Decay for solutions to a plate type equation

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Abstract

In a similar spirit to the papers [7, 8], existence and decay for a plate type equation is obtained. The result is a generalization of the work for the linear equation in the paper [7].

Keywords: Decay estimates; Plate type equation; Semilinear; Memory term.

Section 1 : Preliminary

We study the following equation in $R^n \times [0,+\infty)$ ($n \geq 1$)

$$\begin{cases} \partial_t^2 \chi + (\Delta^2 + 1)\chi - \kappa *_t \chi = g(\partial_x^2 \chi, \partial_t \chi), \\ \chi(0) = \chi_0(x), \qquad \partial_t \chi(0) = \chi_1(x). \end{cases} \tag{P}$$

The memory term $\kappa *_t \chi$ is defined by

$$(\kappa *_t \chi) := \int_0^t \kappa(t-s)\chi(x,s)ds.$$

We assume the memory kernel $\kappa(t)$ satisfies the following assumption.

$$\begin{aligned} &(a) \quad 0 < \kappa \in C^2([0,\infty)), \\ &(b) \quad -C_1 \leq \kappa'(t)/g(t) \leq -C_2, \quad |\kappa''(t)/\kappa(t)| \leq C_3 \text{ for } t \geq 0, \\ &(c) \quad \int_0^\infty \kappa(s)ds \leq 1, \end{aligned}$$

with $C_j > 0\,(j = 1,2,3)$ are constants.

And $g(\partial_x^2 \chi, \partial_t \chi)$ satisfies the following assumption.

$$g(\lambda \partial_x^2 \chi, \lambda \partial_t \chi) = \lambda^\alpha g(\partial_x^2 \chi, \partial_t \chi), \forall \lambda > 0,$$

here α is an integer satisfying $\alpha > \alpha_n$ with $\alpha_n := \begin{cases} 5-n, & n \leq 3, \\ 1 + \frac{2}{n}, & n \geq 4. \end{cases}$

For $k \in Z^+$, we denote

$$a(k,n) = 2k + \left\lfloor \frac{n+1}{2} \right\rfloor, n \geq 1.$$

Theorem1.1 Let $s \in Z^+$, $s \geq \max\{n+1, 3\}$. Suppose $\chi_0 \in H^{s+2} \cap L^1$ and $\chi_1 \in H^s \cap L^1$. Put

$$E_0 := \|\chi_0\|_{H^{s+2}} + \|\chi_1\|_{H^s} + \|\chi_0\|_{L^1} + \|\chi_1\|_{L^1}.$$

Then there exists uniquely a solution $\chi \in C^0([0,\infty); H^{s+2}) \cap C^1([0,\infty); H^s)$ of (P) satisfying the following estimates:

$$\|\partial_x^{k+2}\chi(t)\|_{H^{s-\alpha(k,n)}} \leq CE_0(1+t)^{-(\frac{n}{8}+\frac{k}{4})},$$

$$\|\partial_x^k\partial_t\chi(t)\|_{H^{s-\alpha(k,n)}} \leq CE_0(1+t)^{-(\frac{n}{8}+\frac{k}{4})}.$$

Here $k \geq 0$ satisfying $\alpha(k,n) \leq s$.

We recall some related work. Da Luz-Charão (see [6]) studied the following dissipative plate equation in a bounded domain in R^n $(1 \leq n \leq 5)$

$$(1-\Delta)\partial_t^2\chi + \Delta^2\chi + \partial_t\chi = g(\chi).$$

Here $\partial_t\chi$ is the linear dissipative term. Sugitani-Kawashima (see [17]) studied this problem in R^n and extend the results to general n. Subsequently, Liu-Kawashima (see [9, 10]) studied a more complex equation

$$(1-\Delta)\partial_t^2\chi + \sum_{i,j=1}^{n} a^{ij}(\partial_x^2\chi)_{x_i x_j} + \partial_t\chi = 0.$$

In [11], Liu-Kawashima studied the following memory type equation

$$\partial_t^2\chi + (\Delta^2+1)\chi + \kappa *_t \Delta\chi = g(\chi).$$

Liu (see [8], also [12] for related results) further studied the following Cauchy problem

$$(1-\Delta)\partial_t^2\chi + (\Delta^2+1)\chi + \kappa *_t \Delta\chi = g(\chi, \partial_t\chi, \nabla\chi).$$

Mao-Liu (see [15]) generalized the results of plate-type equation (see [8, 11]) with memory to higher order equations. They studied fractional order of derivatives. They also (see [16]) studied equations of variable coefficients.

In these papers, the memory term under consideration is $\kappa *_t \Delta\chi$. Recently, Liu-Ueda (see [7]) studied a type of linear plate equation with some different memory term $\kappa *_t \chi$. They obtained some decay estimates and asymptotical behavior for solutions under suitable assumption.

Similar results also holds for Timoshenko system (see [13, 14]) and hyperbolic-elliptic system (see [5]). For more related results, we refer to [1, 2, 3, 4, 9, 18].

In section 2, we will prove Theorem 1.1, which extends the result in [7] to the case of semi-linear perturbations.

Section 2 Proof of Theorem 1.1

We note that the solution can be formally expressed as

$$\chi(t)=G(t)*_x\chi_0+H(t)*_x\chi_1+\int_0^t H(t-s)*_x g(\partial_x^2\chi,\partial_t\chi)ds.$$

Here G, H are the fundamental solutions of the corresponding linear equation, and the notation $*_x$ denotes the convolution with respect to x.

We recall several lemmas.

Lemma 1 (see [7]**).** Let $s\ge 0$, $1\le p\le 2$. Then the following estimates hold for $0\le k+l\le s$, $\varphi\in S$ (the class of Schwartz functions):

$$(1)\ \|\partial_x^{k+2}G(t)*_x\varphi\|_{L^2}+\|\partial_t\partial_x^k G(t)*_x\varphi\|_{L^2}\le C_{q,k}(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{2}+k\right)}\|\varphi\|_{L^p}+C_\mu(1+t)^{-\frac{\mu}{4}}\|\partial_x^{k+\mu+2}\varphi\|_{L^2},$$

$$(2)\ \|\partial_x^{k+2}H(t)*_x\varphi\|_{L^2}+\|\partial_t\partial_x^k H(t)*_x\varphi\|_{L^2}\le C_{q,k}(1+t)^{-\frac{n}{4}\left(\frac{1}{p}-\frac{1}{2}+k\right)}\|\varphi\|_{L^p}+C_\mu(1+t)^{-\frac{\mu}{4}}\|\partial_x^{k+\mu}\varphi\|_{L^2}.$$

By a little modification of the theorem 2.7 in [7], we have the following

Lemma 2 (see [7]**).** Let $s\ge\left\lfloor\frac{n+1}{2}\right\rfloor$ be an integer, $\chi(t):=G(t)*_x\chi_0+H(t)*_x\chi_1$ and $E_0:=\|\chi_0\|_{H^{s+2}}+\|\chi_1\|_{H^s}+\|\chi_0\|_{L^1}+\|\chi_1\|_{L^1}$. Suppose $\chi_0\in H^{s+2}\cap L^1$, $\chi_1\in H^s\cap L^1$. Then

$$\|\partial_x^{k+2}\chi(t)\|_{H^{s-\omega(k,n)}}+\|\partial_x^k\partial_t\chi(t)\|_{H^{s-\omega(k,n)}}\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}E_0.$$

Proof. Let $k,m\in Z^+$.

Let $p=1$ in Lemma 1, we have

$$\begin{aligned}\|\partial_x^{k+2}\chi(t)\|_{H^m}&\le\|\partial_x^{k+2}G(t)*_x\chi_0(t)\|_{H^m}+\|\partial_x^{k+2}H(t)*_x\chi_1(t)\|_{H^m}\\&\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\|\chi_0\|_{L^1}+C(1+t)^{-\frac{\mu_1}{4}}\|\partial_x^{k+\mu_1+2}\chi_0\|_{H^m}\\&\quad+C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}\|\chi_1\|_{L^1}+C(1+t)^{-\frac{\mu_2}{4}}\|\partial_x^{k+\mu_2}\chi_1\|_{H^m}\\&\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}(\|\chi_0\|_{L^1}+\|\chi_1\|_{L^1})+C(1+t)^{-\frac{\mu_1}{4}}\|\chi_0\|_{H^{m+k+\mu_1+2}}\\&\quad+C(1+t)^{-\frac{\mu_2}{4}}\|\chi_1\|_{H^{m+k+\mu_2}}.\end{aligned}$$

Here $0\le k+m+\mu_i\le s,\ (i=1,2).$

Choose the smallest integers μ_i satisfying

$$\frac{\mu_i}{4}\ge\frac{n}{4}(k+\frac{1}{2}).$$

Then we have

$$\|\partial_x^{k+2}\chi(t)\|_{H^{s-\omega(k,n)}}\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}E_0.$$

Similarly, we have

$$\|\partial_x^k\partial_t\chi(t)\|_{H^{s-\omega(k,n)}}\le C(1+t)^{-\frac{n}{4}(k+\frac{1}{2})}E_0.$$

That is the conclusion.

Lemma 3 (see [8]**).** Let $1\le p,q,r\le\infty$, $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$ and $k\ge 0$, $m\ge 1$, $n\ge 1$ be integers. Then

$$\left\|\partial_x^k\left(\xi^m\eta^n\right)\right\|_{L^p}\le C\|\xi\|_{L^\infty}^{m-1}\|\eta\|_{L^\infty}^{n-1}\left(\|\xi\|_{L^q}\|\partial_x^k\eta\|_{L^r}+\|\eta\|_{L^q}\|\partial_x^k\xi\|_{L^r}\right).$$

Just by a direct computation, we get

Proposition 1 (cf. [8]**).** Let $a\ge 0$ and $b\ge 0$ be real numbers. If $a+b\ge 1$, then there exists $C>0$ such that $\int_0^t(1+t-s)^{-a}(1+s)^{-b}ds\le C.$

Now we come to prove Theorem 1.1, we mimic the argument in [8], and define

$$\Upsilon:=\left\{\chi\in C^0(R^+;H^{s+2})\bigcap C^1(R^+;H^s),\|\chi\|_Y<\infty\right\}$$

with

$$\|\chi\|_Y:=\sup_{t\ge 0,0\le(k,n)\le s}(1+t)^{\frac{n}{8}+\frac{k}{4}}\left(\|\partial_x^{k+2}\chi(t)\|_{H^{s-\alpha(k,n)}}+\|\partial_x^k\partial_t\chi(t)\|_{H^{s-\alpha(k,n)}}\right).$$

Denote

$$T[\chi](t):=G(t)*_x\chi_0+\mathrm{H}(t)*_x\chi_1+\int_0^t\mathrm{H}(t-\tau)*_x g(\Xi)(\tau)d\tau,\quad \text{with}\ \Xi:=(\partial_x^2\chi,\partial_t\chi).$$

By the assumption of f and Lemma 3, we have the following inequalities

$$\begin{aligned}\left\|\partial_x^k\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{L^1}\le C(\|\Xi(\tau)\|_{L^\infty}+\|\Upsilon(\tau)\|_{L^\infty})^{\alpha-2}\Big((\|\Xi(\tau)\|_{L^2}+\|\Upsilon(\tau)\|_{L^2})\left\|\partial_x^k\left(\Xi-\Upsilon\right)(\tau)\right\|_{L^2}\\+(\|\partial_x^k\Xi(\tau)\|_{L^2}+\|\partial_x^k\Upsilon(\tau)\|_{L^2})\|\left(\Xi-\Upsilon\right)(\tau)\|_{L^2}\Big),\end{aligned}$$

and

$$\begin{aligned}\left\|\partial_x^k\left(g(\Xi)-g(\Upsilon)\right)(\tau)\right\|_{L^2}\le C(\|\Xi(\tau)\|_{L^\infty}+\|\Upsilon(\tau)\|_{L^\infty})^{\alpha-2}\Big((\|\Xi(\tau)\|_{L^\infty}+\|\Upsilon(\tau)\|_{L^\infty})\left\|\partial_x^k\left(\Xi-\Upsilon\right)(\tau)\right\|_{L^2}\\+(\|\partial_x^k\Xi(\tau)\|_{L^2}+\|\partial_x^k\Upsilon(\tau)\|_{L^2})\|\left(\Xi-\Upsilon\right)(\tau)\|_{L^\infty}\Big).\end{aligned}$$

Now we will prove that the mapping $\chi\to T[\chi]$ is contraction on $B_\varepsilon:=\left\{\chi\in Y;\|\chi\|_Y\le\varepsilon\right\}$ for some $\varepsilon>0$. This will be done in the following (S1)—(S4).

(S1). Set $s_n=\left\lfloor\frac{n}{2}\right\rfloor+1,\ \theta_n=\frac{n}{2s_n}$. Taking $\chi\in Y$ and by Nirenberg's inequality, we have

$$\|\Xi(t)\|_{L^\infty}\le C\|\Xi(t)\|_{L^2}^{1-\theta_n}\|\partial_x^{s_n}\Xi(t)\|_{L^2}^{\theta_n}.$$

(i) For $n=1$, by assumption of s, we get $\|\Xi(t)\|_{L^2}\le C(1+t)^{-\frac{1}{8}}\|\chi\|_Y$ and $\|\partial_x^{s_n}\Xi(t)\|_{L^2}\le C(1+t)^{-\frac{3}{8}}\|\chi\|_Y$. It yields $\|\Xi(t)\|_{L^\infty}\le C(1+t)^{-\frac{1}{4}}\|\chi\|_Y$.

(ii) For $n\ge 2$, since $s-\alpha(0,n)\ge s_n$, we obtain $\|\Xi(t)\|_{L^2}\le C(1+t)^{-\frac{n}{8}}\|\chi\|_Y$ and $\|\partial_x^{s_n}\Xi(t)\|_{L^2}\le C(1+t)^{-\frac{n}{8}}\|\chi\|_Y$. Then

$$\|\Xi(t)\|_{L^\infty}\le C(1+t)^{-\frac{n}{8}}\|\chi\|_Y.$$

(S2). Take any $\chi,\eta\in Y$, and denote $\Xi:=(\partial_x^2\chi,\partial_t\chi),\ \Upsilon:=(\partial_x^2\eta,\partial_t\eta)$. Then we have

$$T[\chi](t)-T[\eta](t)=\int_0^t\mathrm{H}(t-\tau)*_x\left(g(\Xi)-g(\Upsilon)\right)(\tau)d\tau.$$

Assume $s \geq \alpha(k,n)$, then we have

$$\begin{aligned}\|\partial_x^{k+2}(T[\chi](t)-T[\eta](t))\|_{H^m} &\leq C\int_0^{\frac{t}{2}}\|\partial_x^{k+m+2}\mathrm{H}(t-\tau)*_x\left(g(\Xi)-g(\Upsilon)\right)(\tau)\|_{H^0}\,d\tau \\ &+C\int_{\frac{t}{2}}^{t}\|\partial_x^{k+m+2}\mathrm{H}(t-\tau)*_x\left(g(\Xi)-g(\Upsilon)\right)(\tau)\|_{H^0}\,d\tau \\ &=: I+II.\end{aligned}$$

Let $p=1$ in Lemma 1, we have

$$\begin{aligned}I &\leq C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\|\left(g(\Xi)-g(\Upsilon)\right)(\tau)\|_{L^1}\,d\tau \\ &+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{\mu}{4}}\|\partial_x^{k+m+\mu}\left(g(\Xi)-g(\Upsilon)\right)(\tau)\|_{L^2}\,d\tau.\end{aligned}$$

And

$$\begin{aligned}II &\leq C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\|\left(g(\Xi)-g(\Upsilon)\right)(\tau)\|_{L^1}\,d\tau \\ &+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{\mu}{4}}\|\partial_x^{k+m+\mu}\left(g(\Xi)-g(\Upsilon)\right)(\tau)\|_{L^2}\,d\tau.\end{aligned}$$

Then by a similar way as in [8], we can obtain

$$I+II \leq C(1+t)^{-(\frac{n}{8}+\frac{k}{4})}(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y$$

with $0 \leq m \leq s-\alpha(k,n)$. That is,

$$\|\partial_x^{k+2}(T[\chi](t)-T[\eta](t))\|_{H^m} \leq C(1+t)^{-(\frac{n}{8}+\frac{k}{4})}(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y$$

for $0 \leq m \leq s-\alpha(k,n)$.

So we have that

$$\sup_{t\geq 0}(1+t)^{\frac{n}{8}+\frac{k}{4}}\|\partial_x^{k+2}(T[\chi](t)-T[\eta](t))\|_{H^{s-\alpha(k,n)}} \leq C(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y.$$

(S3). In a similar way as in the part (S2), we can prove that

$$\sup_{t\geq 0}(1+t)^{\frac{n}{8}+\frac{k}{4}}\|\partial_x^{k}\partial_t(T[\chi](t)-T[\eta](t))\|_{H^{s-\alpha(k,n)}} \leq C(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y.$$

(S4). The estimates in (S2) and (S 3) imply that

$$\|T[\chi]-T[\eta]\|_Y \leq C(\|\chi\|_Y+\|\eta\|_Y)^{\alpha-1}\|\chi-\eta\|_Y.$$

So far we proved that $\|T[\chi]-T[\eta]\|_Y \leq C_1\varepsilon^{\alpha-1}\|\chi-\eta\|_Y$, if $\chi,\eta \in B_\varepsilon$. By Lemma 2, we know that $\|G(t)*_x\chi_0+H(t)*_x\chi_1\|_Y \leq C_2E_0$. So if E_0 and ε are sufficiently small, then we have

$$\|T[\chi]-T[\eta]\|_Y \leq \frac{1}{2}\|\chi-\eta\|_Y.$$

It then yields that

$$\|T[\chi]\|_Y \leq \|G(t)*_x\chi_0+H(t)*_x\chi_1\|_Y+\frac{1}{2}\|\chi\|_Y \leq \varepsilon.$$

Hence the mapping $\chi \to T[\chi]$ is contraction on B_ε. Then the fixed point theorem imply that there exists a unique fixed point $\chi \in B_\varepsilon$ satisfying $T[\chi]=\chi$. That is, this $\chi \in B_\varepsilon$ satisfies the equation

$$\chi(t)=G(t)*_x\chi_0+H(t)*_x\chi_1+\int_0^t H(t-s)*_x g(\partial_x^2\chi,\partial_t\chi)ds.$$

So it is the solution to the semi-linear problem (P), and satisfies the corresponding decay estimates in Theorem 1.1.

Remark In the proof above, we just sketched in some parts and left the details. The reader can refer to the paper [8] for similar argument.

Section 3 Conclusion

We studied the Cauchy problem of a class of semi-linear plate type equation. We obtained the global existence (in time t) under the assumption of smallness of initial data, and some decay for solutions to this equation in terms of fixed point theorem. Our result is a generalization of the decay for the linear equation in [7].

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