On a Question of Constructing Möbius Transformations via Spheres and Rigid Motions

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ABSTRACT

A Möbius Transformation or a Fractional Linear Transformation is a complex-valued function that maps points in the extended complex plane into itself either by translations, dilations, inversions, or rotations or even as a combination of the four mappings. Such a mapping can be constructed by a stereographic projection of the complex plane on to a sphere, followed by a rigid motion of the sphere, and a projection back onto the plane. Both Möbius transformations and Stereographic projections are abundantly used in diverse fields such as map making, brain mapping, image processing etc. In 2008, Arnold and Rogness created a short video named as *Möbius Transformation Revealed* and made it available on YouTube which became an instant hit. In answering a question posted in the accompanied paper by the same name, Siliciano in 2012 shows that for any given Möbius transformation and an admissible sphere, there is exactly one rigid motion of the sphere with which the transformation can be constructed. The present work is prepared on a suggestion posted by Silciano in characterizing rigid motions in constructing a specific Möbius transformation. We show that different admissible spheres under a unique Möbius transformation would require different rigid motions.

Keywords: Admissible Sphere; Möbius Transformation; Rigid Motion; Stereographic Projection.

1. INTRODUCTION

One of the aesthetic appeals of mathematics is through visualization of the transformation (see 1, 2) between objects in which a major role is played by a special type of mapping called *Möbius transformations*or fractional linear transformations which are used in areas such as map making, brain mapping, image processingetc (see [3], [4], [5], [6], [7] and references therein). A Möbius transformation is a complex-valued function from the extended complex plane onto itself of the form ([8],[9],[10])

$$f(z) = \frac{az+b}{cz+d} \text{ ---------------- (1),}$$

where the complex numbers $a, b, c,$ and d satisfy the relationship$ad - bc \neq 0$, which is to guarantee that the mapping is not a constant.

A function of the form $f(z) = az + b$ (for $a \neq 0$) is known as an affine transformation; two special cases$z \mapsto az + b$and $z \mapsto az$are respectively called translations and dilations. The mapping $z \mapsto 1/z$ is called an inversion. One of the salient properties about Möbius transformations is that one such transformation can be represented as a composition of translations, dilations, and inversion mapping. For an instance, if $c = 0$ in (1), then The extended complex plane is the complex plane ($\mathbb{C}$) together with the point at infinity (∞), denoted by $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$. To visualize the point at infinity, one can think of $\mathbb{C}$ as passing through the equator of a unit sphere centered at the origin:

i.e., $\{(z_1, z_2, z_3) \in \mathbb{R}^3 | \, z_1^2 + z_2^2 + z_3^2 = 1\}$.

This sphere is called a *Riemann sphere.*A given Möbius transformation is uniquely determined by three distinct points on the Riemann sphere. In particular, a mapping that sends three distinct points z_1, z_2, and z_3 on $\mathbb{C}_\infty$ to three distinct pointsw_1, w_2, and $w_3$on $\mathbb{C}_\infty$respectively, has the explicit form

$$\frac{(w - w_1)(w_2 - w_3)}{(w - w_3)(w_2 - w_1)} = \frac{(z - z_1)(z_2 - z_3)}{(z - z_3)(z_2 - z_1)} - - - - - - (2).$$

A convenient way to visualize $\mathbb{C}_\infty$ is through a *stereographic projection*, which is a special type of correspondence between the points of $\mathbb{C}_\infty$ and the Riemann sphere. As in [11], we identify $\mathbb{R}^3$ with $\mathbb{C} \times \mathbb{R}$. Accordingly,a point in $\mathbb{R}^3$ is expressed as an ordered pairs rather than an ordered triple. We'll stick to the following definition as in [11].To a visual illustration of a formulation of the stereographic projection, refer to [8, pp. 8-9] and [10, p.11-13].

Definition1 (Stereographic Projection) : Given an admissible sphere Scentered at $(\gamma, c) \in \mathbb{R}^3$, the Stereographic projection from $S$to$\mathbb{C}$ is the function$P_S : S \to \mathbb{C}_\infty$ which maps the top of S,$(\gamma, c + 1)$ to ∞, and maps any other point on the sphere to the intersection of the complex plane with the line extending from$(\gamma, c + 1)$through the point.

A sphere $S \in \mathbb{R}^3$ is called as *admissible* if it has radius 1 and is centered at $(\alpha, c) \in \mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3$with $c > -1$. Geometrically this means S is a unit sphere whose "north pole", i.e., the point $(0,0,1) \in \mathbb{R}^3$is above the complex plane.A *rigid motion* of $\mathbb{R}^3$ is an isometry from $\mathbb{R}^3$ into itself that preserves orientation. When using an admissible sphere S, we will call a rigid motion T admissible if the sphere $T(S)$ is also admissible.The term *rigid* is used in the sense that an object that undergoes a rigid motion is not broken or distorted during the process.

2. METHODOLOGY

In 2008, D. Arnold and J. Rogness created an appealing short video demonstrating the visual representations of Möbius transformations. They use a colorful grid and a moving sphere to illustrate the Möbius transformations and depicted what happens to lines, circles, and angles as a flat surface is deformed. This signified that a Möbius transformation can be constructed using a sphere, stereographic projection, and rigid motions of a sphere. In a follow-up article [12] that accompanied the video, they posted an open question; *in how many different wayscan the transformation be constructed using a sphere?* In 2012, R. Siliciano answered this question [11] by characterizing the rigid motions required to construct a specific Möbius transformation for a given admissible sphere, but a different admissible sphere would require a different rigid motion. In the present work, we show that different admissible spheres under a unique Möbius transformation would require different rigid motions.

Although in [11], Silicianoanswered the main open question raised in [12], there are other questions which remained unanswered. For example, in the existence proof in [11], Siliciano characterized the rigid motion required to construct a specific Möbius transformation for a given admissible sphere, but a different admissible sphere would require a different rigid motion. Here we show that there exist different admissible spheres for different rigid motions under a unique Möbius transformation.

We shall use the following definition from [11] to introduce the notations.

Definition 2: Given an admissible sphere S centered at $(\gamma, c) \in \mathbb{R}^3$, the Stereographic Projection from S to $\mathbb{C}$ is the function $P_S: S \rightarrow \mathbb{C}_\infty$ which maps the top of S, $(\gamma, c+1)$, to ∞, and maps any other point on the sphere to the intersection of $\mathbb{C}$ with the line extending from $(\gamma, c+1)$ through the point.

The key representation in this work comes from [1]; given any admissible sphere S, and any admissible rigid motion T, the function $f = P_{T(S)} \circ T \circ P_S^{-1}$ is a Möbius transformation.

Now we are ready to present our main result and the proof.

3. RESULTS AND DISCUSSION

Theorem: Let S,$\hat{S}$ be different admissible spheres and $T, \widehat{T}$ be the respective rigid motions. Then there exist different admissible spheres for different rigid motions under a unique Möbius transformation.

Proof: Let f be the desired rigid motion. Then from the standard construction of f, we can write

$f = P_{\hat{T}(\hat{S})} \circ \widehat{T} \circ P_{\hat{S}}^{-1}$ and $f = P_{T(S)} \circ T \circ P_S^{-1}$,
which implies that
$P_{\hat{T}(\hat{S})} \circ \widehat{T} \circ P_{\hat{S}}^{-1} = P_{T(S)} \circ T \circ P_S^{-1} = f$.

Assume there exist a unique admissible sphere for different rigid motions under a unique Möbiustransformation.
i.e., $P_{\hat{T}(S)} \circ \widehat{T} \circ P_S^{-1} = P_{T(S)} \circ T \circ P_S^{-1} = f$

or equivalently
$P_{\hat{T}(S)}(\widehat{T}(P_S^{-1}(\alpha))) = P_{T(S)}(T(P_S^{-1}(\alpha)))$.

Now consider a cross-sectional view of a vertical translation as illustrated in Figure 1.The spheres S, $T(S)$, and $\widehat{T(S)}$ are centered above the point $\gamma \equiv (\gamma, 0,0)$ in the finite complex plane.

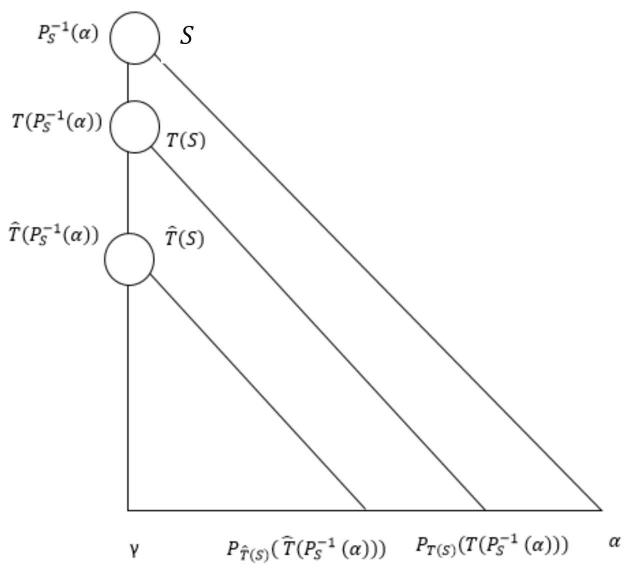

Figure 1: A vertical translation

By similarity of triangles we can conclude that,

$$T(P_S^{-1}(\alpha)) = \hat{T}(P_S^{-1}(\alpha))$$

i.e.,$T = \hat{T}$.

This contradicts the assumption where there exists a unique rigid motion for different admissible spheres under a unique Möbius transformation.

Now consider a cross-sectional view of a horizontal translation.

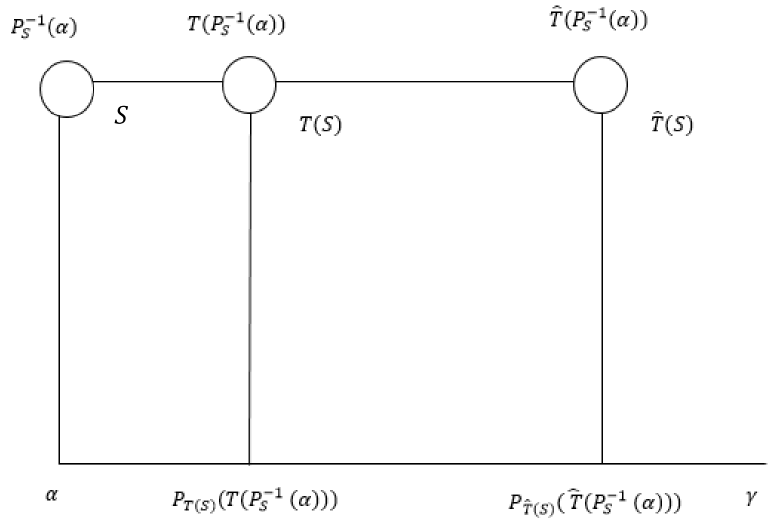

Figure 2: A horizontal translation

Similarly, as in vertical translation, $T = \hat{T}$ and contradicts the assumption where there exists a unique rigid motion for different admissible spheres under a unique Möbius transformation.Therefore, we can conclude that under a unique Möbius transformation, there exists different rigid motions for different admissible spheres.

In stereographic projections, the angle between lines on the surface of the sphere is equal to the angle between the projections of those lines and the circles on the surface of the sphere project as circles on the plane of projection. These are two existing results on stereographic projections and in the present work we have proved that under a unique Möbius transformation, there exists different rigid motions for different admissible spheres. We can combine all these results and use it for map making purposes.

4. CONCLUSION

The present work was first originated in the resolutory eye-soothing video clip and the accompanied article by Douglas Arnold and Jonathan Rogness [12]. Even non-mathematicians became curious about the salient features of a Möbius transformation. The two authors concluded their paper by leaving out an open question: *given a specific Mobius transformation, in how many different ways* can the transformation be constructed using a rigid sphere ? After the span of about four years, Rob Silciano answered the question in [11] using geometrical arguments. Continuing with both references, in the present work, we show that there exist different admissible spheres for different rigid motions under a unique Möbius transformation.Specifically, a well-known result in the theory of Möbius transformations is that such a transformation f can be represented using a Stereographic projection, an admissible sphere and a rigid motion. Using this representation as the tool, we prove that, under a unique Möbius transformation, there exists different rigid motions for different admissible spheres.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

AUTHORS' CONTRIBUTIONS

All authors read and approved the final manuscript.

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