Classification of Units of Five Radical Zero Completely Primary Finite Rings with Variant Orders of Second Galois Ring Module Generators

Abstract

Let R be a commutative completely primary finite ring with a unique maximal ideal $Z(R)$ such that $(Z(R))^5 = (0); (Z(R))^4 \neq (0)$. Then $R/Z(R) \cong GF(p^r)$ is a finite field of order p^r. Let $R_0 = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and of characteristic p^k for some prime number p and positive integers $k,\ r$ so that $R = R_0 \bigoplus U \bigoplus V \bigoplus W \bigoplus Y$, where $U,\ V,\ W$ and Y are R_0/pR_0 - spaces considered as R_0 modules generated by $e,\ f,\ g$ and h elements respectively. Then R is of characteristic p^k where $1 \leq k \leq 5$. In this paper, we investigate and determine the structures of the unit groups of some classes of commutative completely primary finite ring R with $pu_i = p^{\xi}v_j = pw_k = py_l = 0$, where $\xi = 2, 3; 1 \leq i \leq e,\ 1 \leq j \leq f,\ 1 \leq k \leq g$, and $1 \leq l \leq h$.

Keywords: Completely Primary Finite Rings, Five Radical Zero, Unit Groups

1 Introduction

Completely primary finite ring is a ring R with identity $1 \neq 0$ whose subset of all its zero divisors forms a unique maximal ideal. Recall that completely primary finite rings are necessary for the classification of finite rings, which is still inconclusive with just some few expositions on the structures of group of units as well as the zero divisors of the finite rings that have been constructed. Chikunji in [1, 2] obtained the structures of group of units of classes of completely primary finite rings in which the product of any three zero divisors is zero. In [3], the authors determined the structure of the unit groups of completely primary finite rings in which the product of any four zero divisors is zero. Were et al in [4] obtained the structures of the group of units of a completely primary finite rings in which the product of any five zero divisors is zero satisfying $p^{\xi}u_i = pv_j = pw_k = py_l = 0$, where $\xi = 1, 2, 3, 4; 1 \leq i \leq e,\ 1 \leq j \leq f,\ 1 \leq k \leq g$, and $1 \leq l \leq h$. Some of the previously studied related work can be obtained from [5, 1, 6]. Unless otherwise stated, R shall denote a finite ring, $Z(R)$ its Jacobson radical and R^* the group of units of R. If a is an element of R^*, then $< a >$ denotes the cyclic group generated by a. The rest of the notations are standard and reference can be made to [1, 2, 7, 8].

The rest of this paper is presented as follows. In section 2, we give the preliminary to the main result in this work which is basically the construction of five radical zero completely primary finite rings. The conditions necessary for a class of rings for each characteristic $p^k, 1 \leq k \leq 5$ is given. In section 3, we determine the structure of the group of units R^* of R for all characteristics $p^k,\ 1 \leq k \leq 5$ restricted to the conditions $pu_i = p^{\xi}v_j = pw_k = py_l = 0$, where $\xi = 2, 3; 1 \leq i \leq e,\ 1 \leq j \leq f,\ 1 \leq k \leq g$, and $1 \leq l \leq h$. Finally section 4 gives the conclusion of this research and what future researchers may study.

2 Preliminaries

2.1 Construction of Five Radical Zero Commutative Completely Primary Finite Rings

Let $R_0 = GR(p^{kr}, p^k)$ be a Galois ring of order p^{kr} and characteristic p^k where p is a prime integer, $1 \leq k \leq 5$ and $r \in \mathbb{Z}^+$. Suppose U, V, W and Y are R_0/pR_0 - spaces considered as R_0 modules generated by e, f, g and h elements, respectively, such that the corresponding generating sets are $\{u_1, \ldots, u_e\}$, $\{v_1, \ldots, v_f\}$, $\{w_1, \ldots, w_g\}$ and $\{y_1, \ldots, y_h\}$ so that $R = R_0 \bigoplus U \bigoplus V \bigoplus W \bigoplus Y$ is an additive abelian group. Then on the additive group, we define multiplication by the following relations:

(i) If $k = 1$, then

$$u_i u_{i'} = u_{i'} u_i = v_j, \quad u_i v_j = v_j u_i = w_k, \quad u_i w_k = w_k u_i = y_l, \quad u_i y_l = y_l u_i = 0,$$

$$v_j v_{j'} = v_{j'} v_j = y_l, \quad v_j w_k = w_k v_j = 0, \quad v_j y_l = y_l v_j = 0, \quad w_k w_{k'} = w_{k'} w_k = 0,$$

$$w_k y_l = y_l w_k = 0, \quad y_l y_{l'} = y_{l'} y_l = 0$$

(ii) If $k = 2$, then

$$u_i u_{i'} = u_{i'} u_i = pr_0 + pu_i + v_j, \ u_i v_j = v_j u_i = pu_i + w_k, u_i w_k = w_k u_i = pu_i + y_l,$$

$$u_i y_l = y_l u_i = pu_i, v_j v_{j'} = v_{j'} v_j = y_l, v_j w_k = w_k v_j = 0, v_j y_l = y_l v_j = 0, \ w_k w_{k'} = w_{k'} w_k = 0$$

$$w_k y_l = y_l w_k = 0, \quad y_l y_{l'} = y_{l'} y_l = 0$$

(iii) If $3 \leq k \leq 5$, then

$$u_i u_{i'} = u_{i'} u_i = p^2 r_0 + pu_i + v_j, \quad u_i v_j = v_j u_i = p^2 r_0 + pu_i + pv_j + w_k,$$

$$u_iw_k = w_ku_i = p^2r_0 + pu_i + pw_k + y_l, \quad u_iy_l = y_lu_i = p^2r_0 + pu_i,$$

$$v_jv_{j'} = v_{j'}v_j = p^2r_0+pv_j+y_l, \; v_jw_k = w_kv_j = p^2r_0+pv_j+pw_k, v_jy_l = y_lv_j = p^2r_0+pv_j,$$

$$w_kw_{k'} = w_{k'}w_k = p^2r_0 + pw_k, \quad w_ky_l = y_lw_k = p^2r_0 + pw_k, \quad y_ly_{l'} = y_{l'}y_l = p^2r_0.$$

Further $u_iu_{i'}u_{i''}u_{i'''}u_{i^{iv}} = 0, \quad u_ir_0 = r_0u_i, \quad v_jr_0 = r_0v_j, \quad w_kr_0 = r_0w_k, y_lr_0 = r_0y_l,$ where $r_0 \in R_0$ and $1 \leq i, \quad i' \leq e, \; 1 \leq j, \quad j' \leq f, \quad 1 \leq k, \; k' \leq g, 1 \leq l, \; l' \leq h.$ From the given multiplication in R, we see that if $r_0 + \sum_{i=1}^{e} r_iu_i + \sum_{j=1}^{f} s_jv_j + \sum_{k=1}^{g} t_kw_k + \sum_{l=1}^{h} z_ly_l$ and $r'_0 + \sum_{i=1}^{e} r'_iu_i + \sum_{j=1}^{f} s'_jv_j + \sum_{k=1}^{g} t'_kw_k + \sum_{l=1}^{h} z'_ly_l$ are any two elements of R, then

$$\left(r_0 + \sum_{i=1}^{e} r_iu_i + \sum_{j=1}^{f} s_jv_j + \sum_{k=1}^{g} t_kw_k + \sum_{l=1}^{h} z_ly_l\right)\left(r'_0 + \sum_{i=1}^{e} r'_iu_i + \sum_{j=1}^{f} s'_jv_j + \sum_{k=1}^{g} t'_kw_k + \sum_{l=1}^{h} z'_ly_l\right)$$
$$= r_0r'_0 + p^a \sum_{i,m=1}^{e} (r_ir'_m + pR_0)$$
$$+ \sum_{i=1}^{e} [r_0r'_i + r_ir'_0 + pR_0]\, u_i + \sum_{j=1}^{f} \left[(r_0 + pR_0)s'_j + s_j(r'_0 + pR_0) + \sum_{\nu,\ \mu=1}^{e} (r_\nu r'_\mu + pR_0)\right] v_j$$
$$+ \sum_{k=1}^{g} \left[(r_0 + pR_0)t'_k + t_k(r'_0 + pR_0) + \sum_{i,j}(r_i + pR_0)s'_j + s_j(r'_i + pR_0)\right] w_k$$
$$+ \sum_{l=1}^{h} \left[(r_0 + pR_0)z'_l + z_l(r'_0 + pR_0) + \sum_{i,k}(r_i + pR_0)t'_k + t_k(r'_i + pR_0) + \sum_{\kappa,\tau=1}^{f} (s_\kappa s'_\tau + pR_0)\right] y_l$$

where $a = 1, 2, 3,$ or 4 depending on whether $Char\ R_0 = p^2,\ p^3,\ p^4$ or p^5. It can be verified that this multiplication turns R into a commutative ring with identity 1.

Notice that if $R_0 = GR(p^r, p)$ where $Char\ R = p$, then the above multiplication reduces to

$$\left(r_0 + \sum_{i=1}^{e} r_i u_i + \sum_{j=1}^{f} s_j v_j + \sum_{k=1}^{g} t_k w_k + \sum_{l=1}^{h} z_l y_l\right)\left(r'_0 + \sum_{i=1}^{e} r'_i u_i + \sum_{j=1}^{f} s'_j v_j + \sum_{k=1}^{g} t'_k w_k + \sum_{l=1}^{h} z'_l y_l\right)$$
$$= r_0 r'_0 + \sum_{i=1}^{e} \left[r_0 r'_i + r_i r'_0\right] u_i + \sum_{j=1}^{f} \left[(r_0) s'_j + s_j (r'_0) + \sum_{\nu,\ \mu=1}^{e} (r_\nu r'_\mu)\right] v_j$$
$$+ \sum_{k=1}^{g} \left[(r_0) t'_k + t_k (r'_0) + \sum_{i,j} (r_i) s'_j + s_j (r'_i)\right] w_k$$
$$+ \sum_{l=1}^{h} \left[(r_0) z'_l + z_l (r'_0) + \sum_{i,k} (r_i) t'_k + t_k (r'_i) + \sum_{\kappa,\tau=1}^{f} (s_\kappa s'_\tau)\right] y_l$$

Since the unique maximal ideal of R is

$$Z(R) = pR_0 + \sum_{i=1}^{e} R_0 u_i + \sum_{j=1}^{f} R_0 v_j + \sum_{k=1}^{g} R_0 w_k + \sum_{l=1}^{h} R_0 y_l$$

and

$$1 + Z(R) = 1 + pR_0 + \sum_{i=1}^{e} R_0 u_i + \sum_{j=1}^{f} R_0 v_j + \sum_{k=1}^{g} R_0 w_k + \sum_{l=1}^{h} R_0 y_l$$

We use the ideas of Raghavendran [7] and Chikunji [2] to classify the unit groups of the rings constructed in this section.

$$R^* = (R^*/1 + Z(R)) \times (1 + Z(R)) = < b > \times (1 + Z(R))$$

where

$$< b >= (R^*/1 + Z(R)) = (R/Z(R))^* = \mathbb{F}_{p^r}^* \cong \mathbb{Z}_{p^r - 1}$$

Proposition 2.1. *Let R be the ring described by the above construction and of charac-*

teristic p with $pu_i = pv_j = pw_k = py_l = 0$. *Then its group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_2^r)^g, & \text{if } p = 2 \\ \mathbb{Z}_{3^r-1} \times (\mathbb{Z}_9^r)^e \times (\mathbb{Z}_3^r)^f \times (\mathbb{Z}_3^r)^h, & \text{if } p = 3 \\ \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_p^r)^e \times (\mathbb{Z}_p^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & \text{if } p > 3 \end{cases}$$

Proof. See proof of Proposition 2.1 in [4]. □

3 Main Results

Proposition 3.1. *Let R be the ring described by the above construction and of characteristic* p^2 *with* $pu_i = p^2v_j = pw_k = py_l = 0$. *Then its group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_2^r)^g \times (\mathbb{Z}_2^r)^h, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_p^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^h, & \text{if } p \geq 3 \end{cases}$$

Proof. Since R is commutative, $R^* =< b > \cdot (1 + Z(R)) \cong< b > \times (1 + Z(R))$ a direct product of the p - group $1+Z(R)$ by the cyclic group $< b >$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the group of units R^* . Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider the two cases separately.

Case (i): For $p = 2$, $1 + Z(R)$ contains subgroups $< 1 + 2\varepsilon_t >$ of order 2, $< 1 + \varepsilon_t u_i >$ of order 8, $< 1 + \varepsilon_t w_k >$ of order 2 and $< 1 + \varepsilon_t y_l >$ of order 2 for every $t = 1, \ldots, r$. Since the intersection of any pair of the cyclic subgroups $< 1 + 2\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1 + \varepsilon_t w_k >$ and $< 1 + \varepsilon_t y_l >$ $(1 \leq i \leq e,\ 1 \leq k \leq g,\ 1 \leq l \leq h)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+2\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\varepsilon_t w_k>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\varepsilon_t y_l>$$
$$\cong\mathbb{Z}_2^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_2^r)^g\times(\mathbb{Z}_2^r)^h$$

Case(ii): For $p\geq 3$, $1+Z(R)$ contains subgroups $<1+p\varepsilon_t>$ of order p, $<1+\varepsilon_t u_i>$ of order p^2, $<1+\varepsilon_t v_j>$ of order p^2 and $<1+\varepsilon_t y_l>$ of order p for every $t=1,\ldots,r$. Since the intersection of any pair of the cyclic subgroups $<1+p\varepsilon_t>$, $<1+\varepsilon_t u_i>$, $<1+\varepsilon_t v_j>$ and $<1+\varepsilon_t y_l>$ $(1\leq i\leq e,\ 1\leq j\leq f,\ 1\leq l\leq h)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+p\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\varepsilon_t v_j>\times\prod_{l=1}^{h}\prod_{t=1}^{r}<1+\varepsilon_t y_l>$$
$$\cong\mathbb{Z}_p^r\times(\mathbb{Z}_{p^2}^r)^e\times(\mathbb{Z}_{p^2}^r)^f\times(\mathbb{Z}_p^r)^h$$

□

Proposition 3.2. *Let R be the ring described by the above construction and of characteristic p^3 with $pu_i=p^2v_j=pw_k=py_l=0$. Then its group of units*

$$R^*\cong\begin{cases}\mathbb{Z}_{2^r-1}\times\mathbb{Z}_2^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_4^r)^f\times(\mathbb{Z}_2^r)^g, & \text{if } p=2\\ \mathbb{Z}_{p^r-1}\times\mathbb{Z}_{p^2}^r\times(\mathbb{Z}_{p^2}^r)^e\times(\mathbb{Z}_{p^2}^r)^f\times(\mathbb{Z}_p^r)^g, & \text{if } p\geq 3\end{cases}$$

Proof. Since R is commutative, $R^*=<b>\cdot(1+Z(R))\cong<b>\times(1+Z(R))$, a direct product of the $p-$group $1+Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the unit group R^*. Let $\varepsilon_1,\ldots,\varepsilon_r$ be elements of R_0 with $\varepsilon_1=1$ such that $\bar{\varepsilon}_1,\ldots,\bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider two cases separately:

Case(i): For $p=2$, $1+Z(R)$ contains subgroups $<1+2\varepsilon_t>$ of order 2, $<1+\varepsilon_t u_i>$

of order 8, $< 1 + \varepsilon_t v_j >$ of order 4 and $< 1 + \varepsilon_t w_k >$ of order 2 for every $t = 1, \ldots, r$. Since the intersection of any pair of the cyclic subgroups $< 1 + 2\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$ and $< 1+\varepsilon_t w_k >$ $(1 \leq i \leq e,\ 1 \leq j \leq f, 1 \leq k \leq g)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^{r} < 1 + 2\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k > \\ &\cong \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g \end{aligned}$$

Case(ii): For $p \geq 3$, $1 + Z(R)$ contains subgroups $< 1 + p\varepsilon_t >$ of order p^2, $< 1 + \varepsilon_t u_i >$ of order p^2, $< 1 + \varepsilon_t v_j >$ of order p^2 and $< 1 + \varepsilon_t w_k >$ of order p for every $t = 1, \ldots, r$. Since the intersection of any pair of the cyclic subgroups $< 1 + p\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$ and $< 1+\varepsilon_t w_k >$ $(1 \leq i \leq e,\ 1 \leq j \leq f, 1 \leq k \leq g)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1 + Z(R)|$, it follows that

$$\begin{aligned} 1 + Z(R) &= \prod_{t=1}^{r} < 1 + p\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k > \\ &\cong \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g \end{aligned}$$

□

Proposition 3.3. *Let R be the ring described by the above construction and of characteristic p^3 with $pu_i = p^3 v_j = pw_k = py_l = 0$. Then its group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_2^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_8^r)^f \times (\mathbb{Z}_2^r)^g, & \textit{if}\ \ p = 2 \\ \mathbb{Z}_{p^r-1} \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^3}^r)^f \times (\mathbb{Z}_p^r)^g \times (\mathbb{Z}_p^r)^h, & \textit{or} \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^2}^r \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^3}^r)^f, & \textit{if}\ \ p \geq 3 \end{cases}$$

Proof. Since R is commutative, $R^* =< b > \cdot (1 + Z(R)) \cong < b > \times (1 + Z(R))$, a direct product of the $p-$group $1+Z(R)$ by the cyclic group $< b >$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the unit group R^*. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2$, $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, the generators are $1+2\varepsilon_t$ of order 2, $1+\varepsilon_t u_i$ of order 8, $1+\varepsilon_t v_j$ of order 8, and $1+\varepsilon_t w_k$ of order 2. The rest of the proof follows a similar argument and maybe deduced from that of Proposition 3.2.

Case(ii): For $p \geq 3$, $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, $1 \leq l \leq h$, the generators are $1+\varepsilon_t u_i$ of order p^3, $1+\varepsilon_t v_j$ of order p^3, $1+\varepsilon_t w_k$ of order p, and $1+\varepsilon_t y_l$ of order p **or** $1+p\varepsilon_t$ of order p^2, $1+\varepsilon_t u_i$ of order p^3, and $1+\varepsilon_t v_j$ of order p^3. The rest of the proof follows a similar argument and maybe deduced from that of Proposition 3.2. □

Proposition 3.4. *Let R be the ring described by the above construction and of characteristic p^4 with $pu_i = p^2 v_j = pw_k = py_l = 0$. Then its group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_4^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_4^r)^f \times (\mathbb{Z}_2^r)^g, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^3}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g, & \text{if } p \geq 3 \end{cases}$$

Proof. Since R is commutative, $R^* =< b > \cdot (1 + Z(R)) \cong< b > \times (1 + Z(R))$, a direct product of the $p-$group $1+Z(R)$ by the cyclic group $< b >$. Then it suffices to determine the structure of the subgroup $1 + Z(R)$ of the unit group R^*. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider two cases separately:

Case(i): For $p = 2$, $1 + Z(R)$ contains subgroups $< 1 + 2\varepsilon_t >$ of order 4, $< 1 + \varepsilon_t u_i >$ of order 8, $< 1 + \varepsilon_t v_j >$ of order 4 and $< 1 + \varepsilon_t w_k >$ of order 2 for every $t = 1, \ldots, r$. Since the intersection of any pair of the cyclic subgroups $< 1 + 2\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$ and $< 1+\varepsilon_t w_k >$ $(1 \leq i \leq e,\ 1 \leq j \leq f, 1 \leq k \leq g)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with

$|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+2\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\varepsilon_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\varepsilon_t w_k>$$
$$\cong \mathbb{Z}_4^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_4^r)^f\times(\mathbb{Z}_2^r)^g$$

Case(ii): For $p\geq 3$, $1+Z(R)$ contains subgroups $<1+p\varepsilon_t>$ of order p^3, $<1+\varepsilon_t u_i>$ of order p^2, $<1+\varepsilon_t v_j>$ of order p^2 and $<1+\varepsilon_t w_k>$ of order p for every $t=1,\dots,r$. Since the intersection of any pair of the cyclic subgroups $<1+p\varepsilon_t>$, $<1+\varepsilon_t u_i>$, $<1+\varepsilon_t v_j>$ and $<1+\varepsilon_t w_k>$ $(1\leq i\leq e,\ 1\leq j\leq f, 1\leq k\leq g)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1+Z(R)|$, it follows that

$$1+Z(R)=\prod_{t=1}^{r}<1+p\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\varepsilon_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\varepsilon_t w_k>$$
$$\cong \mathbb{Z}_{p^3}^r\times(\mathbb{Z}_{p^2}^r)^e\times(\mathbb{Z}_{p^2}^r)^f\times(\mathbb{Z}_p^r)^g$$

□

Proposition 3.5. *Let R be the ring described by the above construction and of characteristic p^4 with $pu_i=p^3v_j=pw_k=py_l=0$. Then its group of units*

$$R^*\cong\begin{cases}\mathbb{Z}_{2^r-1}\times\mathbb{Z}_4^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_8^r)^f\times(\mathbb{Z}_2^r)^g, & \text{if } p=2\\ \mathbb{Z}_{p^r-1}\times\mathbb{Z}_{p^3}^r\times(\mathbb{Z}_{p^3}^r)^e\times(\mathbb{Z}_{p^3}^r)^f, & \text{if } p\geq 3\end{cases}$$

Proof. Since R is commutative, $R^*=<b>\cdot(1+Z(R))\cong<b>\times(1+Z(R))$, a direct product of the $p-$group $1+Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the unit group R^*. Let $\varepsilon_1,\dots,\varepsilon_r$ be elements of R_0 with $\varepsilon_1=1$ such that $\bar{\varepsilon}_1,\dots,\bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p=2,\ 1\le t\le r,\quad 1\le i\le e,\ 1\le j\le f,\ 1\le k\le g$, the generators are $1+2\varepsilon_t$ of order 4, $1+\varepsilon_t u_i$ of order 8, $1+\varepsilon_t v_j$ of order 8, and $1+\varepsilon_t w_k$ of order 2. The rest of the proof is similar to that of Proposition 3.4.

Case(ii): For $p\ge 3$, $1\le t\le r,\quad 1\le i\le e,\ 1\le j\le f,$ the generators are $1+p\varepsilon_t$ of order p^3, $1+\varepsilon_t u_i$ of order p^3, and $1+\varepsilon_t v_j$ of order p^3, The rest of the proof is similar to that of Proposition 3.4. □

Proposition 3.6. *Let R be the ring described by the above construction and of characteristic p^5 with $pu_i=p^2v_j=pw_k=py_l=0$. Then its group of units*

$$R^*\cong\begin{cases}\mathbb{Z}_{2^r-1}\times\mathbb{Z}_8^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_4^r)^f\times(\mathbb{Z}_2^r)^g, & \text{if } p=2\\ \mathbb{Z}_{p^r-1}\times\mathbb{Z}_{p^4}^r\times(\mathbb{Z}_{p^2}^r)^e\times(\mathbb{Z}_{p^2}^r)^f\times(\mathbb{Z}_p^r)^g, & \text{if } p\ge 3\end{cases}$$

Proof. Since R is commutative, $R^*=<b>\cdot(1+Z(R))\cong<b>\times(1+Z(R))$, a direct product of the $p-$group $1+Z(R)$ by the cyclic group $<b>$. Then it suffices to determine the structure of the subgroup $1+Z(R)$ of the unit group R^*. Let $\varepsilon_1,\dots,\varepsilon_r$ be elements of R_0 with $\varepsilon_1=1$ such that $\bar{\varepsilon}_1,\dots,\bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. We consider two cases separately:

Case(i): For $p=2$, $1+Z(R)$ contains subgroups $<1+2\varepsilon_t>$ of order 8, $<1+\varepsilon_t u_i>$ of order 8, $<1+\varepsilon_t v_j>$ of order 4 and $<1+\varepsilon_t w_k>$ of order 2 for every $t=1,\dots,r$. Since the intersection of any pair of the cyclic subgroups $<1+2\varepsilon_t>$, $<1+\varepsilon_t u_i>$, $<1+\varepsilon_t v_j>$ and $<1+\varepsilon_t w_k>$ $(1\le i\le e,\ 1\le j\le f, 1\le k\le g)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1+Z(R)|$, it follows that

$$\begin{aligned}1+Z(R)&=\prod_{t=1}^{r}<1+2\varepsilon_t>\times\prod_{i=1}^{e}\prod_{t=1}^{r}<1+\varepsilon_t u_i>\times\prod_{j=1}^{f}\prod_{t=1}^{r}<1+\varepsilon_t v_j>\times\prod_{k=1}^{g}\prod_{t=1}^{r}<1+\varepsilon_t w_k>\\&\cong\mathbb{Z}_8^r\times(\mathbb{Z}_8^r)^e\times(\mathbb{Z}_4^r)^f\times(\mathbb{Z}_2^r)^g\end{aligned}$$

Case(ii): For $p\ge 3$, $1+Z(R)$ contains subgroups $<1+p\varepsilon_t>$ of order p^4, $<1+\varepsilon_t u_i>$ of order p^2, $<1+\varepsilon_t v_j>$ of order p^2 and $<1+\varepsilon_t w_k>$ of order p for every $t=1,\dots,r$.

Since the intersection of any pair of the cyclic subgroups $< 1 + p\varepsilon_t >$, $< 1 + \varepsilon_t u_i >$, $< 1+\varepsilon_t v_j >$ and $< 1+\varepsilon_t w_k >$ $(1 \leq i \leq e,\ 1 \leq j \leq f, 1 \leq k \leq g)$ is trivial, and that the order of the group generated by direct product of these cyclic subgroups coincides with $|1 + Z(R)|$, it follows that

$$1 + Z(R) = \prod_{t=1}^{r} < 1 + p\varepsilon_t > \times \prod_{i=1}^{e}\prod_{t=1}^{r} < 1 + \varepsilon_t u_i > \times \prod_{j=1}^{f}\prod_{t=1}^{r} < 1 + \varepsilon_t v_j > \times \prod_{k=1}^{g}\prod_{t=1}^{r} < 1 + \varepsilon_t w_k >$$
$$\cong \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^2}^r)^e \times (\mathbb{Z}_{p^2}^r)^f \times (\mathbb{Z}_p^r)^g$$

$\square$

Proposition 3.7. *Let R be the ring described by the above construction and of characteristic p^4 with $pu_i = p^3 v_j = pw_k = py_l = 0$. Then its group of units*

$$R^* \cong \begin{cases} \mathbb{Z}_{2^r-1} \times \mathbb{Z}_8^r \times (\mathbb{Z}_8^r)^e \times (\mathbb{Z}_8^r)^f \times (\mathbb{Z}_2^r)^g, & \text{if } p = 2 \\ \mathbb{Z}_{p^r-1} \times \mathbb{Z}_{p^4}^r \times (\mathbb{Z}_{p^3}^r)^e \times (\mathbb{Z}_{p^3}^r)^f, & \text{if } p \geq 3 \end{cases}$$

Proof. Since R is commutative, $R^* =< b > \cdot (1 + Z(R)) \cong< b > \times (1 + Z(R))$, a direct product of the $p-$group $1+Z(R)$ by the cyclic group $< b >$. Then it suffices to determine the structure of the subgroup $1 + Z(R)$ of the unit group R^*. Let $\varepsilon_1, \ldots, \varepsilon_r$ be elements of R_0 with $\varepsilon_1 = 1$ such that $\bar{\varepsilon}_1, \ldots, \bar{\varepsilon}_r$ form a basis for R_0/pR_0 regarded as a vector space over its prime subfield $\mathbb{F}_p$. Then the generators with their respective orders are as indicated below:

Case(i): For $p = 2$, $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, $1 \leq k \leq g$, the generators are $1 + 2\varepsilon_t$ of order 8, $1 + \varepsilon_t u_i$ of order 8, $1 + \varepsilon_t v_j$ of order 8, and $1 + \varepsilon_t w_k$ of order 2. The rest of the proof is similar to that of Proposition 3.6.

Case(ii): For $p \geq 3$, $1 \leq t \leq r$, $1 \leq i \leq e$, $1 \leq j \leq f$, the generators are $1 + p\varepsilon_t$ of order p^4, $1 + \varepsilon_t u_i$ of order p^3, and $1 + \varepsilon_t v_j$ of order p^3, The rest of the proof is similar to that of Proposition 3.6. $\square$

4 Conclusion

This study has classified the group of units of a class of five radical zero commutative completely primary finite rings with variant orders of second Galois ring module generators. This has been achieved through isolation of the set of units from the set of zero divisors followed by the use of fundamental theorem of finitely generated abelian groups. The results are noted to be in piece when the prime integer p is even and odd. Further research will focus on the classification of the group of units of classes of five radical zero commutative completely primary finite rings with variant orders of third Galois ring module generators as well as mixed variant orders of first and second Galois ring module generators.

Acknowledgment

The first author would like to thank the second author for his valuable comments and suggestions especially about the title of this paper and the ranks of the groups.

Conflict of Interest

The authors declare that they have no conflicts of interest

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