Algebraic points of degree at most 5 on the affine curve $y^2 = x^5 - 243$

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Abstract. In this work, we determine the set of algebraic points of degree at most 5 on the affine curve $y^2 = x^5 - 243$. This result extends a result of J.TH Mulholland who described in [4] the set of $\mathbb{Q}-$rational points i.e the set of points of degree one over $\mathbb{Q}$ on this curve.

Keywords : Planes curves - Degree of algebraic points - Rationals points - Algebraic extensions - Linear system - Jacobian

1 Introduction

Let $\mathcal{C}$ be a smooth algebraic curve defined over $\mathbb{Q}$. Let K be a numbers field. We note by $\mathcal{C}$(K) the set of points of C with coordinates in K and $\bigcup_{[K:\mathbb{Q}]\leq d} \mathcal{C}(K)$ the set of points of $\mathcal{C}$ with coordinates in K of degree at most d over $\mathbb{Q}$.
The goal is to determine the set of algebraic points of given degree over $\mathbb{Q}$ on the curve $\mathcal{C}$ given by the affine equation

$$y^2 = x^5 - 243 \tag{1}$$

We denote by J the jacobian of $\mathcal{C}$ and by $j(P)$ the class $[P - \infty]$ of $P - \infty$, that is to say that j is the Jacobian diving $\mathcal{C} \longrightarrow J(\mathbb{Q})$. The Mordell-Weil group $J(\mathbb{Q})$ of rational points of the jacobian is a finite set (refer to [4]).
We denote by : $P = (3, 0)$ and $\infty = (0, 1, 0)$. In [4], J.TH Mulholland gave a description of the rational points on $\mathbb{Q}$ on this curve. This description is as follows :
Proposition. The rational points on $\mathcal{C}$ are given by $\mathcal{C}(\mathbb{Q}) = \{\infty, P\}$.
In this note, we determine the set of algebraic points of degree at most five on the affine curve $y^2 = x^5 - 243$.
Our essential tools are :

- The Mordell-Weil group $J(\mathbb{Q})$ of rational points of the jacobian (refer to [4]),
- Abel Jacobi's theorem (refer to [3]),
- Linear systems on the curve $\mathcal{C}$.

Our main result is given by the following theorem :

Theorem.

1. The set of quadratic points on $\mathcal{C}$ are given by

$$\mathcal{S} = \left\{ \left(\alpha, \pm\sqrt{\alpha^5 - 243} \right),\ \alpha \in \mathbb{Q} \right\}$$

2. The set of cubic points on $\mathcal{C}$ is empty.

3. The set of quartic points on $\mathcal{C}$ are given by $\mathcal{C}_1 \cup \mathcal{C}_2$ with

$$\mathcal{C}_1 = \left\{ \left(x, \pm\sqrt{\alpha^5 - 243} \right) \mid x \in \mathbb{Q},\ [\mathbb{Q}(x) : \mathbb{Q}] = 2 \right\}$$

$$\mathcal{C}_2 = \left\{ \begin{array}{l} (x, (x-3)(\lambda_1 + \lambda_2(x+3))) \ \mid \lambda_1, \lambda_2 \in \mathbb{Q} \text{ and } x \text{ root of} \\ \\ A(x) = x^4 + 3x^3 + 9x^2 + 27x + 81 - (x-3)(\lambda_1 + \lambda_2(x+1))^2 \end{array} \right\}$$

4. The set of quintic points on $\mathcal{C}$ are given by $\mathcal{A}_1 \cup \mathcal{A}_2$ with

$$\mathcal{A}_1 = \left\{ \begin{array}{l} (x, \alpha_1 + \alpha_2 x + \alpha_3 x^2) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}^\star \text{ and } x \text{ root of} \\ \\ B(x) = x^5 - \alpha_3^2 x^4 - 2\alpha_2\alpha_3 x^3 - (\alpha_2^2 + 2\alpha_1\alpha_2)x^2 - 2\alpha_1\alpha_2 x - (\alpha_1^2 + 243) \end{array} \right\}$$

$$\mathcal{A}_2 = \left\{ \begin{array}{l} (x, (x-3)\left[n_1 + n_2(x+3) + n_3(x^2+3x+9)\right]) \mid n_1, n_2, n_3 \in \mathbb{Q}^\star \text{ and } x \text{ root of} \\ \\ C(x) = (x-3)\left(n_1 + n_2(x+3) + n_3(x^2+3x+9)\right)^2 - (x^4 + 3x^3 + 9x^2 + 27x + 81) \end{array} \right\}$$

2 Auxiliary results

For a divisor D on $\mathcal{C}$, we note $\mathcal{L}(D)$ the $\overline{\mathbb{Q}}-$ -vector space of rational functions F defined on $\mathbb{Q}$ such that $F = 0$ or $div(F) \geq -D$; $l(D)$ designates $\overline{Q}-$ of $\mathcal{L}(D)$.
In [4], the Mordell-Weil group $J(\mathbb{Q})$ of $\mathcal{C}$ is isomorph to $\mathbb{Z}/2\mathbb{Z}$ and $\mathcal{C}$ is a hyperelliptic curve of genus $g = 2$. Let x, y be two rational functions on $\mathbb{Q}$ defined as follow :

$$x(X,Y,Z) = \frac{X}{Z} \ et \ y(X,Y,Z) = \frac{Y}{Z}$$

The projective equation of $\mathcal{C}$ is

$$\mathcal{C} : Y^2Z^3 = X^5 - 243Z^5 = X^5 - (3Z)^5 \tag{2}$$

We denote by $\eta_1 = e^{\frac{i\Pi}{2}}$ and let's put $A_k = (0,\ 9\sqrt{3}\eta_1^{2k+1})$ for $k \in \{0, 1\}$.
We denote by $\eta_2 = e^{\frac{2i\Pi}{5}}$ and let's put $B_k = (3\eta_2^k,\ 0)$ for $k \in \{0, 1, 2, 3, 4\}$.
Let us designate by $\mathcal{D}.\mathcal{C}$ the intersection cycle of algebraic curve $\mathcal{D}$ defined on $\mathbb{Q}$ and $\mathcal{C}$.

Lemma 1.

- $div(x-3) = 2P - 2\infty$
- $div(y) = B_0 + B_1 + B_2 + B_3 + B_4 - 5\infty$
- $div(x) = A_0 + A_1 - 2\infty$

Proof $\mathcal{C} : Y^2Z^3 = X^5 - (3Z)^5$ (projective equation)

- $div(x-3) = (X - 3Z = 0).\mathcal{C} - (Z = 0).\mathcal{C}$
  For $X = 3Z$, we have $Y^2 = 0$ with $Z = 1$ or $Z^3 = 0$ with $Y = 1$. We obtain the point $P = (3, 0, 1)$ with multiplicity 2 and the point $\infty = (0, 1, 0)$ with multiplicity 3. Hence $(X - 3Z = 0).\mathcal{C} = 2P + 3\infty$ $(*)$.
  Even if $Z = 0$, then $X^5 = 0$; and for $Y = 1$, we have the point $\infty = (0, 1, 0)$ with multiplicity 5. Hence $(Z = 0).\mathcal{C} = 5\infty$ $(**)$.
  The relations $(*)$ and $(**)$ imply that $div(x-3) = 2P - 2\infty$.
- Similarly we show that $div(y) = B_0 + B_1 + B_2 + B_3 + B_4 - 5\infty$ and $div(x) = A_0 + A_1 - 2\infty$.

consequence of lemma 1 : $2j(P) = 0$.

Lemma 2.

- $\mathcal{L}(\infty) = \langle 1 \rangle$
- $\mathcal{L}(2\infty) = \langle 1,\ x \rangle = \mathcal{L}(3\infty)$
- $\mathcal{L}(4\infty) = \langle 1,\ x,\ x^2 \rangle$
- $\mathcal{L}(5\infty) = \langle 1,\ x,\ x^2,\ y \rangle$
- $\mathcal{L}(6\infty) = \langle 1,\ x,\ x^2,\ y,\ x^3 \rangle$

Proof Results from lemma 1 and from the fact that according to the theorem of Riemann-Roch we have $l(m\infty) = m - 1$ as soon as $m \geq 3$.

Lemma 3. $J(\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z} = \langle [P - \infty] \rangle = \{a\,[P - \infty]\,, a \in \{0, 1\}\}$.

Proof Refer to [4].

3 Proof of theorem

3.1 Quadratic points (algebraic points of degree 2) on $\mathcal{C}$

The set of quadratic points on $\mathcal{C}$ are given by

$$\mathcal{S} = \left\{ \left(\alpha, \pm\sqrt{\alpha^5 - 243} \right),\ \alpha \in \mathbb{Q} \right\}$$

Proof : Given $R \in \mathcal{C}(\bar{\mathbb{Q}})$ with $[\mathbb{Q}[R] : \mathbb{Q}] = 2$. Note that R_1, R_2 are the Galois conjugates of R. Let's work with $t = [R_1 + R_2 - 2\infty] \in J(\mathbb{Q})$, according to lemma 3 we have $t = a\,[P - \infty]\,,\ 0 \leq a \leq 1$. So we have $[R_1 + R_2 - 2\infty] = a\,[P - \infty]$.

For $a=0$, we have $[R_1+R_2-2\infty]=0$; then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F)=R_1+R_2-2\infty$, then $F\in\mathcal{L}(2\infty)$ and according to lemma 2 we have $F(x,y)=a_1+a_2x$ with $a_2\neq 0$ otherwise one of the R_i should be ∞.
For the points R_i, we have $a_1+a_2x=0$ hence $x=-\dfrac{a_1}{a_2}=\alpha\in\mathbb{Q}$.
By replacing x by α in (1), we have :

$$y^2=\alpha^5-243 \tag{3}$$

and then we have

$$y=\pm\sqrt{\alpha^5-243} \tag{4}$$

So we find a family of quadratic points

$$\mathcal{S}=\left\{\left(\alpha,\pm\sqrt{\alpha^5-243}\right),\ \alpha\in\mathbb{Q}\right\}$$

For $a=1$, we have $[R_1+R_2+P-3\infty]=0$, then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F)=R_1+R_2+P-3\infty$, then $F\in\mathcal{L}(3\infty)$ and as $\mathcal{L}(2\infty)=\mathcal{L}(3\infty)$ then one of the R_i should be equal to ∞, we obtain a contradiction.
Conclusion : The set of quadratic points on $\mathcal{C}$ are given by

$$\mathcal{S}=\left\{\left(\alpha,\pm\sqrt{\alpha^5-243}\right),\ \alpha\in\mathbb{Q}\right\}$$

3.2 Cubic points (algebraic points of degree 3) on $\mathcal{C}$

There are no cubic points on $\mathcal{C}$.

Proof : Given $R\in\mathcal{C}(\bar{\mathbb{Q}})$ with $[\mathbb{Q}[R]:\mathbb{Q}]=3$. Note that R_1,R_2,R_3 are the Galois conjugates of R. Let's work with $t=[R_1+R_2+R_3-3\infty]\in J(\mathbb{Q})$, according to lemma 3 we have $t=a\,[P-\infty]\,,\ 0\leq a\leq 1$. So we have $[R_1+R_2+R_3-3\infty]=a\,[P-\infty]$.

For $a=0$, we have $[R_1+R_2+R_3-3\infty]=0$; then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F)=R_1+R_2+R_3-3\infty$, then $F\in\mathcal{L}(3\infty)$ and as $\mathcal{L}(2\infty)=\mathcal{L}(3\infty)$ then one of the R_i should be equal to ∞, we obtain a contradiction.

For $a=1$, we have $[R_1+R_2+R_3+P-4\infty]=0$, then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F)=R_1+R_2+R_3+P-4\infty$, then $F\in\mathcal{L}(4\infty)$, then $F\in\mathcal{L}(2\infty)$ and according to lemma 2 we have $F(x,y)=a_1+a_2x+a_3x^2\ with\ a_3\neq 0$ otherwise one of the R_i should be ∞.
For the point P, we have $a_1+3a_2+9a_3=0$, so $a_1=-3a_2-9a_3$ and replacing a_1 with its expression in $F(x,y)$ we have :

$$F(x,y)=-3a_2-9a_3+a_2x+a_3x^2 \tag{5}$$

$$F(x,y)=a_2\,(x-3)+a_3\left(x^2-9\right) \tag{6}$$

$$F(x,y)=(x-3)\,[a_2+a_3(x+3)] \tag{7}$$

For the points R_i, we have $(x-3)\,[a_2+a_3(x+3)]=0$, then $x\in\mathbb{Q}$ and therefore the R_i should be of degree ≤ 2.
Conclusion : The set of cubic points on $\mathcal{C}$ is empty.

3.3 Quartic points (algebraic points of degree 4) on $\mathcal{C}$

The set of quartic points on $\mathcal{C}$ are given by $\mathcal{C}_1 \cup \mathcal{C}_2$ with

$$\mathcal{C}_1 = \left\{ \left(x, \pm\sqrt{\alpha^5 - 243} \right) \mid x \in \mathbb{Q},\ [\mathbb{Q}(x) : \mathbb{Q}] = 2 \right\}$$

$$\mathcal{C}_2 = \left\{ \begin{array}{l} (x, (x-3)(\lambda_1 + \lambda_2(x+3))) \mid \lambda_1, \lambda_2 \in \mathbb{Q} \text{ and } x \text{ root of} \\ \\ A(x) = x^4 + 3x^3 + 9x^2 + 27x + 81 - (x-3)(\lambda_1 + \lambda_2(x+1))^2 \end{array} \right\}$$

Proof : Given $R \in \mathcal{C}(\bar{\mathbb{Q}})$ with $[\mathbb{Q}[R] : \mathbb{Q}] = 4$. Note that R_1, R_2, R_3, R_4 are the Galois conjugates of R. Let's work with $t = [R_1 + R_2 + R_3 + R_4 - 4\infty] \in J(\mathbb{Q})$, according to lemma 3 we have $t = a\,[P - \infty]\,,\ 0 \leq a \leq 1$. So we have $[R_1 + R_2 + R_3 + R_4 - 4\infty] = a\,[P - \infty]$. We have the following two cases :
For $a = 0$, we have $[R_1 + R_2 + R_3 + R_4 - 4\infty] = 0$; then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F) = R_1 + R_2 + R_3 + R_4 - 4\infty$, then $F \in \mathcal{L}(4\infty)$ and according to lemma 2 we have $F(x, y) = a_1 + a_2 x + a_3 x^2 \ with\ a_3 \neq 0$ otherwise one of the R_i should be ∞.
For the points R_i, we have : $a_1 + a_2 x + a_3 x^2$; the relation $y^2 = x^5 - 243$ gives

$$y = \pm\sqrt{x^5 - 243}. \tag{8}$$

We find a family of quartic points

$$\mathcal{C}_1 = \left\{ \left(x, \pm\sqrt{x^5 - 243} \right) \mid x \in \mathbb{Q},\ [\mathbb{Q}(x) : \mathbb{Q}] = 2 \right\}$$

For $a = 1$, we have $[R_1 + R_2 + R_3 + R_4 + P - 5\infty] = 0$; then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F) = R_1 + R_2 + R_3 + R_4 + P - 5\infty$, then $F \in \mathcal{L}(5\infty)$ and according to lemma 2 we have $F(x, y) = a_1 + a_2 x + a_3 x^2 + a_4 y \ with\ a_4 \neq 0$.
For the point P, we have : $a_1 + 3a_2 + 9a_3 = 0$; $a_1 + 3a_2 + 9a_3 = 0$ so $a_1 = -3a_2 - 9a_3$ and replacing a_1 with its expression in $F(x, y)$ we have :

$$F(x, y) = -3a_2 - 9a_3 + a_2 x + a_3 x^2 + a_4 y \tag{9}$$

$$F(x, y) = a_2 (x - 3) + a_3 \left(x^2 - 9 \right) + a_4 y \tag{10}$$

$$F(x, y) = (x - 3)(a_2 + a_3(x + 3)) + a_4 y \tag{11}$$

For the points R_i we have $(x - 3)(a_2 + a_3(x + 3)) + a_4 y = 0$, so y is of the form $y = (x - 3)(\lambda_1 + \lambda_2(x + 3))$ with $\lambda_1, \lambda_2 \in \mathbb{Q}$.
The relation $y^2 = x^5 - 243 \Leftrightarrow (x - 3)^2 (\lambda_1 + \lambda_2(x + 3))^2 = x^5 - 243$

$$(x - 3)^2 (\lambda_1 + \lambda_2(x + 3))^2 = (x - 3)(x^4 + 3x^3 + 9x^2 + 27x + 81) \tag{12}$$

Simplifying by $x - 3$ and expanding we get

$$(x - 3)(\lambda_1 + \lambda_2(x + 1))^2 = x^4 + 3x^3 + 9x^2 + 27x + 81 \tag{13}$$

$$x^4 + 3x^3 + 9x^2 + 27x + 81 - (x - 3)(\lambda_1 + \lambda_2(x + 1))^2 = 0 \tag{14}$$

We find a family of quartic points

$$\mathcal{C}_2 = \left\{ \begin{array}{l} (x, (x-3)(\lambda_1 + \lambda_2(x+3))) \ \mid \lambda_1, \lambda_2 \in \mathbb{Q} \text{ and } x \text{ root of} \\ \\ A(x) = x^4 + 3x^3 + 9x^2 + 27x + 81 - (x-3)(\lambda_1 + \lambda_2(x+1))^2 \end{array} \right\}$$

Conclusion : The set of quartic points on $\mathcal{C}$ are given by $\mathcal{C}_1 \cup \mathcal{C}_2$.

3.4 Quintic points (algebraic points of degree 5) on $\mathcal{C}$

The set of quintic points on $\mathcal{C}$ are given by $\mathcal{A}_1 \cup \mathcal{A}_2$ with

$$\mathcal{A}_1 = \left\{ \begin{array}{l} (x, \alpha_1 + \alpha_2 x + \alpha_3 x^2) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}^\star \text{ and } x \text{ root of} \\ \\ B(x) = x^5 - \alpha_3^2 x^4 - 2\alpha_2\alpha_3 x^3 - (\alpha_2^2 + 2\alpha_1\alpha_2)x^2 - 2\alpha_1\alpha_2 x - (\alpha_1^2 + 243) \end{array} \right\}$$

$$\mathcal{A}_2 = \left\{ \begin{array}{l} (x, (x-3)[n_1 + n_2(x+3) + n_3(x^2+3x+9)]) \mid n_1, n_2, n_3 \in \mathbb{Q}^\star \text{ and } x \text{ root of} \\ \\ C(x) = (x-3)(n_1 + n_2(x+3) + n_3(x^2+3x+9))^2 - (x^4 + 3x^3 + 9x^2 + 27x + 81) \end{array} \right\}$$

Proof : Given $R \in \mathcal{C}(\bar{\mathbb{Q}})$ with $[\mathbb{Q}[R] : \mathbb{Q}] = 5$. Note that R_1, R_2, R_3, R_4, R_5 are the Galois conjugates of R. Let's work with $t = [R_1 + R_2 + R_3 + R_4 + R_5 - 5\infty] \in J(\mathbb{Q})$, according to lemma 3 we have $t = a\,[P - \infty]\,,\ 0 \le a \le 1$.
So we have $[R_1 + R_2 + R_3 + R_4 + R_5 - 5\infty] = a\,[P - \infty]$.
We have the following two cases :
For $a = 0$, we have $[R_1 + R_2 + R_3 + R_4 + R_5 - 5\infty] = 0$; then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F) = R_1 + R_2 + R_3 + R_4 + R_5 - 5\infty$, then $F \in \mathcal{L}(5\infty)$ and according to lemma 2 we have $F(x, y) = a_1 + a_2 x + a_3 x^2 + a_4 y \ with\ a_4 \neq 0$.
For the points R_i, we have : $a_1 + a_2 x + a_3 x^2 + a_4 y = 0$, so $y = \alpha_1 + \alpha_2 x + \alpha_3 x^2$ with $\alpha_1 = \frac{-a_1}{a_4}$, $\alpha_2 = \frac{-a_2}{a_4}$ and $\alpha_3 = \frac{-a_3}{a_4}$.
Replacing y by its expression in (1), we have

$$x^5 - 243 = \alpha_1^2 + \alpha_2^2 x + \alpha_3^2 x^4 + 2\alpha_1\alpha_2 x + 2\alpha_1\alpha_3 x^2 + 2\alpha_2\alpha_3 x^3 \tag{15}$$

$$3x^5 - \alpha_3^2 x^4 - 2\alpha_2\alpha_3 x^3 - (\alpha_2^2 + 2\alpha_1\alpha_2)x^2 - 2\alpha_1\alpha_2 x - (\alpha_1^2 + 243) = 0 \tag{16}$$

We find a family of quintic points

$$\mathcal{A}_1 = \left\{ \begin{array}{l} (x, \alpha_1 + \alpha_2 x + \alpha_3 x^2) \mid \alpha_1, \alpha_2, \alpha_3 \in \mathbb{Q}^\star \text{ and } x \text{ root of} \\ \\ B(x) = x^5 - \alpha_3^2 x^4 - 2\alpha_2\alpha_3 x^3 - (\alpha_2^2 + 2\alpha_1\alpha_2)x^2 - 2\alpha_1\alpha_2 x - (\alpha_1^2 + 243) \end{array} \right\}$$

For $a = 1$, we have $[R_1 + R_2 + R_3 + R_4 + R_5 + P - 6\infty] = 0$; then there exist a function F with coefficient in $\mathbb{Q}$ such that $div(F) = R_1 + R_2 + R_3 + R_4 + R_5 + P - 6\infty$, then $F \in \mathcal{L}(6\infty)$ and according to lemma 2 we have $F(x, y) = a_1 + a_2 x + a_3 x^2 + a_4 y + a_5 x^3 \ with\ a_5 \neq 0$.
For the points R_i, we have : $a_1 + 3a_2 + 9a_3 + 27a_5 = 0$, so $a_1 = -3a_2 - 9a_3 - 27a_5$ and Replacing a_1 by its expression in $F(x, y)$, we have :

$$F(x, y) = -3a_2 - 9a_3 - 27a_5 + a_2 x + a_3 x^2 + a_4 y + a_5 x^3 \tag{17}$$

$$F(x,y) = a_2(x-3) + a_3(x^2-9) + a_5(x^3-27) + a_4 y \tag{18}$$

For the points R_i, we have : $a_2(x-3)+a_3(x^2-9)+a_5(x^3-27)+a_4y=0$, so y is of the form $y = n_1(x-3)+n_2(x^2-9)+n_3(x^3-27)$ with $n_1, n_2, n_3 \in \mathbb{Q}^\star$.
Finally we have :

$$y = (x-3)\left(n_1 + n_2(x+3) + n_3(x^2+3x+9)\right) \tag{19}$$

Replacing y by its expression in (1), we have :

$$(x-3)^2\left(n_1 + n_2(x+3) + n_3(x^2+3x+9)\right)^2 = x^5 - 243 \tag{20}$$

$$(x-3)^2\left(n_1 + n_2(x+3) + n_3(x^2+3x+9)\right)^2 = (x-3)(x^4+3x^3+9x^2+27x+81) \tag{21}$$

Simplifying by $x-3$, we obtain

$$(x-3)\left(n_1 + n_2(x+3) + n_3(x^2+3x+9)\right)^2 - (x^4+3x^3+9x^2+27x+81) = 0 \tag{22}$$

Thus we find a family of quintic points

$$\mathcal{A}_2 = \left\{ \begin{array}{l} (x, (x-3)\,[n_1 + n_2(x+3) + n_3(x^2+3x+9)]) \mid n_1, n_2, n_3 \in \mathbb{Q}^\star \text{ and } x \text{ root of} \\ \\ C(x) = (x-3)\,(n_1 + n_2(x+3) + n_3(x^2+3x+9))^2 - (x^4+3x^3+9x^2+27x+81) \end{array} \right\}$$

Conclusion : The set of quintic points on $\mathcal{C}$ are given by $\mathcal{A}_1 \cup \mathcal{A}_2$.

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