

Exact solutions of second-order fractional Fredholm integro-differential equations

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Abstract

The present paper analyzes the second-order fractional Fredholm integro-differential equations by means of the Caputo definition in the fractional calculus (FC). The exact solutions are obtained for two examples utilizing a direct solution method. Furthermore, it is shown that the present solutions in fractional forms reduce to the corresponding classical ones in the relevant literature, with integer derivatives, as special cases.

Keywords: Fractional integral equations; Fredholm; integro-differential equations; exact solution.

1 Introduction

Recently, the fractional calculus (FC) has gained observable interest due to its wide applications in several applied areas of research [1-14]. In the Refs. [15-25], several integro-differential equations have been solved via the FC approach using various numerical and

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analytical methods. In this paper, we are concerned with the second-order fractional Fredholm integro-differential equations of the form:

$${}^C_0D_x^\beta u(x) = g(x) + \lambda \int_{b_1}^{b_2} K(x, \tau)u(\tau) d\tau, \quad 1 < \beta \leq 2, \quad (1)$$

$$u(0) = c_1, \quad u'(0) = c_2, \quad (2)$$

where ${}^C_0D_x^\beta$ stands for the Caputo fractional derivative. The functions $g(x)$ and $K(x, \tau)$ are continuous on $[b_1, b_2]$ and λ , b_1 and b_2 are given constants. It is the aim of this work to obtain the exact solutions for the problem (1)-(2) via a direct method.

The paper is organized as follows. In section 2, basic preliminaries of the FC are introduced. Beside, a basic Lemma is provided for some properties of the Mittag-Leffler functions. Such properties will be implemented in a next step to construct the exact solutions of two examples for the problem (1)-(2). Section 3 is devoted to apply the present direct method on two examples. Furthermore, it will be shown that the present results reduce to the corresponding ones in the literature as $\beta \rightarrow 2$. Finally, section 4 outlines the conclusions.

2 Preliminaries

The Riemann-Liouville fractional integral of order β is defined as [1]:

$$J_0^\beta u(x) = \frac{1}{\Gamma(\beta)} \int_0^x (x - \tau)^{\beta-1} u(\tau) d\tau \quad \beta > 0. \quad (3)$$

The Caputo's fractional derivative of order $\beta \in (1, 2]$ of a function $u(x)$ is defined by

$${}^C_0D_x^\beta u(x) = \frac{1}{\Gamma(2 - \beta)} \int_0^x (x - \tau)^{1-\beta} u''(\tau) d\tau, \quad 1 < \beta \leq 2. \quad (4)$$

The J_0^β and ${}_0^C D_x^\beta$ are related by:

$$J_0^\beta ({}_0^C D_x^\beta u(x)) = u(x) - u(0) - u'(0)x. \quad (5)$$

A basic property of the J_0^β is

$$J_0^\beta (x^r) = \frac{\Gamma(r+1)}{\Gamma(\beta+r+1)} x^{\beta+r}, \quad r > -1. \quad (6)$$

The Mittag-Leffler function (MLF) of one-parameter is defined as

$$E_\alpha(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + 1)}, \quad z \in C, \quad (7)$$

while the two-parameter MLF is defined by

$$E_{\alpha,\beta}(z) = \sum_{m=0}^{\infty} \frac{z^m}{\Gamma(\alpha m + \beta)}, \quad \alpha > 0, \beta > 0. \quad (8)$$

The following properties are also hold [1]:

$$E_{\alpha,\beta}(z) = z E_{\alpha,\beta}(z) + \frac{1}{\Gamma(\beta)}, \quad (9)$$

$$E_{1,k}(z) = \frac{1}{z^{k-1}} \left(e^z - \sum_{m=0}^{k-2} \frac{z^m}{\Gamma(m+1)} \right), \quad (10)$$

$$E_{2,1}(-z^2) = \cos(z), \quad E_{2,2}(-z^2) = \frac{\sin(z)}{z}, \quad (11)$$

and a fundamental integral formula of MLF is given by [1]

$$\int_0^\mu \tau^{\gamma-1} E_{\delta,\gamma}(\sigma \tau^\delta) d\tau = \mu^\gamma E_{\delta,\gamma+1}(\sigma \mu^\delta), \quad \delta > 0, \gamma > 0. \quad (12)$$

2.1 Lemma 1

The MLFs $E_{1,3}(z)$, $E_{1,4}(z)$, $E_{2,3}(-z^2)$, $E_{2,4}(-z^2)$, and $E_{2,5}(-z^2)$ are given by

1. $E_{1,3}(z) = (e^z - 1 - z) / (z^2)$,

$$2. E_{1,4}(z) = (2e^z - 2(1+z) - z^2) / (2z^3),$$

$$3. E_{2,3}(-z^2) = (1 - \cos(z)) / (z^2),$$

$$4. E_{2,4}(-z^2) = (z - \sin(z)) / (z^3),$$

$$5. E_{2,5}(-z^2) = (2 \cos(z) - 2 + z^2) / (2z^4).$$

Proof

The first two formulae follow immediately from Eq. (10) at $k = 3$ and $k = 4$. The third formula follows from the definition (8) as

$$\begin{aligned} E_{2,3}(-z^2) &= \sum_{m=0}^{\infty} \frac{(-z^2)^m}{\Gamma(2m+3)} = \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m+2)!}, \\ &= -\frac{1}{z^2} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} z^{2m+2}}{(2m+2)!}, \\ &= -\frac{1}{z^2} \left(-\frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \right), \\ &= -\frac{1}{z^2} \left(\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \dots \right) - 1 \right), \\ &= \frac{1 - \cos(z)}{z^2}. \end{aligned} \tag{13}$$

Similarly, we have

$$\begin{aligned} E_{2,4}(-z^2) &= \sum_{m=0}^{\infty} \frac{(-1)^m z^{2m}}{(2m+3)!}, \\ &= -\frac{1}{z^3} \sum_{m=0}^{\infty} \frac{(-1)^{m+1} z^{2m+3}}{(2m+3)!}, \\ &= -\frac{1}{z^3} \left(-\frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right), \\ &= -\frac{1}{z^3} \left(\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots \right) - z \right), \\ &= \frac{z - \sin(z)}{z^3}. \end{aligned} \tag{14}$$

Proceeding as above, the fifth formula can be easily proved.

3 Examples

Example 1:

In this example, we consider the following second-order fractional Fredholm integro-differential equation [26]

$${}_0^C D_x^\beta u(x) = 1 - e + e^x + \int_0^1 u(\tau) d\tau, \quad u(0) = u'(0) = 1, \quad 1 < \beta \leq 2. \quad (15)$$

Assume that a_1 is a constant, given by

$$a_1 = \int_0^1 u(\tau) d\tau, \quad (16)$$

then Eq. (15) can be written as

$${}_0^C D_x^\beta u(x) = (1 - e + a_1) + e^x. \quad (17)$$

Expanding e^x as Maclaurin series and then applying J_0^β on both sides of Eq. (17), gives

$$\begin{aligned} u(x) &= u(0) + u'(0)x + (1 - e + a_1) J_0^\beta(1) + J_0^\beta \left(\sum_{m=0}^{\infty} \frac{x^m}{m!} \right), \\ &= 1 + x + \frac{(1 - e + a_1) x^\beta}{\Gamma(\beta + 1)} + \sum_{m=0}^{\infty} \frac{1}{m!} \times \frac{\Gamma(m + 1) x^{\beta+m}}{\Gamma(\beta + m + 1)}, \end{aligned} \quad (18)$$

which can be simplified as

$$u(x) = 1 + x + \frac{(1 - e + a_1) x^\beta}{\Gamma(\beta + 1)} + \sum_{m=0}^{\infty} \frac{x^{\beta+m}}{\Gamma(\beta + m + 1)}, \quad (19)$$

or in terms of the MLF $E_{1,\beta+1}(x)$ as

$$u(x) = 1 + x + \frac{(1 - e + a_1) x^\beta}{\Gamma(\beta + 1)} + x^\beta E_{1,\beta+1}(x). \quad (20)$$

Substituting Eq. (20) into Eq. (16), we obtain

$$a_1 = \frac{3}{2} + \frac{(1 - e + a_1)}{\Gamma(\beta + 2)} + \int_0^1 \tau^\beta E_{1,\beta+1}(\tau) d\tau. \quad (21)$$

From the integral formula given by Eq. (12), we have

$$\int_0^1 \tau^\beta E_{1,\beta+1}(\tau) d\tau = E_{1,\beta+2}(1). \quad (22)$$

Hence, Eq. (21) becomes

$$a_1 = \frac{3}{2} + \frac{(1 - e + a_1)}{\Gamma(\beta + 2)} + E_{1,\beta+2}(1). \quad (23)$$

Solving Eq. (23) for a_1 , yields

$$a_1 = \frac{\frac{3}{2} + \frac{1-e}{\Gamma(\beta+2)} + E_{1,\beta+2}(1)}{1 - \frac{1}{\Gamma(\beta+2)}}. \quad (24)$$

Inserting Eq. (24) into Eq. (20), gives the exact solution of Eq. (15) as

$$u(x) = 1 + x + \left(\frac{\frac{5}{2} - e + E_{1,\beta+2}(1)}{1 - \frac{1}{\Gamma(\beta+2)}} \right) \frac{x^\beta}{\Gamma(\beta + 1)} + x^\beta E_{1,\beta+1}(x). \quad (25)$$

As $\beta \rightarrow 2$, Eq. (25) leads to

$$u(x) = 1 + x + \frac{6}{5} \left(\frac{5}{2} - e + E_{1,4}(1) \right) \frac{x^2}{\Gamma(3)} + x^2 E_{1,3}(x). \quad (26)$$

Implementing the first two properties of Lemma 1, we have

$$u(x) = 1 + x + \frac{6}{5} \left(\frac{5}{2} - e + e - \frac{5}{2} \right) \frac{x^2}{2} + x^2 \left(\frac{e^x - 1 - x}{x^2} \right), \quad (27)$$

which yields

$$u(x) = e^x. \quad (28)$$

Indeed, Eq. (28) is the exact solution of the classical form of the present example, given in

Ref. [26] by $u''(x) = 1 - e + e^x + \int_0^1 u(\tau) d\tau$.

Example 2

Consider the second-order fractional Fredholm integro-differential equation [26]:

$${}_0^C D_x^\beta u(x) = -2x - \sin x + \cos x + \int_0^\pi xu(\tau) d\tau, \quad u(0) = -1, \quad u'(0) = 1, \quad 1 < \beta \leq 2. \quad (29)$$

Suppose that a_2 is the constant given by

$$a_2 = \int_0^\pi u(\tau) d\tau, \quad (30)$$

then Eq. (29) becomes

$${}_0^C D_x^\beta u(x) = (a_2 - 2)x - \sin x + \cos x. \quad (31)$$

Expanding $\sin x$ and $\cos x$ as Maclaurin series and operating with J_0^β on both sides of Eq.

(31), then

$$\begin{aligned} u(x) &= u(0) + u'(0)x + (a_2 - 2) J_0^\beta(x) + J_0^\beta \left(\sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} - \frac{(-1)^m x^{2m+1}}{(2m+1)!} \right), \\ &= -1 + x + \frac{(a_2 - 2)x^\beta}{\Gamma(\beta + 1)} + \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \times \frac{\Gamma(2m+1)x^{\beta+2m}}{\Gamma(\beta + 2m + 1)} - \\ &\quad \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)!} \times \frac{\Gamma(2m+2)x^{\beta+1+2m}}{\Gamma(\beta + 2m + 2)}. \end{aligned} \quad (32)$$

Simplifying Eq. (32), yields

$$u(x) = -1 + x + \frac{(a_2 - 2)x^\beta}{\Gamma(\beta + 1)} + x^\beta \sum_{m=0}^{\infty} \frac{(-x^2)^m}{\Gamma(\beta + 2m + 1)} - x^{\beta+1} \sum_{m=0}^{\infty} \frac{(-x^2)^m}{\Gamma(\beta + 2m + 2)}. \quad (33)$$

In terms of the MLFs $E_{2,\beta+1}(-x^2)$ and $E_{2,\beta+2}(-x^2)$, we have

$$u(x) = -1 + x + \frac{(a_2 - 2)x^\beta}{\Gamma(\beta + 1)} + x^\beta E_{2,\beta+1}(-x^2) - x^{\beta+1} E_{2,\beta+2}(-x^2). \quad (34)$$

Substituting Eq. (34) into Eq. (30), we obtain

$$a_2 = \int_0^\pi \left[-1 + \tau + \frac{(a_2 - 2)\tau^\beta}{\Gamma(\beta + 1)} + \tau^\beta E_{2,\beta+1}(-\tau^2) - \tau^{\beta+1} E_{2,\beta+2}(-\tau^2) \right] d\tau, \quad (35)$$

or

$$a_2 = -\pi + \frac{\pi^2}{2} + \frac{(a_2 - 2)\pi^{\beta+1}}{\Gamma(\beta + 2)} + I_1 - I_2, \quad (36)$$

where I_1 and I_2 are the two integrals:

$$I_1 = \int_0^\pi \tau^\beta E_{2,\beta+1}(-\tau^2) d\tau = \pi^{\beta+1} E_{2,\beta+2}(-\pi^2), \quad (37)$$

and

$$I_2 = \int_0^\pi \tau^{\beta+1} E_{2,\beta+2}(-\tau^2) d\tau = \pi^{\beta+2} E_{2,\beta+3}(-\pi^2). \quad (38)$$

Hence, Eq. (36) becomes

$$a_2 = -\pi + \frac{\pi^2}{2} + \frac{(a_2 - 2)\pi^{\beta+1}}{\Gamma(\beta + 2)} + \pi^{\beta+1} E_{2,\beta+2}(-\pi^2) - \pi^{\beta+2} E_{2,\beta+3}(-\pi^2). \quad (39)$$

Solving Eq. (39) for a_2 , gives

$$a_2 = 2 + \frac{-2 - \pi + \frac{\pi^2}{2} + \pi^{\beta+1} E_{2,\beta+2}(-\pi^2) - \pi^{\beta+2} E_{2,\beta+3}(-\pi^2)}{1 - \frac{\pi^{\beta+1}}{\Gamma(\beta+2)}}. \quad (40)$$

Inserting (40) into (34), we finally get

$$u(x) = -1 + x + \frac{x^\beta}{\Gamma(\beta + 1)} \times \left(\frac{-2 - \pi + \frac{\pi^2}{2} + \pi^{\beta+1} E_{2,\beta+2}(-\pi^2) - \pi^{\beta+2} E_{2,\beta+3}(-\pi^2)}{1 - \frac{\pi^{\beta+1}}{\Gamma(\beta+2)}} \right) + x^\beta E_{2,\beta+1}(-x^2) - x^{\beta+1} E_{2,\beta+2}(-x^2). \quad (41)$$

As $\beta \rightarrow 2$, Eq. (41) reduces to

$$u(x) = -1 + x + \frac{x^2}{2} \left(\frac{-2 - \pi + \frac{\pi^2}{2} + \pi^3 E_{2,4}(-\pi^2) - \pi^4 E_{2,5}(-\pi^2)}{1 - \frac{\pi^3}{6}} \right) + x^2 E_{2,3}(-x^2) - x^3 E_{2,4}(-x^2). \quad (42)$$

Implementing the third, fourth, and fifth properties of Lemma 1:

$$E_{2,3}(-z^2) = (1 - \cos(z)) / (z^2), \quad E_{2,4}(-z^2) = (z - \sin(z)) / (z^3), \quad (43)$$

$$E_{2,5}(-z^2) = (2 \cos(z) - 2 + z^2) / (2z^4), \quad (44)$$

we have

$$E_{2,4}(-\pi^2) = \frac{1}{\pi^2}, \quad E_{2,5}(-\pi^2) = \frac{-4 + \pi^2}{2\pi^4}. \quad (45)$$

Substituting (43-45) into (42) and simplifying yields

$$u(x) = \sin x - \cos x, \quad (46)$$

which is the same exact solution of the corresponding classical form given in Ref. [26] by

$$u''(x) = -2x - \sin x + \cos x + \int_0^\pi xu(\tau) d\tau, \quad u(0) = -1, \quad u'(0) = 1.$$

4 Conclusion

A class of second-order fractional Fredholm integro-differential equations was investigated in this paper by implementing the Caputo definition in the fractional calculus (FC). A direct method was utilized to derive the exact solutions of two examples. The obtained solutions reduce to the corresponding ones in the relevant literature with integer derivatives as a special case.

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