Decay estimate for solutions to a semilinear plate equation with memory

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Abstract

In this paper we consider the initial value problem of a semilinear plate equation with memory term in $R^n (n \geq 1)$. We study the decay estimates for solutions to the equation in the spirit of the papers [7,8]. By using the classical contraction theorem and delicate estimates, we obtained the global existence and decay estimates of solutions to the equation. Our result can be regarded as a generalization of the result in the paper [7].

Keywords: Decay estimates; Plate equation; Semilinear; Memory term.

1 Introduction

In this paper we consider the initial value problem of the following semilinear plate equation with memory term in R^n ($n \geq 1$)

$$u_{tt} + \Delta^2 u + u - g * u = f(\partial_x^2 u, u_t) \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x). \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \cdots, x_n) \in R^n$ and $t > 0$, which represents the transversal displacement of the plate at the point x and the time t. The subscript t in u_t and u_{tt} denotes the time derivative $(i.e., u_t = \partial_t u$ and $u_{tt} = \partial_t^2 u)$. The memory term $g * u$ is defined by

$$(g * u)(x,t) := \int_0^t g(t-\tau) u(x,\tau) d\tau,$$

which means the stress at an instant depends on the whole history of the strains the material has suffered. The memory kernel g is a given function which satisfies the following assumption.

Assumption[A]**:**

(i) $\quad g \in C^2([0,\infty))$,

(ii) $\quad g(t) > 0, -C_1 g(t) \leq g'(t) \leq -C_2 g(t), |g''(t)| \leq C_3 g(t)$ for t ≥ 0,

(iii) $\quad \int_0^t g(\tau) d\tau \leq 1$ for t ≥ 0, where, $C_j\ (j=1,2,3)$ are positive constants.

The nonlinear term $f(\partial_x^2 u, u_t)$ satisfies the following

Assumption[B]: $f(\lambda\partial_x^2 u, \lambda u_t) = \lambda^\alpha f(\partial_x^2 u, u_t), \forall \lambda > 0$,

here α is an integer satisfying $\alpha > \alpha_n$ with $\alpha_n := \begin{cases} 5-n, & n \le 3, \\ 1+\frac{2}{n}, & n \ge 4. \end{cases}$

For an integer $k \ge 0$, we denote

$$\sigma(k,n) = 2k + \left[\frac{n+1}{2}\right], n \ge 1. \tag{1.3}$$

Now our main theorem is stated as follows.

Theorem1.1 (Existence and Decay Estimates). Let s be an integer and $s \ge \max\{n+1, 3\}$. Suppose $u_0 \in H^{s+2}(R^n) \cap L^1(R^n)$ and $u_1 \in H^s(R^n) \cap L^1(R^n)$. Put

$$I_1 := \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s} + \|(u_0, u_1)\|_{L^1}.$$

Then there exists a unique solution $u(x,t)$ of (1.1)--(1.2) with

$$u(x,t) \in C^0([0,\infty); H^{s+2}(R^n)) \cap C^1([0,\infty); H^s(R^n)),$$

satisfying the following decay estimates:

$$\left\|\partial_x^{k+2} u(t)\right\|_{H^{s-\sigma(k,n)}} \le C I_1 (1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{1.4}$$

$$\left\|\partial_x^k u_t(t)\right\|_{H^{s-\sigma(k,n)}} \le C I_1 (1+t)^{-\frac{n}{8}-\frac{k}{4}}. \tag{1.5}$$

Here $k \ge 0$ satisfying $\sigma(k,n) \le s$, and $\sigma(k,n)$ is defined in (1.3).

For the study of the plate equations, there are many results in the literatures. Equations of the fourth-order appear in problems of solid mechanics and in the theory of thin plates and beams, and also in some problems related to the Navier-Stokes equations (see [19]). Da Luz-Charão (see [6]) studied the following semilinear dissipative plate equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = f(u). \tag{1.6}$$

Here u_t is the linear dissipative term. They obtained the global existence of solutions and a polynomial decay of the energy by applying an energy method. However the result is confined to the lower spatial dimension $1 \le n \le 5$. This limitation on the spatial dimension is eliminated by Sugitani-Kawashima (see [17]) by using the fundamental method of energy estimates in the Fourier (or frequency) space and some sharp decay estimates. Subsequently, Liu-Kawashima (see [9, 10]) studied a more complex inertial model for quasilinear dissipative plate equation whose linear part is presented by

$$u_{tt} - \Delta u_{tt} + \sum_{i,j=1}^{n} b^{ij} (\partial_x^2 u)_{x_i x_j} + u_t = 0. \tag{1.7}$$

They obtained the global existence and asymptotic decay of solutions under smallness and some regularity assumptions on the initial data by employing the time-weighted L^2 energy method.

For the case of plate equations with memory term, Liu-Kawashima (see [11]) studied the following semilinear plate equation

$$u_{tt} + \Delta^2 u + u + g * \Delta u = f(u). \tag{1.8}$$

They obtained the global existence and decay estimates of solutions by exploiting the energy method in the Fourier space.

Liu (see [8], also [12] for related results) studied the following initial value problem of rotational plate equations with memory

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u + g * \Delta u = f(u, u_t, \nabla u). \tag{1.9}$$

Due to the rotation term, Liu obtained the global existence and decay estimates of solutions with more general semilinear term.

Mao-Liu (see [15]) considered the following generalized plate-type equation with memory in multi-dimensional space

$$u_{tt} - \Delta u_{tt} + (-\Delta)^p u + u + g * \Delta u = 0. \tag{1.10}$$

Here $p \geq 1$ is a real number. In that paper, the authors extend the order of derivatives from integer to fraction and refine the results of the even-dimensional case in the related literature [8, 11]. Mao-Liu (see [16]) studied plate-type equations with variable coefficients and memory, where the decay and the regularity-loss property is characterized by a function in the spectral space.

In these papers, the dissipation given by the memory term $g * \Delta u$ is relatively weaker compared with the frictional damping term u_t. This dissipative mechanism could be reflected from the decay structure of solutions. Recently, Liu-Ueda (see [7]) studied a type of linear plate equation with some different memory term $g * u$, they obtained some decay estimates and asymptotical behavior for solutions under suitable assumption. The results in these papers and the general dissipative plate equation (see [9, 18]) show that they are of regularity-loss property.

A similar decay structure of the regularity-loss type was also observed for the dissipative Timoshenko system (see [13, 14]) and a hyperbolic-elliptic system related to a radiating gas (see [5]). For more studies on various aspects of dissipation of plate equations, we refer to [1, 2, 3, 4].

The main purpose of this paper is to study the global existence and decay estimates for solutions to the initial value problem (1.1)--(1.2) with semilinear term in the spirit of the papers [7, 8]. We extend the result in [7] to the case of semilinear perturbation.

Before closing this section, we give some notations to be used below. Let $\mathcal{F}[f]$ denote the Fourier transform of f:

$$\hat{f}(\xi) = \mathcal{F}[f](\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-ix\cdot\xi} f(x)dx,$$

Its inverse transform is denoted by $\mathcal{F}^{-1}$.

$L^p = L^p(R^n)$ for $1 \leq p \leq \infty$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. $H^s = H^s(R^n)$ for

$s \in N$ denotes the Sobolev space with the norm

$$\| f\|_{H^s} := \left\| \left(1+|\xi|^2\right)^{\frac{s}{2}} \hat{f} \right\|_{L^2} .$$

$C^k(I; H^s(R^n))$ denotes the space of k -times continuously differentiable functions in the interval I with values in the Sobolev space $H^s = H^s(R^n)$.

Finally, we denote every positive constant by the same symbol C or c without confusion, and $[\cdot]$ the Gauss' symbol.

2 Proof of the Main Theorem

In this section, by virtue of the properties of solution operators, we prove the global existence and decay estimates of solutions to the semilinear problem (1.1)--(1.2) by employing the contraction mapping theorem. Let's recall the fundamental solution formula of the linear problem in the paper (see [7]), which is given by

$$u(t) = u_s(t) := \mathcal{G}(t) * u_0 + \mathcal{H}(t) * u_1 . \tag{2.1}$$

Also, the solution $u(x,t)$ to the problem (1.1)--(1.2) can be formally expressed as

$$u(t) := \mathcal{G}(t) * u_0 + \mathcal{H}(t) * u_1 + \int_0^t \mathcal{H}(t-\tau) * f(\partial_x^2 u, u_t) d\tau . \tag{2.2}$$

To prove theorem 1.1, we need several lemmas.

Lemma 1 (see [7]**).** Assume that $\varphi \in H^{s+2}\left(R^n\right) \bigcap L^q\left(R^n\right)$ and $\psi \in H^s\left(R^n\right) \bigcap L^q\left(R^n\right)$ for $s \ge 0$ and $1 \le q \le 2$. Let k and l be non-negative integers satisfying $k + l \le s$, then the following estimate holds:

(1) $\left\|\partial_x^{k+2} \mathcal{G}(t) * \varphi\right\|_{L^2} \le \sqrt{2} C_{q,k} (1+t)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}} \|\varphi\|_{L^q} + \sqrt{2} C_l (1+t)^{-\frac{l}{4}} \left\|\partial_x^{k+l+2}\varphi\right\|_{L^2}$,

(2) $\left\|\partial_t \partial_x^{k} \mathcal{G}(t) * \varphi\right\|_{L^2} \le \sqrt{2} C_{q,k} (1+t)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}} \|\varphi\|_{L^q} + \sqrt{2} C_l (1+t)^{-\frac{l}{4}} \left\|\partial_x^{k+l+2}\varphi\right\|_{L^2}$,

(3) $\left\|\partial_x^{k+2} \mathcal{H}(t) * \psi\right\|_{L^2} \le C_{q,k} (1+t)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}} \|\psi\|_{L^q} + C_l (1+t)^{-\frac{l}{4}} \left\|\partial_x^{k+l}\psi\right\|_{L^2}$,

(4) $\left\|\partial_t \partial_x^{k} \mathcal{H}(t) * \psi\right\|_{L^2} \le C_{q,k} (1+t)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}} \|\psi\|_{L^q} + C_l (1+t)^{-\frac{l}{4}} \left\|\partial_x^{k+l}\psi\right\|_{L^2}$,

where $C_{q,k}$ is a positive constant depending only on q and k , C_l is a positive constant depending only on l .

By some modification of the theorem 2.7 in the paper of Liu-Ueda [1], we have the following

Lemma 2 (see [7]**).** Let $s \ge 0$ be an integer. Suppose that $u_0 \in H^{s+2}(R^n) \bigcap L^1(R^n)$, $u_1 \in H^s(R^n) \bigcap L^1(R^n)$. Put $I_1 := \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s} + \|(u_0, u_1)\|_{L^1}$. Then the following decay estimate holds for (2.1):

If $s \ge \left[\frac{n+1}{2}\right]$, then

$$\left\|\partial_x^{k+2}u(t)\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\left(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}+\|(u_0,u_1)\|_{L^1}\right),$$

$$\left\|\partial_x^{k}u_t(t)\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\left(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}+\|(u_0,u_1)\|_{L^1}\right).$$

Where $k \ge 0$, $\sigma(k,n) \le s$.

Proof. Let k and m are non-negative integers.

By virtue of (2.1) and (1) (3) in Lemma 1 with $q=1$ and $C=\max\left(\sqrt{2}C_{q,k},\sqrt{2}C_l\right)$, we have

$$\begin{aligned}\left\|\partial_x^{k+m+2}u(t)\right\|_{L^2} &\le \left\|\partial_x^{k+m+2}\mathcal{G}(t)*u_0(t)\right\|_{L^2}+\left\|\partial_x^{k+m+2}\mathcal{H}(t)*u_1(t)\right\|_{L^2}\\
&\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|u_0\|_{L^1}+C(1+t)^{-\frac{l_1}{4}}\left\|\partial_x^{k+m+l_1+2}u_0\right\|_{L^2}\\
&+C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|u_1\|_{L^1}+C(1+t)^{-\frac{l_2}{4}}\left\|\partial_x^{k+m+l_2}u_1\right\|_{L^2}\\
&\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(u_0,u_1)\|_{L^1}+C(1+t)^{-\frac{l_1}{4}}\left\|\partial_x^{k+m+l_1+2}u_0\right\|_{L^2}\\
&+C(1+t)^{-\frac{l_2}{4}}\left\|\partial_x^{k+m+l_2}u_1\right\|_{L^2}.\end{aligned}$$

Here $l_1 \ge 0$, $l_2 \ge 0$, $k+m+l_1 \le s$, $k+m+l_2 \le s$.

Choose the smallest integers l_1 and l_2 satisfying

$$\frac{l_1}{4} \ge \frac{n}{8}+\frac{k}{4}, \qquad \frac{l_2}{4} \ge \frac{n}{8}+\frac{k}{4}.$$

It yields that

$$l_1 \ge \left[\frac{n+1}{2}\right]+k, \qquad l_2 \ge \left[\frac{n+1}{2}\right]+k.$$

Take $l_1=l_2=\sigma(k,n)-k$, where $\sigma(k,n)$ is defined by (1.3). Hence it holds with m satisfying $0 \le m \le s-\sigma(k,n)$. Taking sum with m, we get that

$$\left\|\partial_x^{k+2}u(t)\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\left(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}+\|(u_0,u_1)\|_{L^1}\right).$$

Similarly, we can prove that

$$\left\|\partial_x^{k}u_t(t)\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\left(\|u_0\|_{H^{s+2}}+\|u_1\|_{H^s}+\|(u_0,u_1)\|_{L^1}\right).$$

So far we complete the proof of Lemma 2.

Lemma 3 (see [8]**).** Assume that p,q,r,k,α and β are integers, $1 \le p,q,r \le \infty$, $\frac{1}{p}=\frac{1}{q}+\frac{1}{r}$ and $k \ge 0$, $\alpha \ge 1$, $\beta \ge 1$. Then

$$\left\|\partial_x^k\left(u^\alpha v^\beta\right)\right\|_{L^p} \le C\|u\|_{L^\infty}^{\alpha-1}\|v\|_{L^\infty}^{\beta-1}\left(\|u\|_{L^q}\left\|\partial_x^k v\right\|_{L^r}+\|v\|_{L^q}\left\|\partial_x^k u\right\|_{L^r}\right).$$

Proposition 1. Let $a \ge 0$ and $b \ge 0$ be real numbers. If $a+b \ge 1$, then there exists $C>0$ (independent of $t>0$) such that the following estimate holds,

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}d\tau \le C.$$

Proof. Directly computation!

Now we come to prove theorem 1.1, we define

$$X := \left\{u \in C^0([0,\infty);H^{s+2}((R^n)))\cap C^1([0,\infty);H^s((R^n))), \|u\|_X < \infty\right\}$$

with

$$\|u\|_X := \sum_{\{k;\sigma(k,n)\le s\}} \sup_{t\ge 0}(1+t)^{\frac{n}{8}+\frac{k}{4}} \left\|\partial_x^{k+2}u(t)\right\|_{H^{s-\sigma(k,n)}} + \sum_{\{k;\sigma(k,n)\le s\}} \sup_{t\ge 0}(1+t)^{\frac{n}{8}+\frac{k}{4}} \left\|\partial_x^{k}u_t(t)\right\|_{H^{s-\sigma(k,n)}}.$$

Denote

$$U := (\partial_x^2 u, u_t),$$
$$S_R := \left\{u\in X; \|u\|_X \le R\right\}, \forall R>0,$$
$$\phi[u](t) := \mathcal{G}(t)*u_0 + \mathcal{H}(t)*u_1 + \int_0^t \mathcal{H}(t-\tau)*f(U)(\tau)d\tau,$$
$$\phi_0(t) := \mathcal{G}(t)*u_0 + \mathcal{H}(t)*u_1.$$

Noticing that $f(v)=O(|v|^\alpha)$ and applying lemma 3, we have the following inequalities for $k\ge 0$:

$$\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} \le C\left\|(V,W)(\tau)\right\|_{L^\infty}^{\alpha-2}\Big(\left\|(V,W)(\tau)\right\|_{L^2}\left\|\partial_x^k(V-W)(\tau)\right\|_{L^2} + \left\|\partial_x^k(V,W)(\tau)\right\|_{L^2}\left\|(V-W)(\tau)\right\|_{L^2}\Big), \tag{2.3}$$

$$\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} \le C\left\|(V,W)(\tau)\right\|_{L^\infty}^{\alpha-2}\Big(\left\|(V,W)(\tau)\right\|_{L^\infty}\left\|\partial_x^k(V-W)(\tau)\right\|_{L^2} + \left\|\partial_x^k(V,W)(\tau)\right\|_{L^2}\left\|(V-W)(\tau)\right\|_{L^\infty}\Big). \tag{2.4}$$

Now we will prove that $u\to\phi[u]$ is a contraction mapping on S_R for some $R>0$. We divide the proof into the following four steps.

Step 1. First we give an estimate on the $L^\infty-$norm by using the Gagliardo-Nirenberg inequality which will be regularly used in the succeeding computation.

Set $s_0=\left[\frac{n}{2}\right]+1$, $\theta_n=\frac{n}{2s_0}$. Take $u\in X$, by using the Gagliardo-Nirenberg inequality, we have

$$\|U(t)\|_{L^\infty} \le C\|U(t)\|_{L^2}^{1-\theta_n}\left\|\partial_x^{s_0}U(t)\right\|_{L^2}^{\theta_n}.$$

When $n=1$, since $s\ge 3$, i.e. $s-\sigma(1,1)\ge 0$, hence we get $\|U(t)\|_{L^2}\le C(1+t)^{-\frac{1}{8}}\|u\|_X$ and $\left\|\partial_x^{s_0}U(t)\right\|_{L^2}\le C(1+t)^{-\frac{3}{8}}\|u\|_X$ by the definition of $\|u\|_X$. It yields $\|U(t)\|_{L^\infty}\le C(1+t)^{-\frac{1}{4}}\|u\|_X$.

When $n\ge 2$, since $s\ge n+1$ and $\frac{[n]}{2}+\frac{[n+1]}{2}=n$, i.e. $s-\sigma(0,n)\ge s_0$, we obtain $\|U(t)\|_{L^2}\le C(1+t)^{-\frac{n}{8}}\|u\|_X$ and $\left\|\partial_x^{s_0}U(t)\right\|_{L^2}\le C(1+t)^{-\frac{n}{8}}\|u\|_X$ by the definition of $\|u\|_X$. Then we have $\|U(t)\|_{L^\infty}\le C(1+t)^{-\frac{n}{8}}\|u\|_X$.

Denote

$$d_n = \begin{cases}\frac{1}{4}, & n=1,\\ \frac{n}{8}, & n\ge 2.\end{cases}$$

Therefore we have

$$\|U(t)\|_{L^\infty} \le C\|u\|_X (1+t)^{-d_n}. \tag{2.5}$$

Step 2. Take any $v, w \in X$, and denote $V := (\partial_x^2 v, v_t)$, $W := (\partial_x^2 w, w_t)$. Then we have

$$\phi[v](t) - \phi[w](t) := \int_0^t \mathcal{H}(t-\tau) * \big(f(V) - f(W)\big)(\tau) d\tau.$$

Assume that k, m and l are non-negative integers and $s \ge \sigma(k,n)$, then we have that

$$\begin{aligned} \left\|\partial_x^{k+m+2}(\phi[v](t) - \phi[w](t))\right\|_{L^2} &\le C\left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t\right)\left\|\partial_x^{k+m+2}\mathcal{H}(t-\tau) * \big(f(V) - f(W)\big)(\tau)\right\|_{L^2} d\tau \\ &=: I_1 + I_2. \end{aligned} \tag{2.6}$$

By applying Lemma 1 (3) with $q = 1$ and $C = \max(C_{q,k}, C_l)$, we have that

$$\begin{aligned} I_1 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\big(f(V) - f(W)\big)(\tau)\right\|_{L^1} d\tau \\ &\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l}{4}}\left\|\partial_x^{k+m+l}\big(f(V) - f(W)\big)(\tau)\right\|_{L^2} d\tau \\ &=: I_{11} + I_{12}. \end{aligned} \tag{2.7}$$

By applying (2.3) with $k = 0$, we obtain that

$$\left\|\big(f(V) - f(W)\big)(\tau)\right\|_{L^1} \le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\|(V,W)(\tau)\|_{L^2}\|(V-W)(\tau)\|_{L^2}.$$

By virtue of (2.5), we get that

$$\begin{aligned} \left\|\big(f(V) - f(W)\big)(\tau)\right\|_{L^1} &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X (1+\tau)^{-d_n(\alpha-2)-\frac{n}{4}} \\ &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X \begin{cases} (1+\tau)^{-\frac{\alpha-1}{4}}, & n = 1, \\ (1+\tau)^{-\frac{n\alpha}{8}}, & n \ge 2. \end{cases} \end{aligned} \tag{2.8}$$

In view of Assumption [B], we have that $\frac{\alpha-1}{4} > 1$ and $\frac{n\alpha}{8} > 1$. Therefore, by Proposition 1, we obtain that

$$I_{11} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X \tag{2.9}$$

for $m \ge 0$.

Similarly, if $k+m+l \le s$, by using (2.4) with k replaced by $k+m+l$, we obtain that

$$\begin{aligned} \left\|\partial_x^{k+m+l}\big(f(V) - f(W)\big)(\tau)\right\|_{L^2} &\le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(\|(V,W)(\tau)\|_{L^\infty}\left\|\partial_x^{k+m+l}(V-W)(\tau)\right\|_{L^2} \\ &\quad + \left\|\partial_x^{k+m+l}(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^\infty}\Big). \end{aligned}$$

By virtue of (2.5), it holds that

$$\begin{aligned} \left\|\partial_x^{k+m+l}\big(f(V) - f(W)\big)(\tau)\right\|_{L^2} &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X (1+\tau)^{-d_n(\alpha-1)-\frac{n}{8}} \\ &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X \begin{cases} (1+\tau)^{-\frac{\alpha-1}{4}}, & n = 1, \\ (1+\tau)^{-\frac{n\alpha}{8}}, & n \ge 2. \end{cases} \end{aligned} \tag{2.10}$$

Take $l = \sigma(k,n) - k$, then $\frac{l}{4} \ge \frac{n}{8} + \frac{k}{4}$. By virtue of Assumption [B] and Proposition 1, we have that

$$I_{12} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X \tag{2.11}$$

with $0 \le m \le s-\sigma(k,n)$.

Put the estimates for I_{11} and I_{12} in (2.7), then we obtain

$$I_1 \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X \tag{2.12}$$

with $0 \le m \le s-\sigma(k,n)$.

Also, by using Lemma 1 (3) with $q=1$ and $C=\max(C_{q,k},C_l)$, we have that

$$\begin{aligned} I_2 &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}} \left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} d\tau \\ &+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{l}{4}} \left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} d\tau \\ &=: I_{21}+I_{22}. \end{aligned} \tag{2.13}$$

By using (2.3), we have

$$\begin{aligned} \left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} \le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(&\|(V,W)(\tau)\|_{L^2}\left\|\partial_x^k(V-W)(\tau)\right\|_{L^2} \\ &+\left\|\partial_x^k(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^2}\Big). \end{aligned}$$

Hence we obtain that

$$\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} \le C\|(V,W)(\tau)\|_X^{\alpha-1}\|(V-W)(\tau)\|_X (1+t)^{-d_n(\alpha-2)-\frac{n}{4}-\frac{k}{4}}.$$

It holds that

$$I_{21} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

with $0 \le m \le s-\sigma(k,n)$.

Similarly, by using (2.4) with k replaced by $k+m+l$, we obtain that

$$\begin{aligned} \left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} \le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(&\|(V,W)(\tau)\|_{L^\infty}\left\|\partial_x^{k+m+l}(V-W)(\tau)\right\|_{L^2} \\ &+\left\|\partial_x^{k+m+l}(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^\infty}\Big). \end{aligned}$$

By applying (2.5), it holds that

$$\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} \le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X (1+\tau)^{-d_n(\alpha-2)-\frac{n}{4}-\frac{k}{4}}$$

with $0 \le m \le s-\sigma(k,n)$.

It yields that

$$I_{22} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

for $0 \le m \le s-\sigma(k,n)$.

Put the estimates for I_{21} and I_{22} in (2.13), then we obtain

$$I_2 \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X \tag{2.14}$$

for $0 \le m \le s-\sigma(k,n)$.

Combining the estimates (2.6), (2.12) and (2.14), we obtain that

$$\left\|\partial_x^{k+m+2}\left(\phi[v]-\phi[w]\right)(t)\right\|_{L^2} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

Taking sum with $0 \le m \le s-\sigma(k,n)$, we have that

$$\left\|\partial_x^{k+2}(\phi[v](t)-\phi[w](t))\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

It holds that

$$\sup_{t\ge 0}(1+\tau)^{\frac{n}{8}+\frac{k}{4}}\left\|\partial_x^{k+2}(\phi[v](t)-\phi[w](t))\right\|_{H^{s-\sigma(k,n)}} \le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X . \tag{2.15}$$

Step 3. Assume that k, m and l are non-negative integers and $\sigma(k,n)\le s$, then we get that

$$\begin{aligned}\left\|\partial_x^{k+m}\partial_t(\phi[v](t)-\phi[w](t))\right\|_{L^2} &\le C\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right)\left\|\partial_x^{k+m}\mathcal{H}_t(t-\tau)*\left(f(V)-f(W)\right)(\tau)\right\|_{L^2}d\tau \\ &=: I_3+I_4.\end{aligned} \tag{2.16}$$

By applying Lemma 1 (4) with $q=1$ and $C=\max(C_{q,k},C_l)$, we have that

$$\begin{aligned} I_3 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\left(f(V)-f(W)\right)(\tau)\right\|_{L^1}d\tau \\ &+C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l}{4}}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2}d\tau \\ &=: I_{31}+I_{32}.\end{aligned} \tag{2.17}$$

Similar to the argument of (2.9), we obtain

$$I_{31}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

At the same time, similar to the estimate of I_{12}, we have

$$I_{32}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

Put the estimates for I_{31} and I_{32} in (2.17), then it holds that

$$I_3\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X \tag{2.18}$$

Similarly, by applying Lemma 1 (4) with $q=1$ and $C=\max(C_{q,k},C_l)$, we have that

$$\begin{aligned} I_4 &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1}d\tau \\ &+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{l}{4}}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2}d\tau \\ &=: I_{41}+I_{42}.\end{aligned} \tag{2.19}$$

Since

$$\begin{aligned}\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} &\le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(\|(V,W)(\tau)\|_{L^2}\left\|\partial_x^k(V-W)(\tau)\right\|_{L^2} \\ &+\left\|\partial_x^k(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^2}\Big),\end{aligned}$$

We have

$$\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1}\le C\|(V,W)(\tau)\|_X^{\alpha-1}\|(V-W)(\tau)\|_X(1+t)^{-d_n(\alpha-2)-\frac{n}{4}-\frac{k}{4}}.$$

It yields that

$$I_{41}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

Similar to the estimate of I_{22}, we obtain

$$I_{42}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

Put the estimates for I_{41} and I_{42} in (2.19), then it holds that

$$I_4\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X . \tag{2.20}$$

Combining the estimates (2.16), (2.18) and (2.20), we get that

$$\left\|\partial_x^{k+m}\partial_t\left(\phi[v]-\phi[w]\right)(t)\right\|_{L^2} \le C(1+\tau)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

Taking sum with $0 \le m \le s-\sigma(k,n)$, we have that

$$\left\|\partial_x^k\partial_t(\phi[v](t)-\phi[w](t))\right\|_{H^{s-\sigma(k,n)}} \le C(1+\tau)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

It yields that

$$\sup_{t\ge 0}(1+\tau)^{\frac{n}{8}+\frac{k}{4}}\left\|\partial_x^k\partial_t(\phi[v](t)-\phi[w](t))\right\|_{H^{s-\sigma(k,n)}} \le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X . \qquad (2.21)$$

Step 4. Combining the estimates (2.15) and (2.21), we get that

$$\left\|\left(\phi[v]-\phi[w]\right)\right\|_X \le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X .$$

So far we proved that $\left\|\left(\phi[v]-\phi[w]\right)\right\|_X \le C_1R^{\alpha-1}\|(v-w)\|_X$, if $v,w\in S_R$. On the other hand, since $\phi[0](t)=\phi_0(t)=u_s(t)$, by lemma 2 we know that $\|\phi_0\|_X \le C_2I_1$, if I_1 is suitably small. Taking $R=2C_2I_1$, if I_1 is suitably small such that $R<1, C_1R\le\frac{1}{2}$, then we obtain that

$$\left\|\left(\phi[v]-\phi[w]\right)\right\|_X \le \frac{1}{2}\|(v-w)\|_X .$$

It yields that

$$\|\phi[v]\|_X \le \|\phi_0\|_X + \frac{1}{2}\|v\|_X \le C_2I_1 + \frac{1}{2}R \le R .$$

Hence $v\to\phi[v]$ is a contraction mapping on S_R, and by the fixed point theorem there exists a unique $u\in S_R$ satisfying $\phi[u]=u$, and it is the solution to the semilinear problem (1.1)--(1.2) satisfying the decay estimates (1.4) and (1.5). Thus we complete the proof of Theorem 1.1.

3 Conclusion

In this paper, we studied the initial-value problem of a type of semilinear plate equation. We obtained the global existence and decay estimates for solutions to this equation in terms of contraction theorem. Our results can be regarded as a generalization of the results in the paper [7].

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