmatrix} -r & -\beta k & \frac{-p_1 k}{s+k} \\ 0 & \beta k - d_1 & 0 \\ 0 & 0 & \frac{q_1 k}{s+k} - d_2 \end{pmatrix}$$

Thus the eigen values are:$\lambda_1 = -r < 0$, $\lambda_2 = \beta k - d_1$, $\lambda_3 = \frac{q_1 k}{s+k} - d_2$

Axial steady state point E_A is stable if $\beta k - d_1 < 0$ & $\frac{q_1 k}{s+k} - d_2 < 0$& unstable if $\beta k - d_1 < 0$ & $\frac{q_1 k}{s+k} - d_2 < 0$

Theorem9. The disease-free steady state point $\bar{E}(\bar{x} \ \ 0, \ \bar{y})$ is stable if $\beta \bar{x} - \frac{p_2 \bar{y}}{s+\bar{x}} < 0$ & the quadratic polynomial $\left(r - \frac{2r\bar{x}}{k} - \frac{sp_1\bar{y}}{(s+x)^2} - \lambda\right) * \left(\frac{q_1\bar{x}}{s+\bar{x}} - d_2 - \lambda\right) + \frac{sp_1 q_1 \bar{x}\bar{y}}{(s+\bar{x})^3} = 0$ is stable

Proof: The Jacobian matrix at $\bar{E}$ is obtained as

$$J(\bar{E}) = \begin{pmatrix} r - \frac{2rx}{k} - \frac{sp_1 y}{(s+x)^2} & -\beta x + \frac{p_1 pxy}{(s+x)^2} & \frac{-p_1 x}{s+x} \\ 0 & \beta x - \frac{p_2 y(s+x)}{(s+x)^2} - d_1 & 0 \\ \frac{sq_1 y}{(s+x)^2} & \frac{(q_2 - q_1)xyp}{(s+x)^2} & \frac{q_1 x}{s+x} - d_2 \end{pmatrix}$$

The eigen values of the Jacobian matrix is $\det(J(\bar{E}) - \lambda I_3) = 0$

$$\begin{vmatrix} r - \frac{2rx}{k} - \frac{sp_1 y}{(s+x)^2} - \lambda & -\beta x + \frac{p_1 pxy}{(s+x)^2} & \frac{-p_1 x}{s+x} \\ 0 & \beta x - \frac{p_2 y(s+x)}{(s+x)^2} - d_1 - \lambda & 0 \\ \frac{sq_1 y}{(s+x)^2} & \frac{(q_2 - q_1)xyp}{(s+x)^2} & \frac{q_1 x}{s+x} - d_2 - \lambda \end{vmatrix} = 0$$

$$\left(\beta x - \frac{p_2 y(s+x)}{(s+x)^2} - d_1 - \lambda\right) * \begin{vmatrix} r - \frac{2rx}{k} - \frac{sp_1 y}{(s+x)^2} - \lambda & \frac{-p_1 x}{s+x} \\ \frac{sq_1 y}{(s+x)^2} & \frac{q_1 x}{s+x} - d_2 - \lambda \end{vmatrix} = 0$$

$$\left(\beta x - \frac{p_2 y(s+x)}{(s+x)^2} - d_1 - \lambda\right) * \left\{\left(r - \frac{2rx}{k} - \frac{sp_1 y}{(s+x)^2} - \lambda\right) * \left(\frac{q_1 x}{s+x} - d_2 - \lambda\right) + \frac{sp_1 q_1 \bar{x}\bar{y}}{(s+\bar{x})^3}\right\} = 0$$

is the characteristic polynomial which implies that eigen value are: $\lambda_1 = \beta x - \frac{p_2 y(s+x)}{(s+x)^2} - d_1$ & the remaining eigen values are determined from the quadratic equation

$$\left(\underbrace{r - \frac{2rx}{k} - \frac{sp_1 y}{(s+x)^2}}_{a} - \lambda\right) * \left(\underbrace{\frac{q_1 x}{s+x} - d_2}_{b} - \lambda\right) + \underbrace{\frac{sp_1 q_1 \bar{x}\bar{y}}{(s+\bar{x})^3}}_{c} = 0$$

The disease free steady state point is stable if $\beta x - \frac{p_2 y(s+x)}{(s+x)^2} - d_1 < 0$ and $a + b > 0\,, ab + c > 0$ otherwise unstable, where $x = \frac{sd_2}{q_1 - d_2}$ and $y = \frac{rsq_1(kq_1 - kd_2 - sd_2)}{(q_1 - d_2)^2 p_1 k}$

Theorem10. The predator-free steady state point $\tilde{E}(\tilde{x},\ \tilde{w},\ 0)$ stable if $+b > 0\,, ab + c > 0$, where $a = r - \frac{2rx}{k} - \frac{\beta w}{1+w}$, $b = \frac{\beta x}{(1+w)^2} - d_1$, $c = \frac{\beta^2 xw}{(1+w)^3}$

Proof: Consider the Jacobian matrix at predator -free steady state point $\tilde{E}$ is given by

$$J(\tilde{E}) = \begin{pmatrix} r - \frac{2rx}{k} - \frac{\beta w}{1+w} & \frac{-\beta x}{(1+w)^2} & \frac{-p_1 x}{s+x+pw} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2} - d_1 & \frac{-p_2 w}{s+x+pw} \\ 0 & 0 & \frac{q_1 x + q_2 w}{s+x+pw} - d_2 \end{pmatrix}$$

The eigen values the variation matrix is $\det\left(J(\tilde{E}) - \lambda I_3\right) = 0$

$$\begin{vmatrix} r - \frac{2rx}{k} - \frac{\beta w}{1+w} - \lambda & \frac{-\beta x}{(1+w)^2} & \frac{-p_1 x}{s+x+pw} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2} - d_1 - \lambda & \frac{-p_2 w}{s+x+pw} \\ 0 & 0 & \frac{q_1 x + q_2 w}{s+x+pw} - d_2 - \lambda \end{vmatrix} = 0$$

$$\left(\frac{q_1 x + q_2 w}{s+x+pw} - d_2 - \lambda\right) * \begin{vmatrix} r - \frac{2rx}{k} - \frac{\beta w}{1+w} - \lambda & \frac{-\beta x}{(1+w)^2} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2} - d_1 - \lambda \end{vmatrix} = 0$$

$\left(\frac{q_1x+q_2w}{s+x+pw}-d_2-\lambda\right)*\left\{\left(r-\frac{2rx}{k}-\frac{\beta w}{1+w}-\lambda\right)*\left(\frac{\beta x}{(1+w)^2}-d_1-\lambda\right)+\frac{\beta^2xw}{(1+w)^3}\right\}=0$

is characteristic polynomial. eigen values are: $\lambda_1=\frac{q_1x+q_2w}{s+x+pw}-d_2$ & the remaining eigen values are obtained from the roots of the quadratic equation:

$$\left(\underbrace{r-\frac{2rx}{k}-\frac{\beta w}{1+w}}_{a}-\lambda\right)*\left(\underbrace{\frac{\beta x}{(1+w)^2}-d_1}_{b}-\lambda\right)+\underbrace{\frac{\beta^2xw}{(1+w)^3}}_{c}=0$$

Thus the characteristic polynomial is stable if $\frac{q_1x+q_2w}{s+x+pw}-d_2<0$ and

$\left(\underbrace{r-\frac{2rx}{k}-\frac{\beta w}{1+w}}_{a}-\lambda\right)*\left(\underbrace{\frac{\beta x}{(1+w)^2}-d_1}_{b}-\lambda\right)+\underbrace{\frac{\beta^2xw}{(1+w)^3}}_{c}=0$is stable iff $a+b>0\,,ab+c>0$ by Routh Hourwith criterion

Theorem11. The positive steady state point $E^*(x^*,\quad w^*,\quad y^*)$ is globally asymptotically stable.

Proof: Consider a liapunove function:

$$L(x,\quad w,\quad y)=\frac{\alpha_1}{2}(x-x^*)^2+\frac{\alpha_2}{2}(w-w^*)^2+\frac{\alpha_3}{2}(y-y^*)^2 \qquad (15)$$

Take the derivative of a Liapunove function (15) with respect to time t

$$\frac{dL}{dt}=\alpha_1(x-x^*)\frac{dx}{dt}+\alpha_2(w-w^*)\frac{dw}{dt}+\alpha_3(y-y^*)\frac{dy}{dt}$$

Plug the governing Equations (12)-(14) into (15), as follows:

$$\frac{dL}{dt}=\alpha_1(x-x^*)\left[rx\left(1-\frac{x}{k}\right)-\frac{\beta xw}{1+w}-\frac{p_1xy}{s+x+pw}\right]+\alpha_2(w-w^*)\left[\frac{\beta xw}{1+w}-\frac{\mathrm{p}_2\mathrm{wy}}{s+x+pw}-\mathrm{d}_1w\right]$$
$$+\alpha_3(y-y^*)\left[\frac{\mathrm{q}_1xy}{\mathrm{s}+x+\mathrm{pw}}+\frac{\mathrm{q}_2wy}{\mathrm{s}+x+\mathrm{pw}}-\mathrm{d}_2y\right]$$

Common out $x, w,$ & y from each square bracket & write as a change as follows

$$\frac{dL}{dt}=\alpha_1(x-x^*)^2\left[r\left(1-\frac{x}{k}\right)-\frac{\beta w}{1+w}-\frac{p_1y}{s+x+pw}\right]+\alpha_2(w-w^*)^2\left[\frac{\beta x}{1+w}-\frac{\mathrm{p}_2\mathrm{y}}{s+x+pw}-\mathrm{d}_1\right]$$
$$+\alpha_3(y-y^*)^2\left[\frac{\mathrm{q}_1x}{\mathrm{s}+x+\mathrm{pw}}+\frac{\mathrm{q}_2w}{\mathrm{s}+x+\mathrm{pw}}-\mathrm{d}_2\right]$$

Common out negative symbol from each square bracket

$$\frac{dL}{dt}=-\alpha_1(x-x^*)^2\left[-r\left(1-\frac{x}{k}\right)+\frac{w\beta}{1+w}+\frac{yp_1}{s+x+pw}\right]$$
$$-\alpha_2(w-w^*)^2\left[-\frac{x\beta}{1+w}+\frac{y\mathrm{p}_2}{\mathrm{s}+x+pw}+\mathrm{d}_1\right]$$
$$-\alpha_3(y-y^*)^2\left[-\frac{x\mathrm{q}_1}{\mathrm{s}+x+\mathrm{wp}}-\frac{w\mathrm{q}_2}{\mathrm{s}+x+\mathrm{wp}}+\mathrm{d}_2\right]$$

Choose the values of $\alpha_1, \alpha_2, \alpha_3$ so that $\frac{dL}{dt} < 0$ .The endemic steady state point is globally stable.

Theorem12. The Basic reproduction number (BRN) is obtained as

$$R_0 = \frac{\beta s d_2 p_1 k}{k(q_1 - d_2)(rp_2 + d_1 p_1) - rsd_2 p_2}$$

Proof: take the infected predator equations (5)

$$\frac{\mathrm{dw}}{\mathrm{dt}} = \frac{\beta x w}{1 + w} - \frac{\mathrm{p}_2 \mathrm{wy}}{\mathrm{s} + x + pw} - \mathrm{d}_1 w = \left[\underbrace{\frac{x\beta}{w + 1}}_{\mathrm{F}} - \left(\underbrace{\frac{y\mathrm{p}_2}{\mathrm{s} + x + pw} + \mathrm{d}_1}_{\mathrm{V}}\right)\right] w$$

Suppose the functions $F = \frac{\beta x}{1+w}$,$\mathrm{V} = \frac{\mathrm{p}_2 \mathrm{y}}{\mathrm{s}+x+pw} + \mathrm{d}_1$, Compute the functions at disease free steady state point (DFEP) $\bar{E} = (\bar{x} \quad 0, \quad \bar{y})$ where

$$\bar{x} = \frac{sd_2}{q_1 - d_2}, \qquad \bar{w} = 0, \qquad \bar{y} = \frac{rsq_1(kq_1 - kd_2 - sd_2)}{(q_1 - d_2)^2 p_1 k}$$

$$F(\bar{E}) = \beta \bar{x} = \frac{\beta s d_2}{q_1 - d_2}$$

$$V(\bar{E}) = \frac{\mathrm{p}_2 \bar{\mathrm{y}}}{\mathrm{s} + \bar{x}} + \mathrm{d}_1 = \frac{\mathrm{p}_2 \left[\frac{rsq_1(kq_1 - kd_2 - sd_2)}{(q_1 - d_2)^2 p_1 k}\right]}{\mathrm{s} + \left[\frac{sd_2}{q_1 - d_2}\right]} + \mathrm{d}_1 = \frac{rsq_1 p_2 (kq_1 - kd_2 - sd_2)[q_1 - d_2]}{(q_1 - d_2)^2 p_1 k [sq_1 - sd_2 + sd_2]} + d_1$$

$$= \frac{rsq_1 p_2 (kq_1 - kd_2 - sd_2)}{(q_1 - d_2) p_1 k [sq_1]} + d_1 = \frac{rp_2 (kq_1 - kd_2 - sd_2)}{(q_1 - d_2) p_1 k} + d_1$$

$$V(\bar{E}) = \frac{rp_2 (kq_1 - kd_2 - sd_2) + (q_1 - d_2) d_1 p_1 k}{(q_1 - d_2) p_1 k}$$

Then Basic reproduction number is

$$R_0 = [F(\bar{E})][V(\bar{E})]^{-1} = \left[\frac{\beta s d_2}{q_1 - d_2}\right]\left[\frac{(q_1 - d_2) p_1 k}{rp_2 (kq_1 - kd_2 - sd_2) + (q_1 - d_2) d_1 p_1 k}\right]$$

$$R_0 = \frac{\beta s d_2 p_1 k}{rp_2 (kq_1 - kd_2 - sd_2) + (q_1 - d_2) d_1 p_1 k}$$

$$R_0 = \frac{\beta s d_2 p_1 k}{k(q_1 - d_2)(rp_2 + d_1 p_1) - r s d_2 p_2}$$

5. Numerical Simulations

When there is no predators involved our governing equation becomes a simple 2 by 2 sub system & parametric values r=0.067,k=0.425,Beta=0.0800,d_1=0.6000, & initial conditions for the subsystem variable x_o =10.000, w_o=15.0000 are used for numerical simulation purposes.

dx/dt=r*x*(1-x/k)-(Beta*x*w)/(1+w)

dw/dt=Beta*x*w/(1+w)-d_1*w

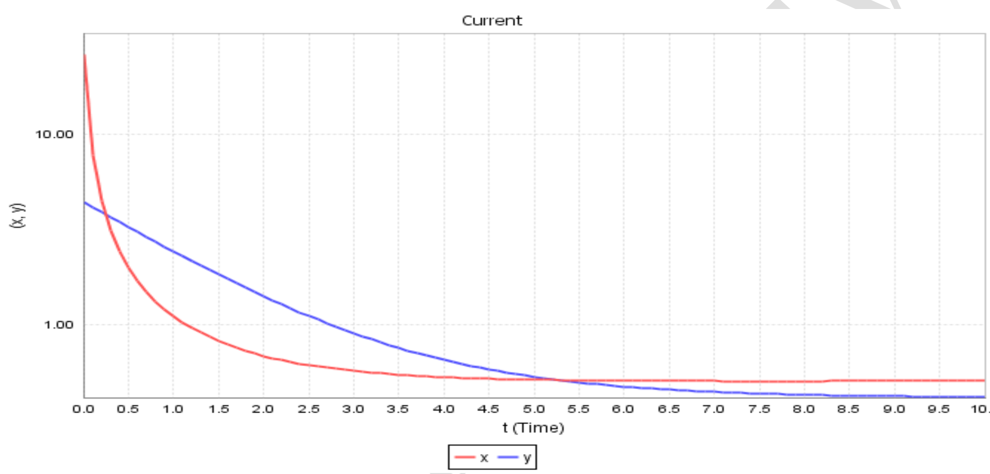

Figure2. predator-prey population with no Infected Prey

There is no infected prey populations in the above numerical simulation.The predators has preference to eat the weaker populaions.Thus the graph is oscilatary with more than two cross and the given data shows graph prey-predator system of lotika voltera.

When there is no infected prey involved the governing equation,the model becomes simpler 2 by 2 subsystems & parametric values r=0.1460,k=0.1990,p_1=0.2120,s=0.2690, q_1=0.2460,d_2=0.7750, and initial conditions model variable x_o=20.0000,y_o=5.0000 are used for numerical simulation purposes.

dx/dt=r*x*(1-x/k)-(p_1*x*y)/(s+x)

dy/dt=(q_1*x*y)/(s+x)-d_2*y

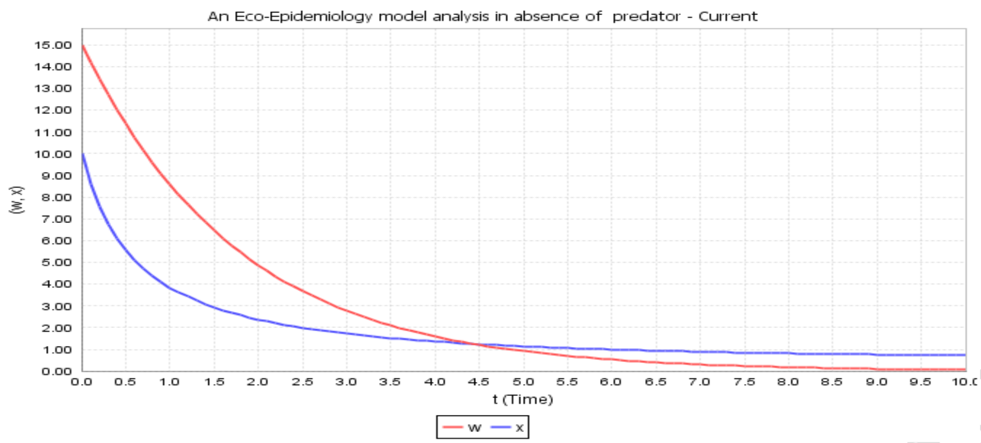

Figure3. Predator-prey population with no predator involvment

Infected prey & susceptible prey decline for some initial time and then infected prey Continue decline more than susceptible prey. This shows predators prefer to consume the weaker populations. Thus the disease removed due to predator consumes more infected preys & susceptible preys have got time to survive and reproduce.

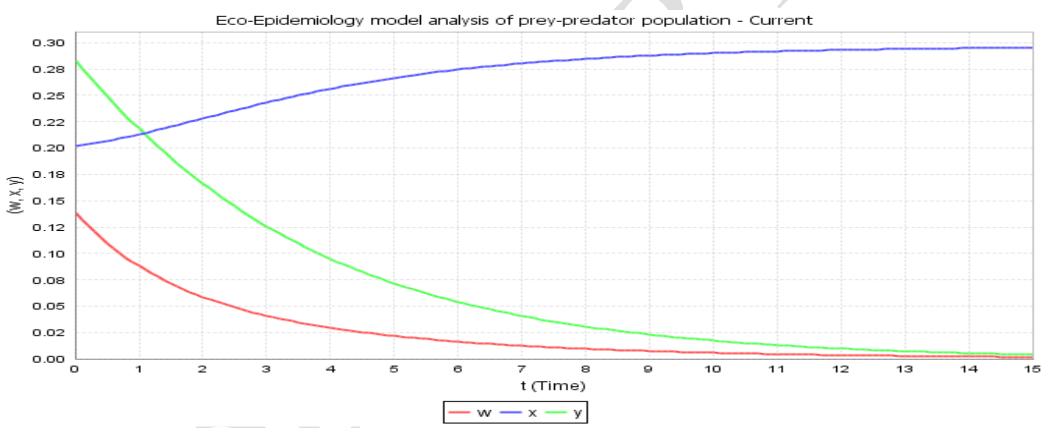

Figure 4. predator-prey population with edemicity

6. Conclusions

This study uses the supposition that an infectious disease is solely spreading among the prey population when analysing eco-epidemiological governing equations that describe the spread of infectious disease in a predator-prey community. Both susceptible and infected prey are consumed by the predators, who have two distinct modified functional responses.

It is demonstrated that positive and bounded solutions to the developed governing equation exist. Different steady state points for the suggested governing equation exist, according to research and calculations. The suggested governing equation's steady state points are subjected to local and global stability analyses using the Jacobian matrix and the liapunove function, respectively. The trivial steady state point $E_o(0,\ 0,\ 0)$ is a saddle point which is locally asymptotically unstable and the positive steady state point $E^*(x^*,\ w^*,\ y^*)$ is globally asymptotically stable.

It has been derived that the threshold value $R_0 = \frac{\beta s d_2 p_1 k}{k(q_1 - d_2)(r p_2 + d_1 p_1) - r s d_2 p_2}$. If the threshold value is less than one, then infectious disease will be removed from the system and if the threshold value is greater than one, then the infectious disease continue to invade the system. We plug values for all parameters & numerical simulations are done using DEDiscover software that approves the analytical results.

Conflicts of Interest: Author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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