One-step Hybrid Block Method for Directly Solving Fifth-order Initial Value Problems of Ordinary Differential Equations

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Abstract

An effective one-step hybrid block for getting the approximate solution of a fifth-order IVP with applications to problems in the sciences and engineering is constructed in this study. The mathematical formulation of the method is based on the principle of interpolation and collocation of the trial solution and its derivatives at the chosen equidistant grid and off-grid points. The basic properties of the derived method are examined, and it has an order greater than one, zero stable, consistent, and hence convergent. The derived method is applied to solve five different linear and nonlinear fifth-order initial value problems. Comparison of the absolute errors obtained using the derived method with a few existing ones in the literature supports its good performance.

Keywords: higher-order, interpolation, collocation, grid and off-grid points, zero stability, consistency

1 Introduction

This paper centers on numerical solution of special and general fifth-order initial value problems (IVPs) of ordinary differential equation of the form

$$y^{(v)} = f(x, y, y', y'', y''', y^{iv}), \tag{1}$$

subject to

$$y(x_0) = y_0, \;\; y'(x_0) = y'_0, \;\; y''(x_0) = y''_0, \;\; y'''(x_0) = y'''_0, \;\; y^{iv}(x_0) = y^{iv}_0,$$

where $x \in [a, b]$, function f is sufficiently differentiable to guarantees the existence and uniqueness of the solution. According to [1]- [10], this type of equation arises frequently in the literature, particularly in aerospace engineering, where the vibration of structures caused by the passage of moving loads under damping forces is seen as critical. The fifth order Korteweg-de Vries (KdV) equation is a partial differential equation used to represent various wave phenomena connected to shallow water surfaces (see [11, 12]), which in some situations can be translated into a higher order IVP system to address. A fifth-order nonlinear differential equation can be used to describe the induction motor [13, 14]. The evolution of charged fluids can also be described using fifth order differential equations [15]. In most instances, it is impossible to find the exact solution to (1). As a result, when the problem in (1) cannot be solved analytically, we rely on numerical approaches to find approximate solutions. Before now, equations of this type were solved by the reduction approach (see [16], [17], [18]) which had been reported to have some shortcomings by [1, 2], [5], [10]. Some scholars have developed various numerical approaches for tackling problems of the type in (1). Among these strategies is the Lacunary interpolation described in [6], the perturbation method presented in [3], and Adomian Decomposition Method (ADM) in [19]. Various academics, including Kayode & Awoyemi [1], Olabode [5], Kayode [2], and Mechee *et al.* [4] have utilised direct methods (which have removed the shortcomings observed with the reduction approach) for solving fifth-order initial value problems. The methods in [1] and [2] are, respectively, implicit and explicit methods, which were both implemented in predictor-corrector mode. The method in [5] shares both the characteristics of block and predictor-corrector methods. Mechee *et al.* [4] approached the direction solution of fifth-order initial value problems using an explicit direct integrator of the RK type, which can only handle the special case of (1). The methods are limited because their implementations require starting values and predictors and are mostly explicit, which reduces their accuracy. Motivated by the success of the direct method and the fact that the implicitness of a linear multistep method can be handled more accurately using a block method with better accuracy (see [21–23]), in the present work, a one-step hybrid block method with four intral grid points is constructed and applied to solve equations of the form (1). The benefit of the present technique is that its implementation eliminates the necessity for starting values, allowing for the generation of approximate solutions without the use of predictors. Development, analysis, and implementation of the derived methods are present in sections 2, 3, and 4, while numerical examples are presented in section 5.

2 Derivation of the Block Method

Assumimg the equation of type (1) is approximated with a power series of a single variable x in the form:

$$y(x) = \sum_{j=0}^{(c+i)-1} a_j x^j, \tag{2}$$

where $a_j, j = 0, 1, 2, \cdots, k$ are the coefficients to be determined, c is the number of collocation points, and i is the number of interpolation points. Equating the fifth derivative of (2) to (1) gives

$$f(x, y, y', y'', y''', y^{iv}) = \sum_{j=5}^{(c+i)-1} j(j-1)(j-2)(j-3)(j-4) a_j x^{j-5}. \tag{3}$$

Collocating (3) at $x = x_{n+j}$, $j = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$, interpolating (2), and evaluating its first, second, third and fourth derivatives at $x = x_n$ lead to systems of algebraic equations that can be written in matrix equation as

$$AX = B \tag{4}$$

where,

$$A=\begin{bmatrix}1&0&0&0&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0&0&0&0\\0&0&2&0&0&0&0&0&0&0&0\\0&0&0&6&0&0&0&0&0&0&0\\0&0&0&0&24&0&0&0&0&0&0\\0&0&0&0&0&120&0&0&0&0&0\\0&0&0&0&0&120&144&\frac{504}{5}&\frac{1344}{25}&\frac{3024}{125}&\frac{6048}{625}\\0&0&0&0&0&120&288&\frac{2016}{5}&\frac{10752}{25}&\frac{48384}{125}&\frac{193536}{625}\\0&0&0&0&0&120&432&\frac{4536}{5}&\frac{36288}{25}&\frac{244944}{125}&\frac{1469664}{625}\\0&0&0&0&0&120&576&\frac{8064}{5}&\frac{86016}{25}&\frac{774144}{125}&\frac{6193152}{625}\\0&0&0&0&0&120&720&2520&6720&15120&30240\end{bmatrix},X=\begin{bmatrix}a_0\\a_1\\a_2\\a_3\\a_4\\a_5\\a_6\\a_7\\a_8\\a_9\\a_{10}\end{bmatrix},B=\begin{bmatrix}y_n\\y_n'\\y_n''\\y_n'''\\y_n^{iv}\\h^5f_n\\h^5f_{n+\frac{1}{5}}\\h^5f_{n+\frac{2}{5}}\\h^5f_{n+\frac{3}{5}}\\h^5f_{n+\frac{4}{5}}\\h^5f_{n+1}\end{bmatrix}.$$

Solving the matrix equation (4) using Gaussian elimination method, yields parameters a_j's which were substituted in (2). After some simplifications, the following continuous one-step scheme was obtained

$$\begin{aligned}y(x) \;=\;& y_n + y'_n x + \frac{1}{2}y''_n x^2 + \frac{1}{6}y'''_n x^3 + \frac{1}{24}y^{iv}_n x^4\\
&-\frac{1}{1451520h^5}\left(-12096\,x^5h^5 + 23016\,x^6h^4 - 27000\,x^7h^3 + 19125\,x^8h^2 - 7500\,x^9h + 1250\,x^{10}\right)f_n\\
&-\frac{1}{1451520h^5}\left(-50400\,x^6h^4 + 92400\,x^7h^3 - 79875\,x^8h^2 + 35000\,x^9h - 6250\,x^{10}\right)f_{n+\frac{1}{5}}\\
&-\frac{1}{1451520}\,\frac{\left(50400\,x^6h^4 - 128400\,x^7h^3 + 132750\,x^8h^2 - 65000\,x^9h + 12500\,x^{10}\right)f_{n+\frac{2}{5}}}{h^5}\\
&-\frac{1}{1451520}\,\frac{\left(-33600\,x^6h^4 + 93600\,x^7h^3 - 110250\,x^8h^2 + 60000\,x^9h - 12500\,x^{10}\right)f_{n+\frac{3}{5}}}{h^5}\\
&-\frac{1}{1451520}\,\frac{\left(12600\,x^6h^4 - 36600\,x^7h^3 + 46125\,x^8h^2 - 27500\,x^9h + 6250\,x^{10}\right)f_{n+\frac{4}{5}}}{h^5}\\
&-\frac{1}{1451520}\,\frac{\left(-2016\,x^6h^4 + 6000\,x^7h^3 - 7875\,x^8h^2 + 5000\,x^9h - 1250\,x^{10}\right)f_{n+1}}{h^5}\end{aligned}\qquad(5)$$

Evaluating the continuous one-step scheme (5) and its first, second, third, fourth derivatives at point $x = x_{n+j}$, j= $\frac{1}{5},\frac{2}{5},\frac{3}{5},\frac{4}{5}$, and 1, yielded a block of discrete and additional schemes which are presented in table (1) below:

Table 1: The main and additional formulas.

	i	y_n	y'_n	y''_n	y'''_n	$y_n^{(iv)}$	f_n	$f_{n+\frac{1}{5}}$	$f_{n+\frac{2}{5}}$	$f_{n+\frac{3}{5}}$	$f_{n+\frac{4}{5}}$	f_{n+1}
$y_{n+\frac{i}{5}}$	1	1	$\frac{h}{5}$	$\frac{h^2}{50}$	$\frac{h^3}{750}$	$\frac{h^4}{15000}$	$\frac{42157h^5}{22680000000}$	$\frac{34845h^5}{22680000000}$	$\frac{-29530h^5}{22680000000}$	$\frac{18830h^5}{22680000000}$	$\frac{-6915h^5}{22680000000}$	$\frac{1093h^5}{22680000000}$
	2	1	$\frac{2h}{5}$	$\frac{2h^2}{25}$	$\frac{4h^3}{375}$	$\frac{2h^4}{1875}$	$\frac{3853h^5}{88593750}$	$\frac{6035h^5}{88593750}$	$\frac{-4110h^5}{88593750}$	$\frac{2570h^5}{88593750}$	$\frac{-935h^5}{88593750}$	$\frac{147h^5}{88593750}$
	3	1	$\frac{3h}{5}$	$\frac{9h^2}{50}$	$\frac{9h^3}{250}$	$\frac{27h^4}{5000}$	$\frac{71253h^5}{280000000}$	$\frac{152685h^5}{280000000}$	$\frac{-78570h^5}{280000000}$	$\frac{52110h^5}{280000000}$	$\frac{-19035h^5}{280000000}$	$\frac{2997h^5}{280000000}$
	4	1	$\frac{4h}{5}$	$\frac{8h^2}{25}$	$\frac{32h^3}{375}$	$\frac{32h^4}{1875}$	$\frac{38336h^5}{44296875}$	$\frac{97920h^5}{44296875}$	$\frac{-35840h^5}{44296875}$	$\frac{29440h^5}{44296875}$	$\frac{-10560h^5}{44296875}$	$\frac{1664h^5}{44296875}$
	5	1	h	$\frac{h^2}{2}$	$\frac{h^3}{6}$	$\frac{h^4}{24}$	$\frac{3205h^5}{1451520}$	$\frac{9125h^5}{1451520}$	$\frac{-2250h^5}{1451520}$	$\frac{2750h^5}{1451520}$	$\frac{-875h^5}{1451520}$	$\frac{141h^5}{1451520}$
$y'_{n+\frac{i}{5}}$	1	0	1	$\frac{h}{5}$	$\frac{h^2}{50}$	$\frac{h^3}{750}$	$\frac{49126h^4}{1134000000}$	$\frac{49045h^4}{1134000000}$	$\frac{-40160h^4}{1134000000}$	$\frac{25430h^4}{1134000000}$	$\frac{-9310h^4}{1134000000}$	$\frac{1469h^4}{1134000000}$
	2	0	1	$\frac{2h}{5}$	$\frac{2h^2}{25}$	$\frac{4h^3}{375}$	$\frac{4264h^4}{8859375}$	$\frac{7960h^4}{8859375}$	$\frac{-4910h^4}{8859375}$	$\frac{3080h^4}{8859375}$	$\frac{-1120h^4}{8859375}$	$\frac{176h^4}{8859375}$
	3	0	1	$\frac{3h}{5}$	$\frac{9h^2}{50}$	$\frac{9h^3}{250}$	$\frac{25488h^4}{14000000}$	$\frac{63315h^4}{14000000}$	$\frac{-26460h^4}{14000000}$	$\frac{19170h^4}{14000000}$	$\frac{-7020h^4}{14000000}$	$\frac{1107h^4}{14000000}$
	4	0	1	$\frac{4h}{5}$	$\frac{8h^2}{25}$	$\frac{32h^3}{375}$	$\frac{40448h^4}{8859375}$	$\frac{116480h^4}{8859375}$	$\frac{-29440h^4}{8859375}$	$\frac{33280h^4}{8859375}$	$\frac{-11360h^4}{8859375}$	$\frac{1792h^4}{8859375}$
	5	0	1	h	$\frac{h^2}{2}$	$\frac{h^3}{6}$	$\frac{3346h^4}{362880}$	$\frac{10525h^4}{362880}$	$\frac{-1400h^4}{362880}$	$\frac{3350h^4}{362880}$	$\frac{-850h^4}{362880}$	$\frac{149h^4}{362880}$
$y''_{n+\frac{i}{5}}$	1	0	0	1	$\frac{h}{5}$	$\frac{h^2}{50}$	$\frac{3929h^3}{5040000}$	$\frac{4975h^3}{5040000}$	$\frac{-3862h^3}{5040000}$	$\frac{2422h^3}{5040000}$	$\frac{-883h^3}{5040000}$	$\frac{139h^3}{5040000}$
	2	0	0	1	$\frac{2h}{5}$	$\frac{2h^2}{25}$	$\frac{317h^3}{78750}$	$\frac{734h^3}{78750}$	$\frac{-380h^3}{78750}$	$\frac{244h^3}{78750}$	$\frac{-89h^3}{78750}$	$\frac{14h^3}{78750}$
	3	0	0	1	$\frac{3h}{5}$	$\frac{9h^2}{50}$	$\frac{5481h^3}{560000}$	$\frac{16119h^3}{560000}$	$\frac{-4374h^3}{560000}$	$\frac{4230h^3}{560000}$	$\frac{-1539h^3}{560000}$	$\frac{243h^3}{560000}$
	4	0	0	1	$\frac{4h}{5}$	$\frac{8h^2}{25}$	$\frac{712h^3}{39375}$	$\frac{2336h^3}{39375}$	$\frac{-224h^3}{39375}$	$\frac{704h^3}{39375}$	$\frac{-200h^3}{39375}$	$\frac{32h^3}{39375}$
	5	0	0	1	h	$\frac{h^2}{2}$	$\frac{233h^3}{8064}$	$\frac{815h^3}{8064}$	$\frac{10h^3}{8064}$	$\frac{310h^3}{8064}$	$\frac{-35h^3}{8064}$	$\frac{11h^3}{8064}$
$y'''_{n+\frac{i}{5}}$	1	0	0	0	1	$\frac{h}{5}$	$\frac{2462h^2}{252000}$	$\frac{4315h^2}{252000}$	$\frac{-3044h^2}{252000}$	$\frac{1882h^2}{252000}$	$\frac{-682h^2}{252000}$	$\frac{107h^2}{252000}$
	2	0	0	0	1	$\frac{2h}{5}$	$\frac{355h^2}{15750}$	$\frac{1088h^2}{15750}$	$\frac{-370h^2}{15750}$	$\frac{272h^2}{15750}$	$\frac{-101h^2}{15750}$	$\frac{16h^2}{15750}$
	3	0	0	0	1	$\frac{3h}{5}$	$\frac{984h^2}{28000}$	$\frac{3501h^2}{28000}$	$\frac{-72h^2}{28000}$	$\frac{870h^2}{28000}$	$\frac{-288h^2}{28000}$	$\frac{45h^2}{28000}$
	4	0	0	0	1	$\frac{4h}{5}$	$\frac{376h^2}{7875}$	$\frac{1424h^2}{7875}$	$\frac{176h^2}{7875}$	$\frac{608h^2}{7875}$	$\frac{-80h^2}{7875}$	$\frac{16h^2}{7875}$
	5	0	0	0	1	h	$\frac{122h^2}{2016}$	$\frac{475h^2}{2016}$	$\frac{100h^2}{2016}$	$\frac{250h^2}{2016}$	$\frac{50h^2}{2016}$	$\frac{11h^2}{2016}$
$y^{(iv)}_{n+\frac{i}{5}}$	1	0	0	0	0	1	$\frac{475h}{7200}$	$\frac{1427h}{7200}$	$\frac{-798h}{7200}$	$\frac{482h}{7200}$	$\frac{-173h}{7200}$	$\frac{27h}{7200}$
	2	0	0	0	0	1	$\frac{28h}{450}$	$\frac{129h}{450}$	$\frac{14h}{450}$	$\frac{14h}{450}$	$\frac{-6h}{450}$	$\frac{h}{450}$
	3	0	0	0	0	1	$\frac{51h}{800}$	$\frac{219h}{800}$	$\frac{114h}{800}$	$\frac{114h}{800}$	$\frac{-21h}{800}$	$\frac{3h}{800}$
	4	0	0	0	0	1	$\frac{14h}{225}$	$\frac{64h}{225}$	$\frac{24h}{225}$	$\frac{64h}{225}$	$\frac{14h}{225}$	
	5	0	0	0	0	1	$\frac{19h}{288}$	$\frac{75h}{288}$	$\frac{50h}{288}$	$\frac{50h}{288}$	$\frac{75h}{288}$	$\frac{19h}{288}$

3 Analysis of the method

In this secion, the analysis of the basic properties of the derived method are presented.

3.1 Local truncation error and order

Assuming that $J(x)$ is a sufficiently differentiable function, the linear difference operators associated with the formulas in Table 1 can be written in the form

$$\mathcal{L}_{\frac{i}{5}}\left[J(x);h\right] \equiv J(x+\frac{i}{5}h)-\left[\sum_{m=0}^{4}\alpha_m(\frac{i}{5})J^{(m)}(x)h^m+h^5\sum_{m=0}^{5}\beta_m(\frac{i}{5})J^{(5)}(x+\frac{i}{5}h)\right], \tag{6}$$

where $i=1,2,\ldots,5$. Taking $J(x)$ as the true solution of the problem in (1), after using in (6) the Taylor series about x yields the formulas for the truncation errors as

$$\mathcal{L}_{\frac{i}{5}}\left[y(x);h\right]=c_0^i y(x)+c_1^i hy'(x)+c_2^i h^2y''(x)+\cdots+c_j^i h^j y^{(j)}(x)+O\left(h^{j+1}\right) \tag{7}$$

where c_j^i are constants. It is important to note that the first $p+5$ constants will be equal to zero, which means that

$$c_0^i=c_1^i=c_2^i=\cdots=c_{p+4}^i=0, \text{ and } c_{p+5}^i\neq 0,$$

which implies that

$$\mathcal{L}_{\frac{i}{2}}\left[y(x);h\right]=c_{p+5}^i h^{p+5}y^{p+5}(x)+O\left(h^{p+6}\right)$$

where p and c_{p+5}^i are respectively known as the order and local principal error constant of the corresponding formula. The calculation of the local truncation errors of the formulas in Table (1) results in:

$$\begin{aligned} c_{p+5} = \Big(&-\frac{57397}{23388750000000000},-\frac{2323}{35437500000000},-\frac{9809}{7087500000000},-\frac{199}{9450000000},\\ &-\frac{863}{4725000000},-\frac{7571}{91362304687500},-\frac{137}{138427734375},-\frac{491}{55371093750},-\frac{19}{369140625},\\ &-\frac{37}{295312500},-\frac{51543}{9450000000},-\frac{1737}{437500000000},-\frac{1917}{87500000000},-\frac{141}{1750000000},-\frac{29}{1750000000},\\ &-\frac{43072}{22840576171875},-\frac{1408}{138427734375},-\frac{1136}{27685546875},-\frac{8}{73828125},-\frac{8}{73828125},\\ &-\frac{293}{59875200000},-\frac{47}{2268000000},-\frac{149}{2268000000},-\frac{11}{75600000},-\frac{11}{37800000}\Big) \end{aligned} \tag{8}$$

$p=(6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6,6)$ Hence, the block method has an uniform order 6.

3.2 Zero-stability and convergence

As stated in [17], a block method is said to be zero stable if the roots $det[\lambda G_a - G_b]=0$, of it's first characteristic polynomial, satisfies $|\lambda|\leq 1$ and for the roots with $|\lambda|=1$, the multiplicity must not exceed the order of the differential equation. According to [20], zero-stability is a kind of stability that tells us the behaviour of the block method when the step-size h tends to zero ($h \to 0$). The formulas in Table 1 reduces to equations that can be written in the form of

$$G_a^r Y_n = G_b^r Y_{n-1}, r = 1, 2, \ldots, 5 \tag{9}$$

where,

$$Y_n = \left(Y_n^0, Y_n^1,\ Y_n^2,\ Y_n^3,\ Y_n^4\right)^T, \qquad Y_{n-1} = \left(Y_{n-1}^0, Y_{n-1}^1,\ Y_{n-1}^2,\ Y_{n-1}^3,\ Y_{n-1}^4\right)^T,$$

$$Y_n^0 = \left(y_{n+\frac{1}{5}},\ y_{n+\frac{2}{5}},\ y_{n+\frac{3}{5}},\ y_{n+\frac{4}{5}},\ y_{n+1}\right), \qquad Y_{n-1}^0 = \left(y_{n-\frac{1}{5}},\ y_{n-\frac{2}{5}},\ y_{n-\frac{3}{5}},\ y_{n-\frac{4}{5}},\ y_n\right),$$

$$Y_n^1 = \left(y'_{n+\frac{1}{5}},\ y'_{n+\frac{2}{5}},\ y'_{n+\frac{3}{5}},\ y'_{n+\frac{4}{5}},\ y'_{n+1}\right), \qquad Y_{n-1}^1 = \left(y'_{n-\frac{1}{5}},\ y'_{n-\frac{2}{5}},\ y'_{n-\frac{3}{5}},\ y'_{n-\frac{4}{5}},\ y'_n\right),$$

$$\vdots \qquad\qquad \vdots$$

$$Y_n^4 = \left(y^{(4)}_{n+\frac{1}{5}},\ y^{(4)}_{n+\frac{2}{5}},\ y^{(4)}_{n+\frac{3}{5}},\ y^{(4)}_{n+\frac{4}{5}},\ y^{(4)}_{n+1}\right), \qquad Y_{n-1}^4 = \left(y^{(4)}_{n-\frac{1}{5}},\ y^{(4)}_{n-\frac{2}{5}},\ y^{(4)}_{n-\frac{3}{5}},\ y^{(4)}_{n-\frac{4}{5}},\ y^{(4)}_n\right),$$

G_a^r's are identity matrices and $G_b^r = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \forall \quad r.$

Since $\lambda_r = \lambda^4 (\lambda - 1) = 0, \quad \forall \ \ r, \lambda_r = \{0, 0, 0, 0, 1\}$ the derived is zero-stable.

Theorem 1 (Henrici [18]) *A numerical linear method which is consistent (with order $p \geq 1$) and zero-stable it is convergent.* Since the conditions stipulated in **theorem 1** are meet by the block method, it is convergent.

4 Implementation

The derived formulas in Table 1 are combined and excuted in block form. The solutions are considered in the interval $[x_n, x_{n+1}]$, $n = 0,\ 1,\ 2,\ \ldots,\ N-1$ where N is the number of blocks. The formulas in Table 1 are written in the form of $M(y) = 0$ with the following unknown values to be obtained

$$\bar{\mathbf{Y}} = \Big(y_0, y'_0, y''_0, y'''_0, y_0^{(4)}, y_{\frac{1}{5}}, y'_{\frac{1}{5}}, y''_{\frac{1}{5}}, y'''_{\frac{1}{5}}, y^{(4)}_{\frac{1}{5}}, y_{\frac{2}{5}}, y'_{\frac{2}{5}}, y''_{\frac{2}{5}}, y'''_{\frac{2}{5}}, y^{(4)}_{\frac{2}{5}}, \ldots, y_1, y'_1, y''_1, y'''_1, y_1^{(4)},$$
$$y_{\frac{6}{5}}, y'_{\frac{6}{5}}, y''_{\frac{6}{5}}, y'''_{\frac{6}{5}}, y^{(4)}_{\frac{6}{5}}, \ldots, y_N, y'_N, y''_N, y'''_N, y_N^{(4)}\Big)$$

The resulting system is solved using Newton's method given as

$$\bar{\mathbf{Y}}^{i+1} = \bar{\mathbf{Y}}^i - \frac{M^i}{\mathbf{J}^i}$$

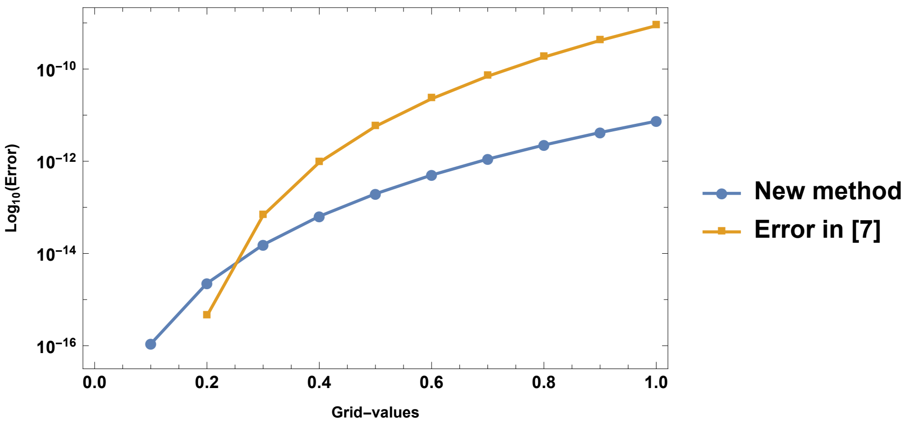


Figure 1: Efficiency curves for problem 1

where **J** is the Jacobian matrix of M. The following Taylor's approximations are considered as starting values to be used with the Newton's method.

$$\begin{aligned}
y_{n+\frac{j}{5}} &= y_n + j\frac{h}{2}y'_n + \frac{1}{2}\left(j\frac{h}{2}\right)^2 y''_n + \frac{1}{3!}\left(j\frac{h}{2}\right)^3 y'''_n + \frac{1}{4!}\left(j\frac{h}{2}\right)^4 y^{(4)}_n + \frac{1}{5!}\left(j\frac{h}{2}\right)^5 f_n, \\
y'_{n+\frac{j}{5}} &= y'_n + j\frac{h}{2}y''_n + \frac{1}{2}\left(j\frac{h}{2}\right)^2 y'''_n + \frac{1}{3!}\left(j\frac{h}{2}\right)^3 y^{(4)}_n + \frac{1}{4!}\left(j\frac{h}{2}\right)^4 f_n, \\
y''_{n+\frac{j}{5}} &= y''_n + j\frac{h}{2}y'''_n + \frac{1}{2}\left(j\frac{h}{2}\right)^2 y^{(4)}_n + \frac{1}{3!}\left(j\frac{h}{2}\right)^3 f_n, \\
y'''_{n+\frac{j}{5}} &= y'''_n + j\frac{h}{2}y^{(4)}_n + \frac{1}{2}\left(j\frac{h}{2}\right)^2 f_n \\
y^{(4)}_{n+\frac{j}{5}} &= y^{(4)}_n + j\frac{h}{2}f_n,
\end{aligned}$$

5 Numerical Examples

In this section, some examples are presented as test problems to measure the accuracy and establish the usefulness of the derived method. The performance of the method is measured via the absolute errors obtained by using the relation $|y(x_i) - y_i|$, where $y(x_i)$ is the exact solution and y_i is the numerical solution.

Problem 1:

The first example is the linear special fifth-order initial value problem of ordinary differential equation

$$y^v = 32y + cosx - 32sinx,\ y(0) = 1,\ y'(0) = 3,\ y''(0) = 4,\ y'''(0) = 7,\ y^{iv}(0) = 16.$$

whose solution is $y(x) = sinx + e^{2x}$. The solution is considered within the interval $[0, 1]$ over ten (10) iterations. The results are presented in Table 2. It clear from the table that the computed results agreed with the exact solution up to at least tewlve decimal places. Figures 1 and 5a respectively, show the comparison of the efficiency curves and the y-exact and y-computed of problem 1.

Table 2: Comparison of results and absolute errors for problem 1

h	$y - computed$	$y - exact$	Absolute Errors	Errors in [7], $p = 8$
0.1	1.321236174806998	1.321236174806998	1.093543E-16	0.000000E+00
0.2	1.690494028436334	1.690494028436331	2.256920E-15	4.440892E-16
0.3	2.117639007051864	2.117639007051848	1.545934E-14	6.705747E-14
0.4	2.614959270801182	2.614959270801118	6.386567E-14	9.467982E-13
0.5	3.197707367063445	3.197707367063248	1.966987E-13	5.706546E-12
0.6	3.884759396132083	3.884759396131582	5.004079E-13	2.265743E-11
0.7	4.699417654083479	4.699417654082366	1.113542E-12	6.985079E-11
0.8	5.670388515296882	5.670388515294638	2.244978E-12	1.814815E-10
0.9	6.832974374044626	6.83297437404043	4.196309E-12	4.16659E-10
1.0	8.230527083745937	8.230527083738547	7.389423E-12	8.710561E-10

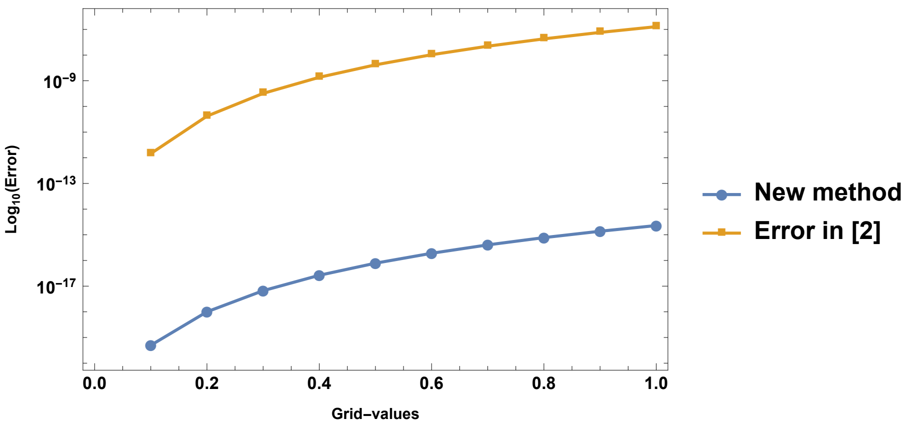


Figure 2: Efficiency curves for problem 2

Problem 2:

Another problem considered is the nonlinear general fifth-order initial value problem of the form:

$$y^{v} = 2y'y'' + yy^{iv} - y'y''' - 8x + (x^2 - 2x - 3)e^x,\ 0 \le x \le 1,$$

with initial conditions $y(0) = 1$, $y'(0) = 1$, $y''(0) = 3$, $y'''(0) = 1$, $y^{iv}(0) = 1$ whose exact solution is obtained as $y(x) = e^x + x^2$. This problem is solved within the interval $[0, 1]$ over a hundred iterations. Table 3 indicates that the exact and computed results agree up to fifteen decimal places. Figures 2 and 5b respectively, show the comparison of the efficiency curves and the y-exact and y-computed of problem 1.

Table 3: Results of problem 2 (with h=0.01)

h	$y-computed$	$y-exact$	Absolute Errors	Errors [2]
0.1	1.115170918075647	1.115170918075647	4.980924E-20	1.459721E-12
0.2	1.26140275816017	1.26140275816017	1.002438E-18	4.187584E-11
0.3	1.439858807576003	1.439858807576003	6.650964E-18	3.221776E-10
0.4	1.651824697641270	1.651824697641270	2.648497E-17	1.365175E-09
0.5	1.898721270700128	1.898721270700128	7.832139E-17	4.166737E-09
0.6	2.182118800390509	2.182118800390509	1.906649E-16	1.033164E-08
0.7	2.503752707470477	2.503752707470476	4.047174E-16	2.218906E-08
0.8	2.865540928492468	2.865540928492468	7.759914E-16	4.288554E-08
0.9	3.269603111156951	3.26960311115695	1.375534E-15	7.645583E-08
1.0	3.718281828459047	3.718281828459045	2.290820E-15	1.278750E-07

Problem 3:

Fifth-order IVP

$$y^{v} = 5y''' + 4y',\ y(0) = 3,\ y'(0) = -5,\ y''(0) = 11,\ y'''(0) = -23,\ y^{iv}(0) = 47$$

with the exact solution $y(x) = 1 - e^{x} + 3e^{-2x}$ is also considered. The solution is obatined within the interval $[0, 1]$ over ten (10) iterations. The results are as shown in Table 4. Figures 3 and 5c respectively, show the comparison of the efficiency curves and the y-exact and y-computed of problem 1.

Table 4: Results of problem 3 (with h=0.1)

h	$y-computed$	$y-exact$	Errors in OHMFOP	Errors in [7]
0.1	3.559037356404862	3.559037356404862	3.286757E-16	2.664535E-15
0.2	4.254071334763648	4.254071334763641	6.812018E-15	1.896261E-13
0.3	5.116497593595571	5.116497593595524	4.694243E-14	2.172484E-12
0.4	6.184798087836328	6.184798087836134	1.954954E-13	1.211808E-11
0.5	7.506124214677616	7.506124214677007	6.082082E-13	4.576428E-11
0.6	9.1382319678207	9.138231967819133	1.566104E-12	1.352662E-10
0.7	11.15184719306708	11.15184719306354	3.534093E-12	3.379645E-10
0.8	13.63355634470011	13.63355634469287	7.238462E-12	7.473215E-10
0.9	16.68933928209566	16.68933928208189	1.376915E-11	1.506349E-09
1	20.44888646835762	20.44888646833290	2.471432E-11	2.824062E-09

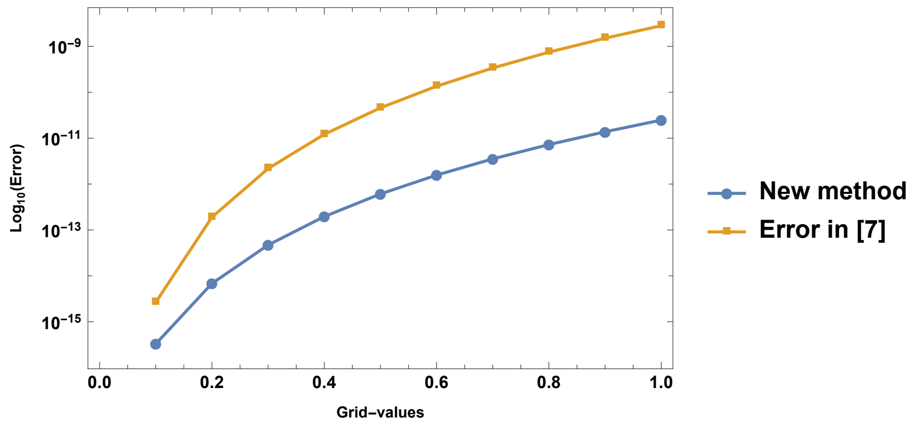


Figure 3: Efficiency curves for problem 3

Problem 4:

Finally, nonlinear general fifth-order IVP of ordinary differential equation

$$y^{v} = 6(2(y')^3) + 6yy'y''' + y^2y''', 1 \leq x \leq 2,$$

$$y(1) = 1,\ y'(1) = -1,\ y''(1) = 2,\ y'''(1) = -6,\ y^{iv}(1) = 24$$

with the exact solution $y(x) = \frac{1}{x}$ is also considered as a test problem. The solution was obtained within the interval $[1, 2]$ over ten (10) iterations. The results are reported in Table 5. Figures 4 and 5d respectively, show the comparison of the efficiency curves and the y-exact and y-computed of problem 1. The results in Tables 2-5 and Figures 1-4show that the proposed method has lower absolute errors than $[1, 2]$ and [7], implying that it is useful and accurate for addressing the problem of type (1) directly.

Table 5: Results of problem 4 (with h=0.1)

h	$y - computed$	$y - exact$	Absolute Errors	Errors in [2]
1.1	0.909090909089724	0.909090909090909	1.184186E-12	6.147755E-10
1.2	0.833333333310120	0.833333333333333	2.321285E-11	1.711526E-08
1.3	0.769230769088278	0.769230769230769	1.424907E-10	1.186268E-07
1.4	0.714285713763587	0.714285714285714	5.221266E-10	4.597295E-07
1.5	0.666666665240363	0.666666666666666	1.426303E-09	1.299344E-06
1.6	0.624999996771376	0.625	3.228624E-09	3.013204E-06
1.7	0.588235287699416	0.588235294117647	6.418231E-09	6.103066E-06
1.8	0.555555543950673	0.555555555555555	1.160488E-08	1.120406E-05
1.9	0.526315769950266	0.526315789473684	1.952342E-08	1.909085E-05
2.0	0.49999996896222	0.5	3.103778E-08	3.068296E-05

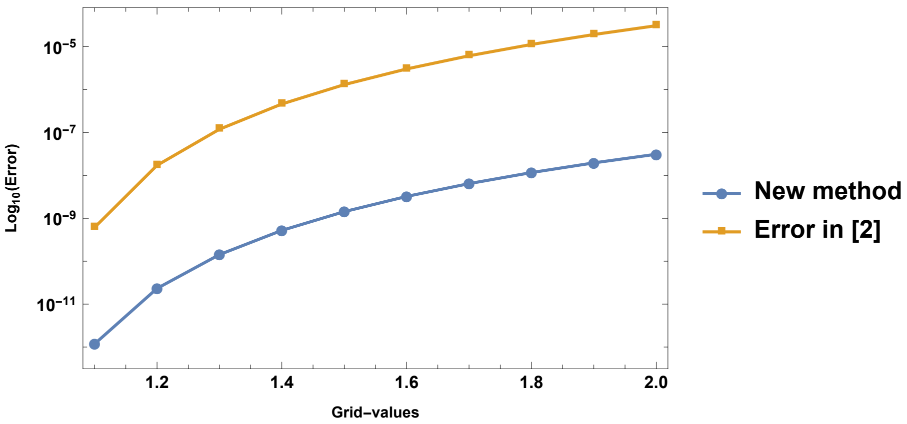
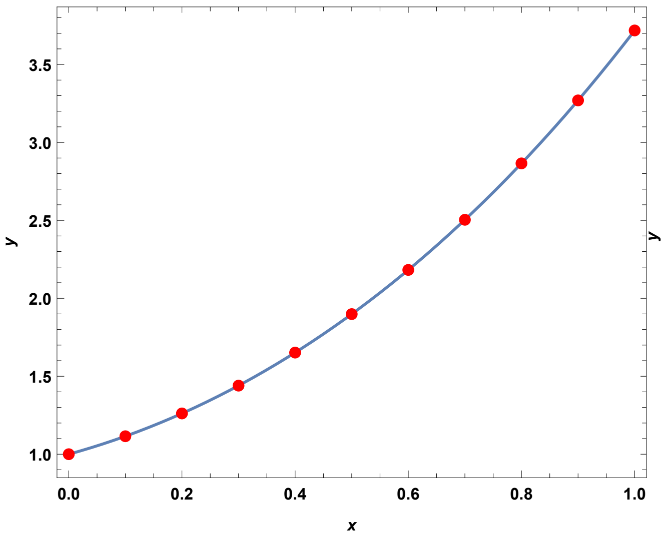
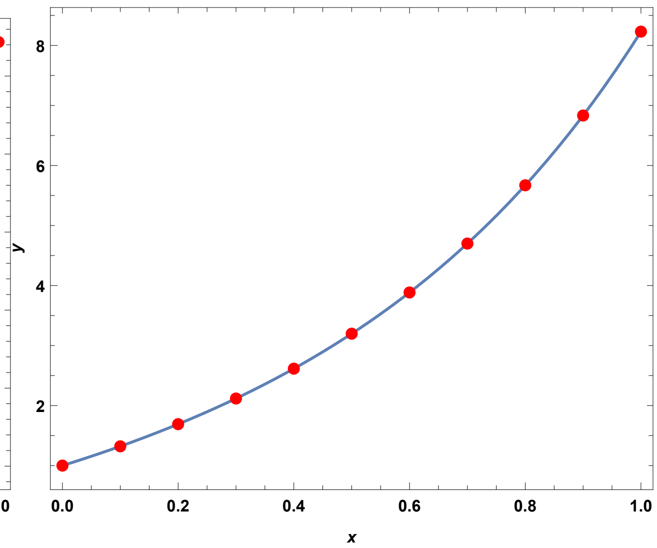


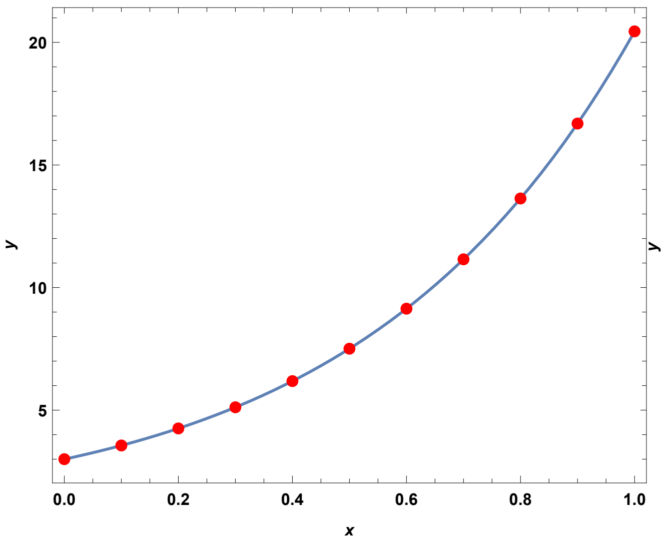
Figure 4: Efficiency curves for problem 4



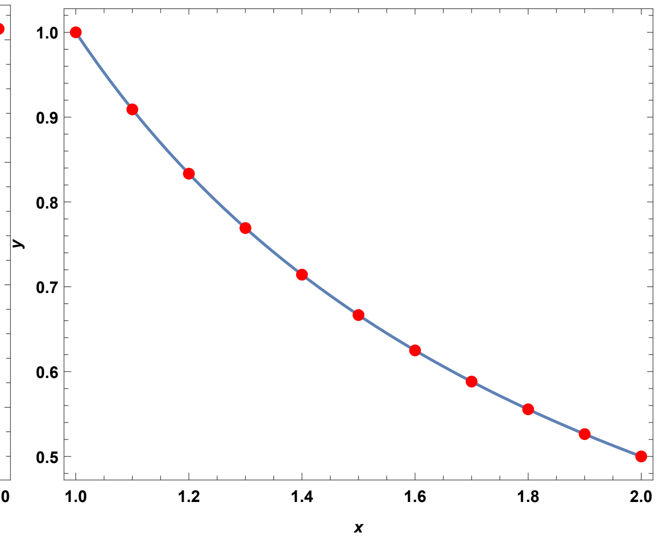
(a) Comparison of y-computed and y-exact for problem 1



(b) Comparison of y-computed and y-exact for problem 2



(c) Comparison of y-computed and y-exact for problem 3



(d) Comparison of y-computed and y-exact for problem 4

Figure 5: Solution of problems 1, 2, 3 and 4

6 Conclusion

A one-step hybrid block method is carefully designed in this work without using the block formular for solving fifth-order initial value problems of ordinary differential equations. The method obtained using the approach in the paper has been shown to be accurate, stable, consistent, and convergent. The method is easy to derive and requires less expertise compared to using a block formula. The results of the numerical experiments attest to the good performance of the proposed method. Implementation of the present method in a variable step-size mode is an area that can be exploited.

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