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SPECTRAL RADIUS OF A NORMAL OPERATOR

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Abstract

The spectral radius $r(T)$ or a square matrix of a bounded linear operator T is the largest absolute value of its eigenvalues (that is supremum among the absolute values of the elements in its spectrum). The spectral radius is a sort of infimum of all norms of a matrix. Indeed, on the one hand $r(T) \leq \|T\|$ for every natural operator norm; and on the other hand, the spectral radius formula states that $r(T) = lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$. The spectral mapping theorem implies that $r(T^n) = (r(T))^n$ for every positive integer n. It frequently turns out that it is easy to compute the spectral radius of an operator even if it is hard to find the spectrum. This is often made easy by the spectral radius formula. Let H be a Hilbert space and T be a bounded linear operator in H. In this paper we show that if T is normal, then T^n is normal for each $n \in \mathbb{N}$ and $\|T^n\| = \|T\|^n$. Consequently, we use the spectral radius formula to show that $r(T) = \|T\|$. Moreover, we show that if X is a Complex Banach space and T is bounded in X then there is a λ belonging to the spectrum of T such that $|\lambda| = r(T)$. Let H be a Complex Hilbert space and T be a bounded operator in H which is normal; we show that $\|T\| = sup\,\{|Tx, x| : x \in H\, and\, \|x\| = 1\}$ and the residual spectrum of T is void.

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1 Introduction

Important results in spectral theory can be found by studying the power series and Laurent developments ([9] and [6]) of the resolvent $(\lambda I - T)^{-1}$. The best theorems relate to the Neumann expansion ([8] and [10]) of the resolvent

$$(\lambda I - T)^{-1} = R(\lambda) = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}$$

which is valid whenever the series is convergent in the uniform topology. By extension of a theorem of a classical function theory [8], about the radius of convergence of a power series, we have convergence of the Neumann series if $|\lambda| > r(T)$ and divergence if $|\lambda| < r(T)$, where

$$r(T) = lim_n sup \|T^n\|^{\frac{1}{n}}.$$

The number $r(T)$ is called the spectral radius of T and this equation was first proved by Gelfand[8] in relation to Banach Algebra. Krein and Rutmann [8] further obtained the result that if T is a compact and positive cone which is total, and if the spectral radius of T is positive, then $r(T)$ is a pole of the resolvent.
In this paper we concern ourselves with the spectral radius of a normal operator. We first prove the spectral radius formula, we then show that if T is normal then T^n is normal for each $n \in \mathbb{N}$ and $\|T^n\| = \|T\|^n$. Consequently, we use the spectral radius formula to show that $r(T) = \|T\|$. Moreover, we show that if $T \in B(X)$ where X is a Complex Banach space, then there is a $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$. Let $T \in B(H)$ be normal, where H is a Complex Hilbert space; we show that $\|T\| = sup\{|Tx, x| : x \in H \, and \, \|x\| = 1\}$ and the residual spectrum of T is void. Most definitions in this paper can be found in ([8], [9], [2] and [1])

1.1 Some results on spectra

Proposition 1. *Let X be a Complex Banach space and $T \in B(X)$ if $S \in B(X)$ is invertible then $\sigma(T) = \sigma(S^{-1}TS)$.*

Proof. We need to prove that $\rho(T) = \rho(S^{-1}TS)$. Let $\lambda \in \rho(T)$. Hence $(\lambda I - T)^{-1}$ exists and is in $B(X)$. Now,
$S^{-1}(\lambda I - T)S = S^{-1}(\lambda S - TS) = \lambda I - S^{-1}TS$.
Note that $S^{-1}, T, S \in B(X)$ implies $S^{-1}TS \in B(X)$. Since S is invertible, $S^{-1} \in B(X)$ and $(S^{-1})^{-1} = S \in B(X)$.
The product $S^{-1}(\lambda I - T)S$ being a product of invertible operators is invertible. Hence
$\lambda I - S^{-1}TS$ is invertible

implies $\lambda \in \rho(S^{-1}TS)$.
Thus
$$\rho(T) \subseteq \rho(S^{-1}TS) \tag{1.1}$$
It remains to establish the reverse inclusion which is a consequence of (1.1) itself. For consider the operator $S^{-1}TS \in B(X)$ instead of T. Then by (1.1)

$$\rho(S^{-1}TS) \subseteq \rho((S^{-1})^{-1}(S^{-1}TS)(S^{-1}))$$
$$\textit{that is}\, \rho(S^{-1}TS) \subseteq \rho(T)$$

Thus $\rho(S^{-1}TS) = \rho(T)$. Taking complements we get
$\sigma(S^{-1}TS) = \sigma(T)$. □

Proposition 2. *Let X be a Banach space and p be a polynomial with complex coefficients. Let $T \in B(X)$. Then*
$\sigma(p(T)) = p(\sigma(T))$
where $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$

Elucidation
Suppose $p(\lambda) = a_o + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n$ where $a_o, a_1, a_2, \ldots, a_n \in \mathbb{C}$ and λ is an indeterminate if $T \in B(X)$, then $p(T)$ stands for the bounded linear operator.
$a_oI + a_1T + a_2T^2 + \ldots + a_nT^n$
where I is the identity operator on X.
So $\sigma(p(T))$ is the spectrum of the bounded linear operator $p(T)$.
Whereas $p(\sigma(T))$ is the set $\{p(\lambda) : \lambda \in \sigma(T)\}$ of complex numbers.
This result is a restricted form of the famous result known as the **Spectral Mapping Theorem** [2].
Application
Suppose $p(\lambda) = \lambda^n$ and $T \in B(X)$. Therefore $p(T) = T^n$. $T \in B(X)$ implies $T^n \in B(X)$. By the Spectral Mapping Theorem[2] we get

$$\sigma(T^n) = p(\sigma(T)) = \{\lambda^n : \lambda \in \sigma(T)\}$$
$$= (\sigma(T))^n$$

(where $(\sigma(T))^n$ stands for the set $\{\lambda^n : \lambda \in \sigma(T)\}$)
In the general form of the spectral mapping theorem, the polynomial p may be replaced by a continuous or even measurable function.

Proof of Proposition 2. Let $\lambda_o \in \mathbb{C}$. Consider the equation $p(\lambda) - p(\lambda_o) = 0$ we can write
$p(\lambda) - p(\lambda_o) = (\lambda - \lambda_o)q(\lambda)$ identically in λ. $q(\lambda)$ being a polynomial $(p(\lambda) - p(\lambda_o)$ vanishes identically when $\lambda = \lambda_o$ therefore, λ_o is a root of the polynomial $p(\lambda) - p(\lambda_o)$ therefore $p(\lambda) - p(\lambda_o) = (\lambda - \lambda_o)q(\lambda))$.
Hence
$p(T) - Ip(\lambda_o) = (T - \lambda_oI)q(T)$

where T is the given operator.
If $\lambda_o \in \sigma(T)$, then $(T-\lambda_o I)$ is not invertible. Hence $S = (T-\lambda_o I)q(T)$ is not invertible (for assume the converse, that is, if S is invertible then

$S^{-1}(T-\lambda_o I)q(T) = I = (T-\lambda_o I)q(T)S$

and this would require that $T-\lambda_o I$ should be invertible that is, $\lambda_o \notin \sigma(T)$, a contradiction!).
Hence $p(T)-p(\lambda_o)I$ is not invertible and consequently $p(\lambda_o) \in \sigma(p(T))$.
Since $\lambda_o \in \sigma(T)$ so $p(\lambda_o) \in p\sigma(T)$. Thus

$p(\sigma(T)) \subseteq \sigma(p(T)))$.

We need to establish the reverse inclusion

$\sigma(p(T)) \subseteq p(\sigma(T))$.

Let $\lambda_o \in \sigma(p(T))$ so $p(T)-\lambda_o I$ is not invertible.
Let $\deg p(t) = n$ over $\mathbb{K}$ and let $\lambda_1, \lambda_2, \lambda_3, \cdots, \lambda_n$ be the n zeros of the polynomial $p(t)-\lambda_o$ (These zeros need not be distinct). We can write

$p(t)-\lambda_o = \alpha(\lambda-\lambda_1)(\lambda-\lambda_2)\cdots(\lambda-\lambda_n)$

where α is the non-zero element of $\mathbb{K}$.
Accordingly,

$p(T)-\lambda_o I = \alpha(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_n I)$

For at least one $j, (1 \leq j \leq n)$,$(T-\lambda_j I)$ must be non-invertible (since if $T-\lambda_k I$ was invertible $for\, all\, k = 1, 2, 3, \cdots n$ then the product $\alpha(T-\lambda_1 I)(T-\lambda_2 I)\cdots(T-\lambda_n I))$ would be invertible, that is, $p(T)-\lambda_o I$ would be invertible, a contradiction to the hypothesis:

$(\lambda_o \in \sigma(p(T)))$.

Thus $\lambda_j \in \sigma(T)$. Therefore

$p(\lambda_j) \in p(\sigma(T))$ but $p(\lambda_j) = \lambda_o$ (for λ_j is a root of $p(t)-\lambda_o$).

Thus

$\lambda_o \in p(\sigma(T))$.

Thus

$\sigma(p(T)) \subseteq p(\sigma(T))$

Hence

$\sigma(p(T)) = p(\sigma(T))$. □

Proposition 3. *Let X be a Complex Banach space and $T \in B(X)$ which is invertible. Then $\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$ where*

$$\{\sigma(T)\}^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}$$

Remark 1. {*Since T is invertible, $0 \notin \sigma(T)$ (for $0 \in \sigma(T)$ implies $T-0I$ is not invertible impliesT is not invertible. Therefore, T is invertible implies $0 \notin \sigma(T)$). Hence if $\lambda \in \sigma(T), \lambda \neq 0$ consequently $\frac{1}{\lambda}$ exists $for\, all\, \lambda \in \sigma(T)$ and $\{\sigma(T)\}^{-1}$ is meaningful*}

Proof. Suppose $\lambda \in \sigma(T)$ (so $\lambda \neq 0$)

Now

$$\lambda I - T = (\lambda T^{-1}T - T)$$
$$= \lambda(T^{-1} - \lambda^{-1}I)T$$

Since $\lambda I - T$ is not invertible and T is invertible and $\lambda \neq 0$, so $(T^{-1} - \lambda^{-1}I)$ must be non invertible, consequently, $\lambda^{-1} \in \sigma(T^{-1})$ that is

$\lambda \in \{\sigma(T^{-1})\}^{-1}$.

Thus

$\sigma(T) \subseteq \{\sigma(T^{-1})\}^{-1}$

or

$$\{\sigma(T^{-1})\}^{-1} \subseteq \sigma(T^{-1}) \tag{1.2}$$

Applying the same result to T^{-1} in place of T (Note T^{-1} is invertible and $(T^{-1})^{-1} = T$). We get

$$\{\sigma(T^{-1})\}^{-1} \subseteq \{\sigma((T^{-1})^{-1})\} = \sigma(T) \tag{1.3}$$

(1.2) implies $\sigma(T^{-1}) \supseteq \{\sigma(T)\}^{-1}$.
(1.3) implies $\sigma(T^{-1}) \subseteq \{\sigma(T)\}^{-1}$

that is

$\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$. □

Proposition 4. *Let H be a Complex Hilbert space and $T \in B(H)$. Then*

$$\sigma(T^*) = \overline{\sigma(T)}$$
$$\textit{where}\ \overline{\sigma(T)} = \{\overline{\lambda}) : \lambda \in \sigma(T)\}$$

Proof. Suppose $\lambda \in \rho(T)$. Then $T - \lambda I$ is invertible. Now $(T - \lambda I)^* = T^* - \overline{\lambda} I$ and from the result that $A \in B(H)$ is invertible if and only if A^* is invertible. We note that $T^* - \overline{\lambda} I$ is also invertible. Thus

$\overline{\lambda} \in \rho(T^*)$.

So $\lambda \in \rho(T)$implies $\overline{\lambda} \in \rho(T^*)$.

implies $\lambda \in \overline{\rho(T^*)}$.

$$\rho(T) \subseteq \overline{\rho(T^*)} \tag{1.4}$$

Applying the result to T^* in place of T and noting that $T^{**} = T$ we get that

$$\rho(T^*) \subseteq \overline{\rho(T)}. \tag{1.5}$$

But (1.4) implies

$$\overline{\rho(T)} \subseteq \rho(T^*). \tag{1.6}$$

(1.5) and (1.6) implies $\rho(T^*) = \overline{\rho(T)}$.
Take complements with respect to $\mathbb{C}$ to get the required result. □

1.2 Spectral radius

Definition 1. *Let X be a normed linear space and $T \in B(X)$. The number $sup\{|\lambda| : \lambda \in \sigma(T)\}$ (note $\sigma(T) \neq \phi$) is called the* ***spectral radius*** *of T and represented by $r(T)$*

It follows that $r(T) \leq \|T\|$
Reason:

$$\lambda I - T = \lambda(I - \frac{T}{\lambda})\, if\, \lambda \neq 0 \tag{1.7}$$

Then $\lambda I - T$ is invertible if and only if $I - \frac{T}{\lambda}$ is invertible. {Since X is a Banach space, we know that $I - T$ is invertible if $\|T\| \leq 1$ [9]}.
Looking at (1.7), we conclude that $I - \frac{T}{\lambda}$ is invertible if $\|\frac{T}{\lambda}\| < 1$, that is, if $|\lambda| > \|\|T\|$.
Hence if $\lambda \in \sigma(T)$; then $|\lambda| \leq \|T\|$. Consequently $sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$.
That is,
$r(T) \leq \|T\|$.
The next result proposition 5 is proved for bounded linear operators in the Hilbert space context.

Proposition 5. *Let H be a complex Hilbert space and $T \in B(H)$.*
Then
$lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$ *exists and equals $r(T)$.*

Proof. It follows from the Spectral Mapping Theorem that
$\{r(T)\}^n = r(T^n)\, forall\, n \in \mathbb{N}$.
To see this, consider the polynomial $p(t) = t^n$ and use the Spectral Mapping Theorem,VIZ.
$\sigma(p(T)) = p(\sigma(T))\, for\, all\, T \in B(H)$
that is
$\sigma(T^n) = (\sigma(T))^n$ where $(\sigma(T))^n = \{\lambda^n : \lambda \in \sigma(T)\}$
therefore

$$\begin{aligned} r(T^n) &= sup\{|\mu| : \mu \in \sigma(T^n)\} \\ &= sup\{|\lambda^n| : \lambda \in \sigma(T)\} for\, \lambda(T^n) = (\sigma(T))^n \\ &= sup\{|\lambda|^n : \lambda \in \sigma(T)\} \\ &= sup\{|\lambda| : \lambda \in \sigma(T)\}^n \\ &= (r(T))^n. \end{aligned}$$

Thus (note $r(T) \geq 0$)
$r(T) = \{r(T^n)\}^{\frac{1}{n}}\ for\, all\ n \in \mathbb{N}$.
We have already seen that for any $A \in B(H)$,
$r(A) \leq \|A\|$.
Since $T^n \in B(H)$ so $r(T^n) \leq \|T^n\|$ and hence we get

$r(T) \leq \|T^n\|^{\frac{1}{n}}$
Letting $n \to \infty$ the last line implies that

$$r(T) \leq lim\,inf\|T^n\|^{\frac{1}{n}} \quad (1.8)$$

(We are just considering the sequence of reals $(\|T^n\|^{\frac{1}{n}})_{n=1}^{\infty}$ on the right hand side and hence it is relevant to invoke the limit infimum of this sequence without consideration of the convergence of the latter).
Let $\lambda \neq 0$ and $\frac{1}{\lambda} \in \rho(T)$. Now $\frac{1}{\lambda}I - T = \frac{1}{\lambda}(I - \lambda T)$. If $\|\lambda T\| \leq 1$ then $\lambda I - T$ is invertible and consequently $\frac{1}{\lambda}$ would be in $\rho(T)$ as asserted.
In such a situation the mapping, given any $x, y \in H$,
$\lambda \mapsto \lambda < (I - \lambda T)^{-1}x, y >$
would be analytic for all λ satisfying $\frac{1}{|\lambda|} > r(T)$. But

$$\lambda < (I - \lambda T)^{-1}x, y >= \lambda < \sum_{n=0}^{\infty}(\lambda T)^n x, y >= \lambda \sum_{n=0}^{\infty} < (\lambda T)^n x, y > \quad (1.9)$$

for
$(I - \lambda T)^{-1} = \sum_{n=0}^{\infty}(\lambda T)^n$ (Neumann series)
therefore
$(I - \lambda T)^{-1}x = \sum_{n=0}^{\infty}(\lambda T)^n x$ (strong convergence in H on the right hand side).
The convergence on the right hand side of (1.9) implies that the sequence of scalars
$(< (\lambda T)^n x, y >)_{n=1}^{\infty}$(for any $x, y \in H$ fixed)
must be bounded. Considering the functionals represented by the elements $(\lambda T)^n x \in H$ we note that this sequence of functionals is bounded pointwise and hence as a consequence of the uniform boundedness principle ([9] and [2]) it follows that there exists a real $M > 0$ such that
$|\lambda|^n \|T^n\| \leq M \;\; for\,all \; n \in \mathbb{N}$
for λ satisfying $|\lambda| \leq \frac{1}{r(T)}$.
{**Explanation**
$f_n y =< y, (\lambda T)^n x >\Rightarrow \|f_n\| = (\|(\lambda T)^n x\|)_{n=1}^{\infty}$ is bounded at each $x \in H \;\Rightarrow\; \|(\lambda T)^n\|$ is bounded $\Rightarrow\;(\|\lambda^n T^n\|)$ is bounded $\Rightarrow\;(|\lambda|^n\|T^n\|)$ is bounded}.
Hence $|\lambda|\|T^n\|^{\frac{1}{n}} \leq M^{\frac{1}{n}}\; for\,all\,\lambda$ and $\;forall\,\lambda$ satisfying $|\lambda| \leq \frac{1}{r(T)}$. Let $n \to \infty$, we know that
$M^{\frac{1}{n}} \to 1$(from calculus)
Hence
$|\lambda| lim\,sup\|T^n\|^{\frac{1}{n}}\| \leq 1$
since this is true $\;for\,all\,|\lambda| \leq \frac{1}{r(T)}$ we obtain
$\frac{1}{r(T)} lim\,sup\|T^n\|^{\frac{1}{n}} \leq 1$
that is

$$lim\,sup\|T^n\|^{\frac{1}{n}} \leq r(T) \quad (1.10)$$

Thus
$r(T) \leq lim\,inf\|T^n\|^{\frac{1}{n}} \leq lim\,sup\|T^n\|^{\frac{1}{n}} \leq r(T)$

hence

$lim\,inf\|T^n\|^{\frac{1}{n}} = lim\,sup\|T^n\|^{\frac{1}{n}} = r(T),$

that is

$lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$ exists and equals $r(T)$. □

Definition 2. *Let X be a normed linear space and T be a linear operator in X. A point $\lambda \in \mathbb{K}$(the underlying field) is called an* ***approximate eigenvalue*** *for T if for each real $\epsilon > 0$ there is an $x \in \mathfrak{D}_T$ such that $x \neq 0$ and*

$\|(T - \lambda I)x\| \leq \epsilon\|x\|.$

The collection of all approximate eigenvalues of T is called the **approximate point spectrum** of T and denoted by $\Pi(T)$.

Alternatively, $\lambda \in \mathbb{K}$ is called the approximate eigenvalue of T if there exists a sequence (x_n) of elements of $\mathfrak{D}_T$ such that $\|x_n\| = 1$ *for all* $n \in \mathbb{N}$ and

$\|(T - \lambda I)x_n\| \to 0$ as $n \to \infty$.

It is clear that, if λ is an eigenvalue, then $\lambda \in \Pi(T)$. For in this case there is an $x \in \mathfrak{D}_T$ such that $x \neq \overline{0}$ and $(T - \lambda I)x = \overline{0}$. Hence

$\|(T - \lambda I)x\| = \|\overline{0}\| = 0.$

Thus for any $\epsilon > 0$ that we choose

$\|(T - \lambda I)x\| < \epsilon\|x\|$

this implies

$\lambda \in \Pi(T).$

Thus $P\sigma(T) \subseteq \Pi(T)$.

Proposition 6. *Let X be a Banach space and $T \in B(X)$ then the following statements are equivalent*

(i) $\lambda \in \Pi(T)$

(ii) $\lambda \in P\sigma(T)$ *or (if* $\lambda \notin P\sigma(T)$*) then* $\lambda I - T$ *has an unbounded inverse on the* $\mathfrak{R}_{\lambda I - T}$

Proof. i)⇒ ii). Since $P\sigma(T) \subseteq \Pi(T)$ a $\lambda \in \Pi(T)$ could belong to $P\sigma(T)$. If $\lambda \notin P\sigma(T)$ then $(\lambda I - T)x \neq \overline{0}$ for all nonzero $x \in \mathfrak{D}_T$. Since $\lambda \in \Pi(T)$ it follows that for each $n \in \mathbb{N}$ there is $x_n \in \mathfrak{D}_T$ such that $\|x_n\| = 1$ and $\|(\lambda I - T)x_n\| \leq \frac{1}{n}$ for all $n \in \mathbb{N}$.

Hence we cannot find an $\epsilon > 0$ such that

$\|(\lambda I - T)x\| < \epsilon\|x\|$ for all nonzero $x \in \mathfrak{D}_T$. Hence $\lambda I - T$ is not bounded from below.

Since $\lambda \notin P\sigma(T)$ it follows that $(\lambda I - T)^{-1}$ exists on $\mathfrak{R}_{\lambda I - T}$ and the result is that $\lambda I - T$ is not bounded from below implies that $(\lambda I - T)^{-1}$ is unbounded.

(ii)⇒ (i): If $\lambda \in P\sigma(T)$ then $\lambda \in \Pi(T)$ for $P\sigma(T) \subseteq \Pi(T)$.

If on the other hand $\lambda \notin P\sigma(T)$ then $\lambda I - T$ has an inverse on the $\mathfrak{R}_{\lambda I - T}$. Since $(\lambda I - T)^{-1}$ is unbounded, it follows that $\lambda I - T$ is not bounded from below. Hence there is no $\epsilon > 0$ such that

$\|(\lambda I - T)x\| \geq \epsilon\|x\|$ forall nonzero $x \in \mathfrak{D}_T$.

Thus we can find a sequence (x_n) of elements of the $\mathfrak{D}_T$ of norm 1 such that

$\|(\lambda I - T)x_n\| \to 0$ that is $\lambda \in \Pi(T)$.
(If X is a Banach space and $T \in B(X)$ then T has a bounded inverse on $\mathfrak{R}_T$ iff T is bounded from below) □

It follows from proposition 6 that:

Corollary 1.1. $\Pi(T) \subseteq \sigma(T)$

Proof. If $\lambda \in \Pi(T)$ then $\lambda \in P\sigma(T)$ or $(\lambda I - T)$ has an unbounded inverse. If $\lambda \in P\sigma(T)$ then $\lambda \in \sigma(T)$. If $(\lambda I - T)$ has an unbounded inverse then $\lambda \in R\sigma(T)$ or $C\sigma(T)$.
That is $\lambda \in \sigma(T)$.
Thus $\Pi(T) \subseteq \sigma(T)$. □

Corollary 1.2. *If $T \in B(X)$, then $|\lambda| \leq \|T\| \, forall \lambda \in \Pi(T)$.*
We have already seen that $|\lambda| \leq \|T\| \, for\, all \, \lambda \in \sigma(T)$.
Since $\Pi(T) \subseteq \sigma(T)$ we get
$|\lambda| \leq \|T\| \, for\, all \, \lambda \in \Pi(T)$.

Proposition 7. *Let X be a Complex Banach space and $T \in B(X)$. Then $\Pi(T)$ is a compact subset of $\mathbb{C}$.*

Proof. That $\Pi(T)$ is a bounded subset of $\mathbb{C}$ is immediate from the result that
$|\lambda| \leq \|T\|$ for all $\lambda \in \Pi(T)$.
We shall show that $(\Pi(T))^c$ is open in $\mathbb{C}$. Let $\lambda_o \in (\Pi(T))^c$ that is, λ_o is not an approximate eigenvalue for T, that is, there is an $\epsilon_o > 0$ such that for all non-zero $x \in X (= \mathfrak{D}_T)$ we have

$$\|(\lambda_o I - T)x\| \geq \epsilon_o \|x\|$$
$$i.e \, \|(\lambda_o I - T)x\| \geq \epsilon_o$$

for all unit vectors $x \in X$.
Take a λ such that $|\lambda - \lambda_o| \leq \frac{\epsilon_o}{2}$.
Then for all $x \in X$ with $\|x\| = 1$

$$\begin{aligned} \|(\lambda I - T)x\| &= \|(\lambda_o I - T)x - (\lambda_o - \lambda)x\| \\ &\geq \|(\lambda_o I - T)x\| - |\lambda_o - \lambda|\|x\| \\ &\geq \|(\lambda_o I - T)x\| - |\lambda_o - \lambda| \\ &\geq \epsilon_o - \frac{\epsilon_o}{2} = \frac{\epsilon_o}{2} \end{aligned}$$

This shows that $\lambda \in (\Pi(T))^c$
Thus for $\lambda_o \in (\Pi(T))^c$ there exists an open neighbourhood $N(\lambda_o, \frac{\epsilon}{2})$ such that $N(\lambda_o, \frac{\epsilon}{2}) \subseteq (\Pi(T))^c$.
Therefore $(\Pi(T))^c$ is open in $\mathbb{C}$ that is $\Pi(T)$ is closed and bounded implies $\Pi(T)$ is a compact subset of $\mathbb{C}$(In Euclidean metric spaces $\mathbb{R}^n, \mathbb{C}^n$ compactness implies closedness and boundedness ([2], [7] and [3])). □

Note: For $T \in B(X)(X$ is a Complex Banach space) $\sigma(T)$ is a compact subset of $\mathbb{C}$. For $\rho(T)$ is open, so $\sigma(T)$ is closed and $|\lambda| \leq \|T\|$ *for all* $\lambda \in \sigma(T)$ implies spectrum of T is bounded. $\sigma(T)$ is bounded and closed implies $\sigma(T)$ is a compact subset of $\mathbb{C}$.

Proposition 8. *Let H be a Hilbert space and $T \in B(H)$. Then the following statements are equivalent.*

(i) *There is a $\lambda \in \Pi(T)$ such that $|\lambda| = \|T\|$*

(ii) $\|T\|=sup\{| < Tx, x > | : x \in H\, and\, \|x\| = 1\}$

Proof. i)$\rightarrow$ ii) Since $\lambda \in \Pi(T)$, it follows that there is a sequence (x_n) of vectors of unit norm in H such that
$\|(T - \lambda I)x_n\| \rightarrow 0$ as $n \rightarrow \infty$.
Now,

$$\begin{aligned}
|\lambda - < Tx, x > | &= |\lambda < x_n, x_n > - < Tx_n, x_n > | \\
&= | < \lambda x_n - Tx_n, x_n > | = | < (\lambda I - T)x_n, x_n > | \\
&\leq \|(\lambda I - T)x_n\|\|x_n\|(by\, C.B.S) \\
&= \|(\lambda I - T)x_n\| \rightarrow 0\, as\, n \rightarrow \infty
\end{aligned}$$

Therefore $< Tx_n, x_n > \rightarrow \lambda$ as $n \rightarrow \infty$.
Thus the sequence $(< Tx_n, x_n >)$ is convergent in $\mathbb{C}$ and hence is bounded.
Moreover by continuity of $| \cdot | : \mathbb{C} \rightarrow \mathbb{R}$ it follows that
$| < Tx_n, x_n > | \rightarrow |\lambda|$. Take $|\lambda| = \|T\|$(since (i) is hypothesis). That is,$(x_n)'s$ are such that $\|x_n\| = 1$ for all $n \in \mathbb{N}$ and $| < Tx_n, x_n > | \rightarrow \|T\|$.
Since for all$x \in H$ with $\|x\| = 1$, we have
$| < Tx, x > | \leq \|Tx\|\|x\| \leq \|T\|\|x\|^2 = \|T\|$.
It follows that
$\{| < Tx, x > | : x \in H\, and\, \|T\| = 1\}$ is bounded above.
Let
M=sup$\{| < Tx, x > | : x \in Hand\, \|x\| = 1\}$
clearly $M \leq \|T\|$. So
$\|T\| \geq M$=sup$\{| < Tx, x > | : x \in Hand\, \|x\| = 1\} \geq sup\{| < Tx_n, x_n > |\}$.
It follows from this that $\|T\| = M$(This is seen thus, since $| < Tx_n, x_n > | \rightarrow \|T\|$, we can choose a subsequence (x_{n_k}) of (x_n)such that
$sup\{| < Tx_{n_k}, x_{n_k} > |\} = \|T\|$)
ii) $\Rightarrow$ i).
Let $\|T\| = sup\{| < Tx, x > | : x \in H\, and\, \|x\| = 1\}$. Hence there is a sequence (x_n) of unit vectors in H such that
$| < Tx_n, x_n > | \rightarrow \|T\|$ as $n \rightarrow \infty$
Therefore the sequence $(< Tx_n, x_n >)$ is bounded in $\mathbb{C}$. Hence by the Bolzano Weirstrass Theorem ([9], [4] and [5]) there is a subsequence (x_{n_k}) of (x_n) such that

$< Tx_{n_k}, x_{n_k} >$ converges in $\mathbb{C}$ to say λ_o that is
$< Tx_{n_k}, x_{n_k} > \to \lambda_o$. By continuity of absolute value function $(|.|)$ we get

$$
\begin{aligned}
| < Tx_{n_k}, x_{n_k} > | &\to |\lambda_o| \\
Since\, | < Tx_{n_k}, x_{n_k} > | &\to \|T\| as\, n \to \infty\, so \\
| < Tx_{n_k}, x_{n_k} > | &\to \|T\|
\end{aligned}
$$

But $| < Tx_{n_k}, x_{n_k} > | \to |\lambda_o|$. Hence by uniqueness of the limit we get
$|\lambda_o| = \|T\|$.
Now we shall compute

$$
\begin{aligned}
0 \leq \|(T - \lambda_o)x_{n_k}\|^2 &= < Tx_{n_k} - \lambda_o x_{n_k}, Tx_{n_k} - \lambda_o x_{n_k} > \\
&= < Tx_{n_k}, Tx_{n_k} > - \lambda_o < x_{n_k}, Tx_{n_k} > - \overline{\lambda_o} < Tx_{n_k}, x_{n_k} > + |\lambda_o|^2 \\
&= \|Tx_{n_k}\|^2 - \lambda_o < x_{n_k}, Tx_{n_k} > - \overline{\lambda_o} < Tx_{n_k}, x_{n_k} > + |\lambda_o|^2 \\
&\leq |\lambda_o|^2 - |\lambda_o|^2 - |\lambda_o|^2 + |\lambda_o|^2 (for \|Tx_{n_k}\|^2 \leq \|T\|^2 \|x_{n_k}\|^2 \leq \|T\|^2 = |\lambda_o|^2) = 0
\end{aligned}
$$

Thus,
$\|(T - \lambda_o)x_{n_k}\| \to 0$ as $k \to \infty$
Therefore
$\lambda_o \in \Pi(T)$.
Thus there is a $\lambda_o \in \Pi(T)$ such that $|\lambda_o| = \|T\|$ and hence i) is proved. □

Proposition 9. *Let H be a Hilbert space and $T \in B(H)$ be normal then $\Pi(T) = \sigma(T)$*

Proof. We have seen that $\Pi(T) \subseteq \sigma(T)$ hence we need to establish the reverse inequality $\sigma(T) \subseteq \Pi(T)$, which is equivalent to showing that
$(\Pi(T))^c \subseteq (\sigma(T))^c$.
Let $\lambda \in (\Pi(T))^c$ that is $\lambda \neq \Pi(T)$. Hence there is a real number $\epsilon_o > 0$ such that

$$
\|(\lambda I - T)x\| \geq \epsilon_o \|x\| for\, all\, x \in H(x \neq \overline{0}). \tag{1.11}
$$

Thus $(\lambda I - T)$ is bounded from below, hence $(\lambda I - T)^{-1}$ exists on $\mathfrak{R}_{\lambda I - T}$ and is bounded. We need to show that $\overline{\mathfrak{R}}_{\lambda I - T} = H$ which is equivalent to showing that if $\mathfrak{R}^{\perp}_{\lambda I - T} = \{\overline{0}\}$ that is $y \perp \mathfrak{R}_{\lambda I - T}$ then $y = \overline{0}$.
Now $T \in B(H)$ is normal $\Rightarrow T - \lambda I$ is normal.
$\{T \in B(H)$ is normal $T \leftrightarrow T^*$ or $\|Tx\| = \|T^* x\|$
$(T - \lambda I)^* = T^* - \overline{\lambda} I$ since $T \leftrightarrow T^*$ so
$(T - \lambda I) \leftrightarrow (T^* - \overline{\lambda} I)$ for

$$
\begin{aligned}
(T - \lambda I)(T^* - \overline{\lambda} I) &= TT^* - \overline{\lambda} T - \lambda T^* + |\lambda|^2 I \\
&= T^* T = \overline{\lambda} T - \lambda T^* + |\lambda|^2 I \\
&= (T^* - \overline{\lambda} I) = (T + \lambda I) \\
i.e\, (T - \lambda I) &\leftrightarrow (T^* - \overline{\lambda} I)\}
\end{aligned}
$$

Hence

$$\|(T-\lambda I)x\| = \|(T^* - \overline{\lambda} I)\| \forall\, x \in H \tag{1.12}$$

If $y \in \mathfrak{R}^{\perp}_{\lambda I - T} = \mathfrak{N}_{\overline{\lambda} I - T^*}$ (standard result).
Hence $(\overline{\lambda} I - T^*)y = \overline{0}$ that is $\|(\overline{\lambda} I - T^*)y\| = 0$. By (1.12)
$\|(T-\lambda I)y\| = \|(T^* - \overline{\lambda} I)y\| = 0$.
Therefore $\|(T-\lambda I)y\| = 0$
By (1.11) we have
$\|(T-\lambda I)y\| \geq \epsilon_o \|y\|$ (put $x = y$ in (1.11))
Thus
$0 = \|(T-\lambda I)y\| \geq \epsilon_o \|y\|$ and $\epsilon_o > 0$

Hence $\|y\| = 0$ that is $y = \overline{0}$. Consequently, $\overline{\mathfrak{R}}_{\lambda I - T} = H$. Therefore,
$\lambda \in \rho(T)$ that is $\lambda \notin \sigma(T)$
$\Rightarrow \lambda \in (\sigma(T))^c$.
Therefore $(\Pi(T))^c \subseteq (\sigma(T))^c$ □

Proposition 10. *Let H be a Hilbert space. $T \in B(H)$ is normal implies T^n is normal for each $n \in \mathbb{N}$.*

Proof. Case I
T is normal implies T^2 is normal.
T is normal implies $TT^* = T^*T$.
$(T^2)^* = (TT)^* = T^*T^* = (T^*)^2$
so

$$\begin{aligned}(T^2)(T^2)^* = (TT)(T^*T^*) = T(TT^*)T^* &= (TT^*)(TT^*)\\ &= (T^*T)(T^*T)\\ &= T^*(TT^*)T\\ &= T^*(T^*T)T\\ &= (T^*T^*)(TT)\\ &= (T^2)^*(T^2)\\ &= (T^*)^2(T^2)\end{aligned}$$

that is, $T^2 \leftrightarrow (T^2)^*$, that is, T^2 is normal.
By induction, it follows that $T \in B(H)$ is normal implies that T^n is normal for each $n \in \mathbb{N}$
Case II
Alternatively, it is easier to proceed thus if T is normal, $T^n \leftrightarrow T^* \; \forall\; n \in \mathbb{N}$. This is true

when $n = 1$ for $T \leftrightarrow T^*$ since T is normal. Suppose for an $r \in \mathbb{N}$ $T^r \leftrightarrow T^*$. Then

$$\begin{aligned} T^{r+1}T^* = (TT^r)T^* &= T(T^rT^*) \\ &= T(T^*T^r) since\, by\, induction\, hypothesis\, T^r \Leftrightarrow T^* \\ &= (TT^*)T^r \\ &= (TT^*)T^r = (T^*T)T^r \\ &= T^*(TT^r) \\ &= T^*T^{r+1}. \end{aligned}$$

Thus $T^r \leftrightarrow T^*$ implies $T^{r+1} \leftrightarrow T^*$. Hence $T^n \leftrightarrow T^*$ for all $n \in \mathbb{N}$. Next we show that $T^n \leftrightarrow (T^*)^n$ for all $n \in \mathbb{N}$. Let $T^n \leftrightarrow (T^*)^r$ for all $r \in \mathbb{N}$. Then

$$\begin{aligned} T^n(T^*)^{r+1} = T^n((T^*)^rT^*) &= (T^n(T^*)^r)T^* \\ &= ((T^*)^rT^n)T^* sinceT^n \leftrightarrow (T^*) \\ &= (T^*)^r(T^nT^*) \\ &= (T^*)^r(T^*T^n) since\, we\, have\, already\, seen\, that\, T^n \leftrightarrow T^* forall\, n \in \mathbb{N} \\ &= ((T^*)^rT^*)T^n \\ &= (T^*)^{r+1}T^n \end{aligned}$$

Thus $T^n \leftrightarrow (T^*)^{r+1}$ if $T^n \leftrightarrow (T^*)^r$. The result is true when $r = 1$ therefore $T^n \leftrightarrow (T^*)^r$ for all $n \in \mathbb{N}$. In particular $T^n \leftrightarrow (T^*)^n$ for all $n \in \mathbb{N}$. But $(T^n)^* = (TT.....T)^* = T^*.....T^* = (T^*)^n$.
Thus we get
$T^n \leftrightarrow (T^n)^*$.
This shows that T^n is normal whenever $T \in B(H)$ is normal. □

We are now in a position to prove;

Proposition 11. *Let H be a Hilbert space. If $T \in B(H)$ is normal then $r(T) = \|T\|$*

Proof. $T \in B(H)$ is normal implies T^n is normal for $n \in \mathbb{N}$. We show that for a normal operator T
$\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$.
Obviously $\|T^n\| = \|T....T\| \leq \|T\|....\|T\| = \|T\|^n$
That is
$\|T^n\| \leq \|T\|^n$ for all $n \in \mathbb{N}$.
Hence we must prove the reverse inequality. Since T is normal, we have
$\|Tx\| = \|T^*x\|$ for all $x \in H$
putting Tx in place of x we get
$\|T(Tx)\| = \|T^*(Tx)\|$ for all $x \in H$
that is
$\|T^2x\| = \|T^*Tx\|$ for all $x \in H$.

Taking the supremum of both sides over all $x \in H$ satisfying $\|x\| \leq 1$ we obtain
$\|T^2\| = \|T^*T\|$
But $\|T^*T\| = \|T\|^2$

$$\therefore \|T^2\| = \|T\|^2 \tag{1.13}$$

if T is normal. Likewise since T^2 is normal, replacing T by T^2 in (1.13) we get
$\|T^4\| = \|T^2\|^2 = (\|T\|^2)^2 = \|T\|^4$.
In general it follows by induction that
$\|T^m\| = \|T\|^m$ for all $m = 2^k$ where $k \in \mathbb{N}$.
Considering $n \in \mathbb{N}$ we can always write
$n = 2^m - r$ for all $r \in \mathbb{N}$
therefore
$n + r = 2^m$.
Then
$\|T^{n+r}\| = \|T\|^{n+r}$
that is,
$\|T\|^{n+r} = \|T^{n+r}\| = \|T^nT^r\| \leq \|T^n\|\|T^r\| \leq \|T^n\|\|T\|^r$
that is
$\|T\|^n\|T\|^r \leq \|T^n\|\|T\|^r$.
Cancelling $\|T\|^r$ from both sides (of course $\|T\| \neq 0$)
we get
$\|T\|^n \leq \|T^n\|$
which is the required reverse inequality. Thus, if $T \in B(H)$ is normal, then
$\|T^n\| = \|T\|^n$ for all $n \in \mathbb{N}$.
Now
$r(T) = lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} = lim_{n\to\infty}(\|T\|^n)^{\frac{1}{n}} = lim_{n\to\infty}\|T\| = \|T\|$. □

Remark 2. *There is an alternative method for arriving at the same result using proposition 5: If $T \in B(X)(X =$ complex Banach space$)$, then,*
$lim_{n\to\infty}\|T^n\|^{\frac{1}{n}}$ *exists.*
Consider a subsequence $(\|T^{n_k}\|^{\frac{1}{n_k}})_{k=1}^{\infty}$ *where* $n_k = 2^k$ *of the given sequence* $(\|T^n\|^{\frac{1}{n}})_{n=1}^{\infty}$. *This subsequence is convergent to the same limit as that of* $\|T^n\|^{\frac{1}{n}}$. *Now we have seen that*
$\|T^{n_k}\| = \|T\|^{n_k}$ *(by proposition 11)*
therefore
$\|T^{n_k}\|^{\frac{1}{n_k}} = \|T\|$.
So $lim_{k\to\infty}\|T^{n_k}\|^{\frac{1}{n_k}} = \|T\|$.
Therefore $lim_{n\to\infty}\|T^n\|^{\frac{1}{n}} = \|T\|$

Proposition 12. *Let X be a Complex Banach space and $T \in B(X)$. Then there is a $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$ (Note: When X is a Complex Banach space then $\sigma(T) \neq \phi$)*

Proof. By definition of $r(T) = sup\{|\lambda| : \lambda \in \sigma(T)\}$. Then there exists a sequence (λ_n) of elements of $\sigma(T)$ such that $|\lambda_n| \longrightarrow r(T)$ as $n \to \infty$. Hence the sequence (λ_n) is a bounded subset of $\mathbb{C}$. Hence by the Bolzano-Weierstrass Theorem ([9], [4] and [5]) there exists a subsequence $(\lambda_{n_k}) of (\lambda_n)$ such that (λ_{n_k}) converges to say $\lambda_o \in \mathbb{C}$. Thus

$\lambda_{n_k} \longrightarrow \lambda_o$

consequently by continuity on the modulus function $|| : \lambda \longmapsto |\lambda|$ we get

$|(\lambda_{n_k})| \longrightarrow |\lambda_o|$.

Since $(\lambda_{n_k}) \in \sigma(T)$ and $\lambda_{n_k} \longrightarrow \lambda_o$, it follows that $\lambda_o \in \overline{\sigma(T)}$. But $\sigma(T)$ is closed in $\mathbb{C}$(in usual topology).

So $\overline{\sigma(T)} = \sigma(T)$

Hence $\lambda_o \in \sigma(T)$.

Since $|\lambda_n| \longrightarrow r(T)$ as $n \to \infty$ so $|(\lambda_{n_k})| \longrightarrow r(T)$ as $k \to \infty$.

Since $(|\lambda_{n_k}|)$ is a subsequence of the convergent sequence $(|\lambda_n|)$. By uniqueness of the limit of a convergent sequence, we obtain

$r(T) = |\lambda_o|$.

Since $\lambda_o \in \sigma(T)$ the assertion of the theorem follows (take λ in the statement of the theorem to be the λ_o obtained in the proof). □

Proposition 13. *Let H be a Complex Hilbert space and $T \in B(H)$ be normal. Then*

$\|T\| = sup\{| < Tx, x > | : x \in H and \|x\| = 1\}$

Proof. We have seen that if $T \in B(H)$ is normal then $r(T) = \|T\|$. We also saw earlier that for a $T \in B(H)$ which is normal,

$\sigma(T) = \Pi(T)$.

In proposition 12, we saw that if $T \in B(X)$ (X is a Banach space), then there is a $\lambda \in \sigma(T)$ such that $|\lambda| = r(T)$.

Hence if H is a Hilbert space and $T \in B(H)$ is normal, then from the result $\sigma(T) = \Pi(T)$ we obtain that there is a $\lambda \in \pi(T)$ such that $|\lambda| = r(T)$.

For such a T we know that $r(T) = \|T\|$. Hence we arrive at the assertion; If $T \in B(H)$ is normal, there is a $\lambda \in \Pi(T)$ such that

$|\lambda| = \|T\|$.

From an earlier result, for $T \in B(H)$ the following statements are equivalent

(i) There exists $\lambda \in \Pi(T)$ such that $|\lambda| = \|T\|$

(ii) $\|T\| = sup\{| < Tx, x > | : x \in H\, and\, \|x\| = 1\}$

It then follows that if $T \in B(H)$ is normal then

$\|T\| = \{| < Tx, x > | : x \in H$and $\|x\| = 1\}$ □

Remark 3. *Since bounded self-adjoint operators and unitary operators are also normal, the result above for $\|T\|$ holds for such operators.*

Proposition 14. *If $T \in B(H)$ is normal, then $R\sigma(T) = \phi$*

Proof. We have already seen that if $T \in B(H)$ is normal, then

$\sigma(T) = \Pi(T)$,

also $P\sigma(T) \subseteq \Pi(T)$. Suppose

$\lambda \in \Pi(T)$. Then $\lambda \in P\sigma(T)$ or $\lambda \notin P\sigma(T)$.

If $\lambda \notin P\sigma(T)$ then it must belong to $C\sigma(T)$ or $R\sigma(T)$. In both of the latter cases, we know that $(\lambda I - T)^{-1}$ would exist as a linear map on $\mathfrak{R}_{\lambda I-T}$, but if λ belonged to $R\sigma(T)$ we would have

$\overline{\mathfrak{R}}_{\lambda I-T} \neq H$

Suppose $\overline{\mathfrak{R}}_{\lambda I-T} \neq H$. In such a case we would have $\lambda \in R\sigma(T)\, or\, P\sigma(T)$. Then there would exist a non-zero $y \in H$ such that

$y \perp \mathfrak{R}_{\lambda I-T}$.

(A linear subspace $(M\, of\, H)$is dense if and only if $x \perp M \Rightarrow x = \overline{0}$) that is

$y \in \mathfrak{R}^{\perp}_{\lambda I-T} = \mathfrak{N}_{\overline{\lambda} I-T^*}$

hence $(\overline{\lambda} I - T^*)y = \overline{0}$

Since T is normal, so is $\lambda I - T$, that is,

$\|(\lambda I - T)x\| = \|(\lambda I - T)^* x\|$ for all $x \in H$.

That is $\|(\lambda I - T)x\| = \|(\overline{\lambda} I - T^*)x\|$ for all $x \in H$. In particular, put $x = y$ we obtain

$\|(\lambda I - T)y\| = \|(\overline{\lambda} I - T^*)y\|$

But $\|(\overline{\lambda} I - T^*)y\| = \|0\| = 0$. Hence $\|(\lambda I - T)y\| = 0$, that is,

$(\lambda I - T) = \overline{0}$.

Since $y \neq \overline{0}$ it follows that λ is an eigenvalue of T, that is

$\lambda \in P\sigma(T)$.

So if $\overline{\mathfrak{R}}_{\lambda I-T} \neq H$, then $\lambda \in P\sigma(T)$ necessarily. Therefore

$R\sigma(T) = \phi$. □

1.3 Conclusion

When one wants to estimate the norms of functions of bounded operators the properties of norms on $B(X)$ usually are very helpful. But imposing assumptions on the spectral radius of an operator is better than imposing on its norm, because the first is intrinsic and the second is not. But for certain sets of bounded operators one can change the norm of the space to an equivalent norm in such a way that the norms of the bounded operators are as near as we wish to their spectral radius. This fact has been widely used by many authors. If we have a finite family of bounded linear operators T_n, such that T_n commutes with T_m for every m, n, then it is possible to define a new norm on the Banach space X, equivalent to the initial norm, in such a way that in the new norm the spectral radius, $r(T_n)$, approximates the norm of the operator T_n, for every n.

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