Original Research Article
S-INDEX OF DIFFERENT GRAPH OPERATIONS

.

ABSTRACT

In this paper, we introduced new index from the Zagreb index family, named as S-index is defined as the sum of degree five of all the vertices of a graph. We derive the S-index of some different graph operations such that Join, Cartesian product, Composition, Corona product, Tensor Product, Strong product, Disjunction, Symmetric difference, Corona join product, Subdivision vertex join are obtained.

Keywords: Zagreb Indices, F-index, S-index, Graph operations

1. INTRODUCTION

Topological indices are a numeric value which is associated with a chemical compound. Graph operations play a important role in chemical mathematics, since some chemically interesting graphs can be derived from some simpler graphs by different graph operations.

Let $G = \{V(G), E(G)\}$ and $G = \{V(H), E(H)\}$ be two connected graphs with vertex sets and edge sets are respectively. The number of vertices and edges of G and H will be denoted by k and j such that $|V(G)| = k_1$, $|V(H)| = k_2$ and $|E(G)| = j_1$, $|E(H)| = j_2$ respectively. The degree of a vertex v is the number of edges incident to v and is denoted by $\gamma_G(v)$.

In 2005, Li and Zheng [12] introduced the first general Zagreb index defined as:

$$M_1^{\alpha+1}(G) = \sum_{v\in V(G)}[\gamma_G^{\alpha+1}(v)] = \sum_{uv\in E(G)}[\gamma_G^{\alpha}(u) + \gamma_G^{\alpha}(v)]$$

In 1972, I. Gutman and N. Trinajstic [13] introduced the first and second Zagreb indices of a graph is defined as:

$$M_1(G) = \sum_{v\in V(G)}[\gamma_G(v)^2] = \sum_{uv\in E(G)}[\gamma_G(u) + \gamma_G(v)]$$

$$M_2(G) = \sum_{uv\in E(G)}[\gamma_G(u)\gamma_G(v)]$$

In 2015, Furtula and Gutman [8] introduced the forgotten index (F-index) is defined as:

$$F(G) = \sum_{v\in V(G)}[\gamma_G(v)^3] = \sum_{uv\in E(G)}[\gamma_G(u)^2 + \gamma_G(v)^2]$$

In 2020, Abdu Alameri and Noman Al-Naggar [7] introduced the Y-index is defined as:

$$Y(G) = \sum_{v \in V(G)} [\gamma_G(v)^4] = \sum_{uv \in E(G)} [\gamma_G(u)^3 + \gamma_G(v)^3]$$

We introduced the new index from the first general Zagreb index [12] ,named as S-index where $\alpha = 4$.

$$S(G) = \sum_{v \in V(G)} [\gamma_G(v)^5] = \sum_{uv \in E(G)} [\gamma_G(u)^4 + \gamma_G(v)^4]$$

Investigators need to study more details on calculating topological indices of some graph operations can be refer [1,3,5,7,9,11,14].

2. MAIN RESULTS

In this section, we study about the S-index of some different graph operations.

Example: 2.1

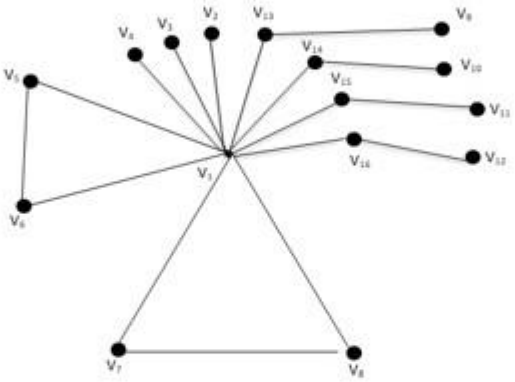

Fig: Firefly Graph F$_{2,3,4}$

The value of S-index for above diagram:

$$S(G) = \sum_{v \in V(G)} [\gamma_G(v)^5] = 1{,}61{,}314$$

Corollary: 2.2

In this part, the S-index of some special graphs are calculated.

1. For a complete graph K_n, with n vertices:

$$S(K_n) = n(n-1)^5 \ , n \geq 3$$

2. For a cycle graph C_n, with n vertices:

$$S(C_n) = 32n\,, n \geq 3$$

3. For a path graph P_n, with n vertices:

$$S(P_n) = 32n - 62\,, n \geq 3$$

4. For a star graph S_n, with n vertices:

$$S(S_n) = (n-1)^5 + (n-1)\,, n \geq 3$$

5. For a wheel graph W_n, with n vertices:

$$S(W_n) = 243n + n^5\,, n \geq 3$$

6. For a ladder graph L_n, with n vertices:

$$S(L_n) = 128 + (2n-4)243\,, n \geq 2$$

The Join of graphs: [2,4]

Definition: 2.3

The **join** $G + H$ of graphs G and H with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G \cup H$ together with all the edges between $V(G)$ and $V(H)$. Obviously, $|V(G+H)| = |V(G)| + |V(H)| = k_1 + k_2$ and $|E(G+H)| = |E(G)| + |E(H)| + |V(G)||V(H)| = j_1 + j_2 + k_1 k_2$.

$$\gamma_{G+H}(v) = \begin{Bmatrix} \gamma_G(v) + k_2, & v \in V(G) \\ \gamma_H(v) + k_1, & v \in V(H) \end{Bmatrix}$$

Theorem: 2.4

The S-index of $G + H$ is given by

$$S(G+H) = S(G) + S(H) + j_1(2k_2^4 + k_1^4) + Y(G)k_2 + k_2^5 k_1 + 4[F(G)k_2^2 + F(H)k_1^2] + 6[M_1(G)k_2^3 + M_1(H)k_1^3] + 8[k_2^4 j_1 + k_1^4 j_2] + Y(H)k_1 + k_1^5 k_2.$$

Proof:

From S-index, we have

$$S(G+H) = \sum_{uv \in E(G+H)} [\gamma_{G+H}(u)^4 + \gamma_{G+H}(v)^4]$$

$$= \sum_{uv \in E(G)} [\gamma_{G+H}(u)^4 + \gamma_{G+H}(v)^4] + \sum_{uv \in E(H)} [\gamma_{G+H}(u)^4 + \gamma_{G+H}(v)^4] + \sum_{u \in V(G)} \sum_{v \in V(H)} [\gamma_{G+H}(u)^4 + \gamma_{G+H}(v)^4]$$

$$= \sum_{uv\in E(G)}[\gamma_G(u)^4 + \gamma_H(v)^4 + 2k_2^4] + \sum_{uv\in E(H)}[\gamma_G(u)^4 + \gamma_H(v)^4 + 2k_1^4] + \sum_{u\in V(G)}\sum_{v\in V(H)}[(\gamma_G(u)+k_2)^4 + (\gamma_H(v)+k_1)^4]$$

$$S(G+H) = S(G) + S(H) + j_1(2k_2^4 + k_1^4) + Y(G)k_2 + k_2^5k_1 + 4[F(G)k_2^2 + F(H)k_1^2] + 6[M_1(G)k_2^3 + M_1(H)k_1^3] + 8[k_2^4 j_1 + k_1^4 j_2] + Y(H)k_1 + k_1^5k_2.$$

Which is complete the proof.

The Cartesian product of graphs: [1,4,14]

Definition: 2.5

The **Cartesian product** $G \times H$ of graphs G and H has the vertex set $V(G\times H) = V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G\times H$ if $uv \in E(G)$ and $x = y$, or $u = v$ and $xy \in E(H)$. Obviously, $|V(G\times H)| = |V(G)||V(H)| = k_1k_2$ and $|E(G\times H)| = |E(G)||V(H)| + |E(H)||V(G)| = j_1k_2 + j_2k_1$.

$$\gamma_{G\times H}(a,b) = \gamma_G(a) + \gamma_G(b)$$

Theorem: 2.6

The S-index of $G\times H$ is given by

$$S(G\times H) = k_2S(G) + k_1S(H) + 10Y(G)j_2 + 10F(G)M_1(H) + 10F(H)M_1(G) + 10j_1Y(H).$$

Proof:

From S-index, we have

$$S(G\times H) = \sum_{(u,v)\in V(G\times H)}[\gamma_{G\times H}(u,v)^5]$$

$$= \sum_{u\in V(G)}\sum_{v\in V(H)}[\gamma_G(u) + \gamma_H(v)]^5$$

$$S(G\times H) = k_2S(G) + k_1S(H) + 10Y(G)j_2 + 10F(G)M_1(H) + 10F(H)M_1(G) + 10j_1Y(H).$$

Which is complete the proof.

The Composition of Graphs: [1,4,9]

Definition: 2.7

The **Composition** $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph with vertex set $V(G)\times V(H)$ and $u = (u_1, v_1)$ is adjacent to $v = (u_2, v_2)$ whenever u_1 is adjacent to u_2 or $u_1 = u_2$ and v_1 is adjacent to v_2. $|V(G[H])| = |V(G)||V(H)| = k_1k_2$, $|E(G[H])| = |E(G)||V(H)|^2 + |V(G)||E(H)| = j_1k_2^2 + k_1j_2$.

$$\gamma_{G[H]}(a,b) = k_2\gamma_G(a) + \gamma_H(b)$$

Theorem: 2.8

The S-index of $G[H]$ is given by

$$S(G[H]) = k_2^6 S(G) + k_1 S(H) + 10k_2^3 F(G)M_1(H) + 10k_2^2 F(H)M_1(G) + 10j_2 k_2^4 Y(G).$$

Proof:

From S-index, we have

$$S(G[H]) = \sum_{(u,v)\in V(G[H])}\left[\gamma_{G[H]}(u,v)^5\right]$$

$$= \sum_{u\in V(G)}\sum_{v\in V(H)}[k_2\gamma_G(u) + \gamma_H(v)]^5$$

$$S(G[H]) = k_2^6 S(G) + k_1 S(H) + 10k_2^3 F(G)M_1(H) + 10k_2^2 F(H)M_1(G) + 10j_2 k_2^4 Y(G).$$

Which is complete the proof.

The Tensor product of graphs: [4,7,14]

Definition: 2.9

The **Tensor product** $G\otimes H$ of graphs G and H has the vertex set $V(G\otimes H) = V(G) \times V(H)$ and $(u,x)(v,y)$ is an edge of $G \oplus H$ if $uv \in E(G)$ and $xy \in E(H)$. Obviously, $|V(G\otimes H)| = |V(G)||V(H)| = k_1k_2$, $|E(G\otimes H)| = 2|E(G)||E(H)| = 2j_1j_2$ and

$$\gamma_{G\otimes H}(u,x) = \gamma_G(u)\gamma_H(x).$$

Theorem: 2.10

The S-index of $G\otimes H$ is given by

$$S(G\otimes H) = S(G)S(H).$$

Proof:

From S-index, we have

$$S(G\otimes H) = \sum_{(u,v)\in V(G\otimes H)}[\gamma_{G\otimes H}(u,v)^5]$$

$$= \sum_{u\in V(G)}\sum_{v\in V(H)}\left[\left(\gamma_G(u)\gamma_H(v)\right)^5\right]$$

$$S(G\otimes H) = S(G)S(H).$$

Which is complete the proof.

The Disjunction of graphs: [1,3,4]

Definition: 2.11

The **Disjunction** $G \vee H$ of a graphs G and H is the graph with vertex set $V(G) \times V(H)$ and u_1v_1 is adjacent with u_2v_2 whenever $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$

$|V(G \vee H)| = |V(G)||V(H)| = k_1k_2$, $|E(G \vee H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 2|E(G)||E(H)| = j_1k_2^2 + j_2k_1^2 - 2j_1j_2$.

$$\gamma_{G\vee H}\big((a,b)\big) = k_2\gamma_G(a) + k_1\gamma_H(b) - \gamma_G(a)\gamma_H(b)$$

Theorem: 2.12

The S-index of $G \vee H$ is given by

$$S(G \vee H) = k_2^5 S(G) + k_1^5 S(H) - S(G)S(H) + 10k_2^4 k_1 Y(G) j_2 - 10k_2^4 S(G) j_2 + 10k_2^3 F(G) k_1^2 M_1(H) + 10k_2^3 S(G) M_1(H) + 10k_2^2 F(H) k_1^3 M_1(G) + 10k_2 j_1 k_1^4 Y(H) - 10 j_1 k_1^4 S(H) + 10k_1^3 S(H) M_1(G) - 10k_2^2 S(G) F(H) - 10k_1^2 S(H) F(G) - 20k_2^3 Y(G) k_1 M_1(H) - 20k_1^3 Y(H) k_2 M_1(G) - 20k_2 k_1 Y(G) Y(H) + 5k_2 S(G) Y(H) + 5k_1 Y(G) S(H) + 30k_2^2 k_1 Y(G) F(H) + 30k_1^2 k_2 F(G) Y(H) - 30k_1^2 k_2^2 F(G) F(H).$$

Proof:

From S-index, we have

$$S(G \vee H) = \sum_{(u_1,u_2) \in V(G \vee H)} [\gamma_{G \vee H}(u_1,u_2)^5]$$

$$= \sum_{u_1 \in V(G)} \sum_{u_2 \in V(H)} [[k_2 \gamma_G(u_1) + k_1 \gamma_H(u_2) - \gamma_G(u_1)\gamma_H(u_2)]^5]$$

$$S(G \vee H) = k_2^5 S(G) + k_1^5 S(H) - S(G)S(H) + 10k_2^4 k_1 Y(G) j_2 - 10k_2^4 S(G) j_2 + 10k_2^3 F(G) k_1^2 M_1(H) + 10k_2^3 S(G) M_1(H) + 10k_2^2 F(H) k_1^3 M_1(G) + 10k_2 j_1 k_1^4 Y(H) - 10 j_1 k_1^4 S(H) + 10k_1^3 S(H) M_1(G) - 10k_2^2 S(G) F(H) - 10k_1^2 S(H) F(G) - 20k_2^3 Y(G) k_1 M_1(H) - 20k_1^3 Y(H) k_2 M_1(G) - 20k_2 k_1 Y(G) Y(H) + 5k_2 S(G) Y(H) + 5k_1 Y(G) S(H) + 30k_2^2 k_1 Y(G) F(H) + 30k_1^2 k_2 F(G) Y(H) - 30k_1^2 k_2^2 F(G) F(H).$$

Which is complete the proof.

The Symmetric Difference of graphs: [1,3,4]

Definition: 2.13

The **Symmetric Difference** $G \oplus H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ and

$E(G \oplus H) = \{(u_1,u_2)(v_1,v_2) / u_1v_1 \in E(G) \; or \; u_2v_2 \in E(H) \; but \; not \; both\}$

$|V(G \oplus H)| = |V(G)||V(H)| = k_1k_2$, $|E(G \oplus H)| = |E(G)||V(H)|^2 + |E(H)||V(G)|^2 - 4|E(G)||E(H)| = j_1k_2^2 + j_2k_1^2 - 4j_1j_2$.

$$\gamma_{G \oplus H}\big((a,b)\big) = k_2\gamma_G(a) + k_1\gamma_H(b) - 2\gamma_G(a)\gamma_H(b)$$

Theorem: 2.14

The S-index of $G \oplus H$ is given by

$$S(G \oplus H) = k_2^5 S(G) + k_1^5 S(H) - 32S(G)S(H) + 10k_2^4 k_1 Y(G) j_2 - 20k_2^4 S(G) j_2 + 10k_2^3 F(G) k_1^2 M_1(H) + 40k_2^3 S(G) M_1(H) + 10k_2^2 F(H) k_1^3 M_1(G) + 10k_2 j_1 k_1^4 Y(H) - 20 j_1 k_1^4 S(H) + 40k_1^3 S(H) M_1(G) - 80k_2^2 S(G) F(H) - 80k_1^2 S(H) F(G) -$$

$40k_2^3Y(G)k_1M_1(H) - 40k_1^3Y(H)k_2M_1(G) - 160k_2k_1Y(G)Y(H) + 80k_2S(G)Y(H) +$
$80k_1Y(G)S(H) + 120k_2^2k_1Y(G)F(H) + 120k_1^2k_2F(G)Y(H) - 60k_1^2k_2^2F(G)F(H).$

Proof:

From S-index, we have

$$S(G \oplus H) = \sum_{(u_1,u_2)\in V(G \oplus H)}[\gamma_{G \oplus H}(u_1,u_2)^5]$$
$$= \sum_{u_1\in V(G)}\sum_{u_2\in V(H)}[[k_2\gamma_G(u_1) + k_1\gamma_H(u_2) - 2\gamma_G(u_1)\gamma_H(u_2)]^5]$$

$S(G \oplus H) = k_2^5S(G) + k_1^5S(H) - 32S(G)S(H) + 10k_2^4k_1Y(G)j_2 - 20k_2^4S(G)j_2 +$
$10k_2^3F(G)k_1^2M_1(H) + 40k_2^3S(G)M_1(H) + 10k_2^2F(H)k_1^3M_1(G) + 10k_2j_1k_1^4Y(H) -$
$20j_1k_1^4S(H) + 40k_1^3S(H)M_1(G) - 80k_2^2S(G)F(H) - 80k_1^2S(H)F(G) -$
$40k_2^3Y(G)k_1M_1(H) - 40k_1^3Y(H)k_2M_1(G) - 160k_2k_1Y(G)Y(H) + 80k_2S(G)Y(H) +$
$80k_1Y(G)S(H) + 120k_2^2k_1Y(G)F(H) + 120k_1^2k_2F(G)Y(H) - 60k_1^2k_2^2F(G)F(H).$

Which is complete the proof.

The Strong product of graphs: [1,3,4]

Definition: 2.15

The **Strong product** $G * H$ of a graphs G and H is a graph with vertex set $V(G) \times V(H)$ and any two vertices (u_p, v_r) and (u_q, v_s) are adjacent if and only if $[u_p = u_q \ and \ v_rv_s \in E(H)]$ or $[v_r = v_s \ and \ u_pu_q \in E(G)]$ or $[u_pu_q \in E(G) \ and \ v_rv_s \in E(H)]$. $|V(G * H)| = |V(G)||V(H)| = k_1k_2$, $|E(G * H)| = |E(G)||V(H)| + |V(G)||E(H)| + 2|E(G)||E(H)| = j_1k_2 + k_1j_2 + 2j_1j_2$.

$$\gamma_{G*H}((a,b)) = \gamma_G(a) + \gamma_H(b) + \gamma_G(a)\gamma_H(b)$$

Theorem: 2.16

The S-index of $G * H$ is given by

$S(G * H) = S(G)k_2 + S(H)k_1 + S(G)S(H) + 10Y(G)j_2 + 10S(G)j_2 + 10Y(H)j_1 +$
$10S(H)j_1 + 5S(G)Y(H) + 5Y(G)S(H) + 10F(G)M_1(H) + 20Y(G)M_1(H) +$
$10S(G)M_1(H) + 10F(H)M_1(G) + 20Y(H)M_1(G) + 10S(H)M_1(G) + 10S(G)F(H) +$
$20Y(G)Y(H) + 10F(G)S(H) + 30F(G)F(H) + 30Y(G)M_1(H) + 30F(G)Y(H).$

Proof:

From S-index, we have

$$S(G * H) = \sum_{(u,v)\in V(G*H)}[\gamma_{G*H}(u,v)^5]$$
$$= \sum_{u\in V(G)}\sum_{v\in V(H)}[\gamma_G(u) + \gamma_H(v) + \gamma_G(u)\gamma_H(v)]^5$$

$S(G*H) = S(G)k_2 + S(H)k_1 + S(G)S(H) + 10Y(G)j_2 + 10S(G)j_2 + 10Y(H)j_1 + 10S(H)j_1 + 5S(G)Y(H) + 5Y(G)S(H) + 10F(G)M_1(H) + 20Y(G)M_1(H) + 10S(G)M_1(H) + 10F(H)M_1(G) + 20Y(H)M_1(G) + 10S(H)M_1(G) + 10S(G)F(H) + 20Y(G)Y(H) + 10F(G)S(H) + 30F(G)F(H) + 30Y(G)M_1(H) + 30F(G)Y(H).$

Which is complete the proof.

The Corona product of graphs: [4,14]

Definition: 2.17

The **Corona product** $G\odot H$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph obtained by one copy of G and k_1 copies of H and joining the i^{th} vertex of G to every vertex in i^{th} copy of H.

Obviously, $|V(G\odot H)| = |V(G)| + |V(G)||V(H)| = k_1 + k_1k_2$, $|E(G\odot H)| = |E(G)| + |V(G)||E(H)| + |V(G)||V(H)| = j_1 + k_1j_2 + k_1k_2$.

$$\gamma_{G\odot H}(v) = \begin{Bmatrix} \gamma_G(v) + k_2, & v \in V(G) \\ \gamma_H(v) + 1, & v \in V(H) \end{Bmatrix}$$

Theorem: 2.18

The S-index of $G\odot H$ is given by

$S(G\odot H) = S(G) + k_1S(H) + 5k_2Y(G) + 5k_1Y(H) + 10k_2^2F(G) + 10k_2^3M_1(G) + 10k_2^4j_1 + k_2^4k_1 + 10k_1F(H) + 10k_1M_1(H) + 10k_1j_2 + k_1k_2$.

Proof:

From S-index, we have

$$S(G\odot H) = \textstyle\sum_{v\in V(G\odot H)}[\gamma_{G\odot H}(v)^5]$$

$$= \textstyle\sum_{v\in V(G)}[(\gamma_G(v) + k_2)^5] + \sum_{v\in V(G)}\sum_{v\in V(H)}[(\gamma_H(v) + 1)^5]$$

$S(G\odot H) = S(G) + k_1S(H) + 5k_2Y(G) + 5k_1Y(H) + 10k_2^2F(G) + 10k_2^3M_1(G) + 10k_2^4j_1 + k_2^4k_1 + 10k_1F(H) + 10k_1M_1(H) + 10k_1j_2 + k_1k_2.$

Which is complete the proof.

Corona join product: [11]

Definition: 2.19

Let $G(k_1, j_1)$ and $H(k_2, j_2)$ be simple connected graphs, and the Corona join graph of G and H is obtained by taking one copy of G, k_1 copies of H , and joining each vertex of the i^{th} copy of H with all vertices of G. The **Corona join product** of G and H is denoted by

$$\gamma_{G \oplus H}(v) = \begin{Bmatrix} \gamma_G(v) + k_1 k_2\,, & if\ v \in V(G) \\ \gamma_H(v) + k_1\,, & if\ v \in V(H) \end{Bmatrix}$$

Theorem: 2.20

The S-index of $G \oplus H$ is given by

$$S(G \oplus H) = S(G) + 5Y(G)k_1k_2 + 10F(G)k_1^2k_2^2 + 10M_1(G)k_1^3k_2^3 + 10j_1k_1^4k_2^4 + k_1^6k_2^5 + k_1S(H) + 5Y(H)k_1^2 + 10F(H)k_1^3 + 10M_1(H)k_1^4 + 10j_2k_1^5 + k_1^6k_2.$$

Proof:

From S-index, we have

$$S(G \oplus H) = \sum_{v \in V(G \oplus H)} [\gamma_{G \oplus H}(v)^5]$$

$$= \sum_{v \in V(G)} [(\gamma_G(v) + k_1k_2)^5] + \sum_{v \in V(G)} \sum_{v \in V(H)} [(\gamma_H(v) + k_1)^5]$$

$$S(G \oplus H) = S(G) + 5Y(G)k_1k_2 + 10F(G)k_1^2k_2^2 + 10M_1(G)k_1^3k_2^3 + 10j_1k_1^4k_2^4 + k_1^6k_2^5 + k_1S(H) + 5Y(H)k_1^2 + 10F(H)k_1^3 + 10M_1(H)k_1^4 + 10j_2k_1^5 + k_1^6k_2.$$

Which is complete the proof.

Subdivision vertex join: [11]

Definition: 2.21

For $G(k_1, j_1)$ and $H(k_2, j_2)$, the **subdivision vertex join** is denoted by $G + H$ and is obtained by joining the each new vertex of $S(G)$ to all vertices of H.

$$\gamma_{G+H}(v) = \begin{Bmatrix} \gamma_G(v), v \in V(G) \\ 2 + k_2, v \in V_s(G) \\ \gamma_H(v) + j_1, v \in V(H) \end{Bmatrix}$$

Theorem: 2.22

The S-index of $G + H$ is given by

$$S(G + H) = S(G) + (2 + k_2)^5 j_1 + S(H) + j_1^5 k_2 + 5Y(H)j_1 + 10F(H)j_1^2 + 10M_1(H)j_1^3 + 10j_1^4 j_2.$$

Proof:

From S-index, we have

$$S(G + H) = \sum_{v \in V(G+H)} [\gamma_{G+H}(v)^5]$$

$$= \sum_{v \in V(G)} [\gamma_G(v)^5] + \sum_{v \in V_s(G)} [(2 + k_2)^5] + \sum_{v \in V(H)} [(\gamma_H(v) + j_1)^5]$$

$$S(G + H) =$$

$$S(G) + (2 + k_2)^5 j_1 + S(H) + j_1^5 k_2 + 5Y(H)j_1 + 10F(H)j_1^2 + 10M_1(H)j_1^3 + 10j_1^4 j_2.$$

Which is complete the proof.

3. CONCLUSION

In this paper, we compute some exact expressions for the S-index of some graph operations such as Join, Cartesian product, Composition, Corona product, Tensor Product, Strong product, Disjunction, Symmetric difference, Corona join product, Subdivision vertex join can be computed.

REFERENCES

1. Khalifeha, M.H.; Yousefi-Azaria, H.; Ashrafi, A.R. The first and second Zagreb indices of some graph operations. Discret. Appl. Math. 2009, 157, 804–811.
2. Das, K.C.; Yurttas, A.; Togan, M.; Cevik, A.S.; Cangul, I.N. The multiplicative Zagreb indices of graph operations. J. Ineql. Appl. 2013, 2013, doi:10.1186/1029-242X-2013-90.
3. De, N.; Nayeem, S.M.A.; Pal, A. F-index of some graph operations. Personal communication, 2015.
4. Mohammed S. Alsharafi. The first and Second Zagreb Index of Complement Graph and Its Applications of Molecular Graph. Asian Jour. Prob. And Stat. 2020, 8(3),15-30.
5. Khalifeh, M.H.; Yousefi-Azari, H.; Ashrafi, A.R. The hyper-Wiener index of graph operations. Comput. Math. Appl. 2008, 56, 1402–1407.
6. Veylaki, M.; Nikmehr, M.J.; Tavallaee, H.A. The third and hyper-Zagreb co indices of some graph operations. J. Appl. Math. Comput. 2015, doi:10.1007/s12190-015-0872-z.
7. A. Alameri, N. Al-Naggar, M. Al-Rumaima, M. Alsharafi, Y-index of some graph operations, Int. J. Appl. Eng. Res. 15 (2)(2020), 173-179.
8. Furtula B, Gutman I. A forgotten topological index. J. Math. Chem. 2015;53(4):1184-1190.
9. Nilanjan D. Reformulated first Zagreb index of some graph operations. Mathematics 2015,3,945-960.doi:10.3390/math.3040945.
10. De N., Nayeem S.M.A., Pal A. The F-co index of some graph operations. SpringerPlus 2016, 5 (221). doi:10.1186/s40064-016-1864-7.
11. M. Bilal., M. Waheed., Three Topological Indices of Two New Varients of Graph Products. Hindawi 2021. doi:10.1155/2021/7724177.

12. Li X, Zheng J. A unified approach to the extremal trees for different indices. MATCH Commun Math Comput Chem. 2005;54:195–208
13. Gutman I., Trinajstic N. Graph theory and molecular orbitals. Total ϕ-electron energy of alternant hydrocarbons. Chemical Physics Letters 1972, 17 (4), 535–538. doi:10.1016/0009-2614(72)85099-1.
14. Hamideh A., Nasrin D. Reformulated F-index of Graph operations. Commun. Comb. Optim. 2017, 87-98. doi:10.22049/CCO.2017.13630.
15. Das, K.C.; Xu, K.; Nam, J. On Zagreb indices of graphs. Front. Math. China 2015, 10, 567-582.