Comparing the Performance of four Sinc Methods for Numerical Indefinite Integration

Abstract

In this paper, we compare the performance of four Sinc methods for the numerical approximation of indefinite integrals with algebraic or logarithmic end-point singularities. The first two quadrature formulas were proposed by Haber based on the sinc method, the third is Stenger's Single Exponential (SE) formula and Tanaka *et al.'s* Double Exponential (DE) sinc method completes the number. Furthermore, an application of the four quadrature formulas on numerical examples, reveals convergence to the exact solution by Tanaka *et al.'s* DE sinc method than by the other three formulas and Haber formula A is the fastest as revealed by the CPU time.

Keywords: Singularities, sinc function, indefinite integrals, quadrature

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1 Introduction

An indefinite integral of a function $f(x)$ is a function $F(x)$ whose derivative $F'(x) = f(x)$ on a certain interval of the x-axis [1]. We assume that $f(x)$ is analytic in a simply connected domain D [8] which we shall define shortly. When we talk of indefinite integrals in the parlance of Numerical Analysis, we mean integrals in which the upper limit of integration is a variable [4],

$$F(x) = \int_c^x f(u)\,\mathrm{d}u. \tag{1.1}$$

This paper will be restricted to computing the numerical indefinite integrals of functions in which the lower limit of integration $c = -1$, and in which the upper limit is x for $-1 < x < 1$, using the sinc method. A treatment of other indefinite integrals over such intervals as $(0, x)$, $0 < x < \infty$ and $(-\infty, x)$ using the double exponential formulas is given by Muhammad and Mori [33] and the references therein. Monegato and Scuderi [12] proposed numerical methods for computing a one-dimensional integral of functions having strong or weak singularities at the endpoints of the interval of integration or complex poles close to the domain of integration. They also computed a four-dimensional integral which arose from the 3 dimensional Galerkin boundary element methods applied to hypersingular boundary integral equations. Nevertheless, they did not use the sinc method. Keller [13] proposed a general method and algorithms for computing indefinite integrals of the form

$$I(x) = \int_0^x f(u)k(u)\mathrm{d}u \qquad (0 \le x \le x_{\max}), \tag{1.2}$$

where f is a smooth function, and k is a function that contains a singular factor or is rapidly oscillatory. While Keller and Wozny [14] presented the convergence and error estimates of the method in Kelly's study [13].

Okayama in [15] gave expressions for the error bounds of sinc quadrature and sinc indefinite integral on semi-infinite and finite intervals, while comparing the SE and DE formulas. Though in this present work, we did not discuss error bounds but the interval of integration is infinite. Numerical methods for the computation of indefinite integrals using the sinc method involve several constants which often times depend on the nature of the integrand, and that is why Okayama *et al.,* [16] revealed explicit forms of all constants in a computable form under the same assumptions of the existing theorems: the function to be approximated is analytic in a suitable region. They also improved some formulas to decrease their computational costs while numerical examples that confirm the theory were given. In the study [17], the authors investigated another Single Exponential (SE) formula obtained by replacing the transformation with Muhammad–Mori's SE transformation. The error bound was theoretically analyzed. Numerical comparisons of Stenger's SE formula with that of Muhammad–Mori's were given as well but without any comparison made with the DE formula. In [18], Hara and Okayama gave an error bound for Mohammed–Mori's DE formula and compared the performance with the SE formula. However, their comparison is on the semi-infinite interval $(0, \infty)$. The work in [20, see, also [19]] is an improvement on Stenger's SE formula by replacing the auxiliary basis functions and SE transformation. They concluded by presenting two different types of error bounds for the modified formula. In a more recent study, Okayama and Kurogi in [21] proposed better selection formulas for the parameters involved in the numerical approximation of quadratures using the DE formulas that reduces its error and also presented computable error bounds of the modified DE formula (see, also [22]).

The word "sinc" is an abbreviation of the phrase "sine–cardinal" [29].

The sinc function sinc(x) arises frequently in Fourier transforms. It is an even function with zeros at $k\pi$ for $k = \pm 1, \pm 2, \cdots$, $\lim_{x \to \pm\infty} \operatorname{sinc}(x) = 0$. Gearhart and Shultz [30] described it as a well–behaved function and also gives some of its properties.

Definition 1.1. The sinc function is defined in [9] as

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin \pi x}{\pi x}, & x \neq 0 \\ 1, & x = 0. \end{cases} \tag{1.3}$$

From the definition above, it is possible to write the complex integral representation [29] for $x \neq 0$,

$$\begin{aligned} \operatorname{sinc}(x) &= \frac{\sin(\pi x)}{\pi x} \\ &= \frac{e^{i\pi x} - e^{-i\pi x}}{2i\pi x} \\ &= \frac{1}{2i\pi x} \left[e^{iwx}\right]_{-\pi}^{\pi} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iwx} \, \mathrm{d}w. \end{aligned}$$

The sinc function has strong relationships with the sine integral. The sine integral is defined in equation (5.2.1) in [31] as

$$\begin{aligned} \operatorname{Si}(x) &= \int_0^x \frac{\sin\ u}{u} \, \mathrm{d}u \\ &= \int_0^x \operatorname{sinc}(u) \, \mathrm{d}u \\ &= \frac{\pi}{2} + \operatorname{si}(x). \end{aligned} \tag{1.4}$$

where, from [31]

$$\operatorname{si}(x) = -\int_x^{\infty} \frac{\sin u}{u} \, \mathrm{d}u. \tag{1.5}$$

The sine integral satisfies the symmetry relation $\operatorname{Si}(-x) = -\operatorname{Si}(x)$, which means that it is an odd function. When one wants to evaluate an integral over an infinite interval $(-\infty, \infty)$, i.e.

$$I = \int_{-\infty}^{\infty} f(u) \, \mathrm{d}u, \tag{1.6}$$

in which the integrand $f(u)$ is analytic over $(-\infty, \infty)$, with a constant step size h, the first thing that comes to mind is the uniformly divided trapezoidal formula [34]

$$I = h \sum_{k=-\infty}^{\infty} f(kh). \tag{1.7}$$

Such a formula is not useful for evaluating integrands with algebraic or logarithmic singularities at one or both ends of the interval of integration.

In approximating $F(x) = \int_{-1}^{x} f(u)\mathrm{d}u$, we shall make two basic transformations: single exponential transformation $w = \phi(z) = \tanh\frac{z}{2}$ and double exponential transformation $w = \phi_1(z) = \tanh\left[\frac{\pi}{2}\sinh(z)\right]$, which maps $(-\infty, \infty)$ to $(-1, 1)$. After the transformation, we will then use the well-known trapezoidal formula in part to derive the four quadrature formulas.

Let the given integral be

$$I = \int_{c}^{d} f(x)\,\mathrm{d}x. \tag{1.8}$$

Throughout the analysis, we shall be making variable transformations of the form

$$x = \phi(u) \;\; \textit{where} \;\; \phi(-\infty) = c, \;\; \phi(\infty) = d \tag{1.9}$$

to (1.8) so as to change the interval from (c, d) to $(-\infty, \infty)$. Hence, after the transformation, we have

$$I = \int_{c}^{d} f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(\phi(u))\phi'(u)\,\mathrm{d}u. \tag{1.10}$$

One or both of the endpoints c and d in the original integral can be finite [34]. We now apply the trapezoidal rule (1.7) to obtain the quadrature formula,

$$I = h\sum_{k=-\infty}^{\infty} f(\phi(kh))\phi'(kh), \tag{1.11}$$

Haber's formulas A and B, which involves the conformal transformation from the interval $(-\infty, \infty)$ to $(-1, 1)$ via the transformation $w = \phi(z) = \tanh\frac{z}{2}$. McNamee, Stenger and Whitney [11] describe the cardinal function as a "function of royal blood whose distinguished properties set it apart from its bourgeois brethren".

Definition 1.2. Let f be a function which is defined on the real line $\mathbb{R}$. Then the formal series

$$\sum_{k=-\infty}^{\infty} f(kh)S(k, h, x), \tag{1.12}$$

is called the cardinal series of the function f with respect to a positive step size h. If the series (1.12) converges, we denote its sum by $C(f, h, x)$, and the function $C(f,h,x)$ is called the cardinal function (or Whittaker cardinal function) of the function f [11]. Where

$$\begin{aligned} S(k,h,x) &= \operatorname{sinc}\left(\frac{x-kh}{h}\right) \\ &= \begin{cases} \dfrac{\sin[(\frac{\pi}{h})(x-kh)]}{(\frac{\pi}{h})(x-kh)}, & x \neq kh; \\ 1, & x = kh, \end{cases} \end{aligned} \tag{1.13}$$

is the k'th sinc function with step size h, evaluated at x. Kearfott [7] calls (1.13) the interpolation property of the sinc function.

The truncated cardinal series is given by

$$C(M,N,f,h,x) = \sum_{k=-M}^{N} f(kh)S(k,h,x),$$

in general $N \neq M$. But as we are assuming that the functions we shall be dealing with are symmetric [9], hence $N = M$, and the truncated series can be written as

$$C(N,f,h,x) = \sum_{k=-N}^{N} f(kh)S(k,h,x). \tag{1.14}$$

An interesting property of the cardinal function is given in Theorem 1.2. We give a property of the Paley-Wiener class of functions $B(h)$ below.

Theorem 1.1. *If $f \in B(h)$, then for all $z \in \mathbb{C}$*

$$f(z) = \frac{1}{h}\int_{-\infty}^{\infty} f(u)\operatorname{sinc}\left(\frac{u-z}{h}\right) \mathrm{d}u. \tag{1.15}$$

Proof. See [9]. □

Definition 1.3. Let h be a positive constant, the Paley-Wiener class of functions (denoted by $B(h)$) is the family of entire functions f such that on the real line $f \in \mathbf{L}^2(\mathbb{R})$ and in the complex plane f is of exponential type $\frac{\pi}{h}$ [9] i.e

$$|f(z)| \leq Ke^{\pi|z|h}, \quad K > 0.$$

We are now in a position to derive an exact interpolation and quadrature formula for functions in $B(h)$. These can be found in the theorem below, coupled with all the results obtained earlier.

Theorem 1.2. *Let $f \in B(h)$, then for all $z \in \mathbb{C}$ [9],*

$$f(z) = \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}\left(\frac{z-kh}{h}\right) \tag{1.16}$$

$f(kh)$ means evaluating $f(x)$ at $x = kh$,

$$f(kh) = \frac{1}{h}\int_{-\infty}^{\infty} f(u)\,\mathrm{sinc}\left(\frac{u-kh}{h}\right)\mathrm{d}u. \tag{1.17}$$

Moreover, according to [25]

$$\lim_{N\to\infty}\int_{-N}^{N} f(x)\,\mathrm{d}x = \lim_{N\to\infty} h\sum_{k=-N}^{N} f(kh). \tag{1.18}$$

Proof. See [9]. □

2 Derivation of Haber's Formula A

We shall start from the contour integral [9]

$$G(z) = \int_{B_{k,\varepsilon}} g(z)\,\mathrm{d}z, \tag{2.1}$$

where

$$g(z) = \frac{\sin\frac{\pi x}{h} f(z)}{(z-x)\sin\frac{\pi z}{h}}.$$

Let h and c be positive constants. From [9] we are given that for each positive integer k, $B_{k,\varepsilon}$ is the rectangular contour with vertical sides $x = \pm\frac{(2k+1)h}{2}$ and horizontal sides $y = \pm(c-\epsilon)$:

$$B_{k,\varepsilon} = \left\{ z = x+iy : -\left(\frac{2k+1}{2}\right)h < x < \left(\frac{2k+1}{2}\right)h, |y| < (c-\varepsilon)\right\}.$$

Assume that

$\mathbf{A}_1$ f is analytic in the strip $|\mathbf{y}| < \mathbf{c}$.

The assumption $\mathbf{A_1}$ above is the same as Haber's condition H_1 in [6].

The real number x [6] is less than or equal to kh in absolute value. The singularities of $g(z)$ in (2.1) are $z = x$ and $z = kh$, where k is all integers between $-n$ and n. From residue theory, at $z = kh$

$$
\begin{aligned}
\operatorname{Res}(g, kh) &= \frac{\sin(\frac{\pi x}{h}) f(kh)}{\left[(z-x)\sin(\frac{\pi z}{h})\right]'_{z=kh}} \\
&= \frac{\sin(\frac{\pi x}{h}) f(kh)}{\left[\sin(\frac{\pi z}{h}) + \frac{\pi}{h}(z-x)\cos(\frac{\pi z}{h})\right]_{z=kh}} \\
&= \frac{\sin(\frac{\pi x}{h}) f(kh)}{\left[\sin(\pi k) + \frac{\pi}{h}(kh-x)\cos(\pi k)\right]} \\
&= \frac{(-1)^k h \sin(\frac{\pi x}{h}) f(kh)}{\pi(kh-x)} \\
&= -\frac{h \sin\left(\pi(x-kh)/h\right) f(kh)}{\pi(x-kh)} \\
&= -f(kh)\operatorname{sinc}\left(\frac{x-kh}{h}\right).
\end{aligned}
$$

Thus

$$
\operatorname{Res}(g, kh) = -f(kh)\operatorname{sinc}\left(\frac{x-kh}{h}\right). \tag{2.2}
$$

The residue at $z = x$ is given by

$$
\operatorname{Res}(g, x) = f(x). \tag{2.3}
$$

For the singularities, the Residue Theorem [8] yields

$$
\begin{aligned}
G(z) &= 2\pi i\left[\operatorname{Res}(g, x) + \sum_{k=-n}^{n} \operatorname{Res}(g, kh)\right] \\
&= 2\pi i\left[f(x) - \sum_{k=-n}^{n} f(kh)\operatorname{sinc}\left(\frac{x-kh}{h}\right)\right].
\end{aligned}
$$

Making $f(x)$ the subject, we have

$$
f(x) = \sum_{k=-n}^{n} f(kh)\operatorname{sinc}\left(\frac{x-kh}{h}\right) + \frac{\sin\frac{\pi x}{h}}{2\pi i}\int_{B_{k,\varepsilon}} \frac{f(z)\,\mathrm{d}z}{(z-x)\sin\frac{\pi z}{h}}
$$

and

$$
\begin{aligned}
f(x) &= \sum_{k=-n}^{n} f(kh)\operatorname{sinc}\left(\frac{x-kh}{h}\right) + R_{k,\varepsilon}(x); \\
R_{k,\varepsilon}(x) &= \frac{\sin\frac{\pi x}{h}}{2\pi i}\int_{B_{k,\varepsilon}} \frac{f(z)\,\mathrm{d}z}{(z-x)\sin\frac{\pi z}{h}}. \qquad (2.4)
\end{aligned}
$$

The above equation holds for $x = \pm nh$, $|n| \le k$ and for all $x \in [-kh, kh]$. Haber [6] also gave another condition

$\mathbf{A}_2$ for a small positive ε, each of the integrals $\int_{-\infty}^{\infty} |f(x - i(c - \varepsilon))|\,\mathrm{d}x$ and $\int_{-\infty}^{\infty} |f(x + i(c - \varepsilon))|\,\mathrm{d}x$ exists, and is bounded in ε;

and

$\mathbf{A}_3$ for each small positive ε, the integral $\int_{\varepsilon - c}^{c-\varepsilon} |f(x + iy)|\,\mathrm{d}y$ is a bounded function of x.

The next section contains some useful identities.

The following Lemma was proved by Stenger ([25], pages 172-173). It contains an important identity that will help us derive Haber's formula A.

Lemma 2.1. ***(Main Result)*** *Let $h > 0,\ k \in \mathbb{Z},\ x \in \mathbb{R}$ and set*

$$J(k, h, x) = \int_{-\infty}^{x} S(k, h, x)\,\mathrm{d}x, \tag{2.5}$$

where $S(k, h, x)$ is as defined in (1.13). Then, for $x \in \mathbb{R}$,

$$|J(k, h, x)| \le 1.1h. \tag{2.6}$$

Proof. See [23]. □

The lemma below is from Haber's paper [6].

Lemma 2.2. *Assuming that f satisfies $\mathbf{A}_1$, $\mathbf{A}_2$ and $\mathbf{A}_3$, then*

$$f(u) = \sum_{k=-\infty}^{\infty} f(kh)\,\mathrm{sinc}\left(\frac{u - kh}{h}\right) + \frac{\sin\frac{\pi u}{h}}{2\pi i}(R_- - R_+), \quad \forall\ u \in \mathbb{R}. \tag{2.7}$$

Conditions $\mathbf{A}_1$ and $\mathbf{A}_2$ are not very explanatory, because if one wants to change to the interval $(-1, 1)$ one needs simpler conditions. We integrate (2.7) with respect to u in the following manner:

$$\begin{aligned}\int_{-C}^{x} f(u)\,\mathrm{d}u = &\sum_{k=-\infty}^{\infty} f(kh)\int_{-C}^{x} \mathrm{sinc}\left(\frac{u - kh}{h}\right)\mathrm{d}u\\ &+\frac{1}{2\pi i}\left\{\int_{-C}^{x} \sin\frac{\pi u}{h}\int_{-\infty}^{\infty}\frac{f(v - i(c-\varepsilon))}{(v - u - i(c-\varepsilon))\sin\frac{\pi(v - i(c-\varepsilon))}{h}}\,\mathrm{d}v\,\mathrm{d}u\right.\\ &\left.-\int_{-C}^{x} \sin\frac{\pi u}{h}\int_{-\infty}^{\infty}\frac{f(v + i(c-\varepsilon))}{(v - u + i(c-\varepsilon))\sin\frac{\pi(v + i(c-\varepsilon))}{h}}\,\mathrm{d}v\,\mathrm{d}u\right\},\end{aligned} \tag{2.8}$$

where

$$G_{\pm} = \int_{-C}^{x} \sin\frac{\pi u}{h}\int_{-\infty}^{\infty}\frac{f(v \pm i(c-\varepsilon))}{(v - u \pm i(c-\varepsilon))\sin\frac{\pi(v \pm i(c-\varepsilon))}{h}}\,\mathrm{d}v\,\mathrm{d}u.$$

Haber [6] points out that the interchange of integration and summation is possible by imposing the following condition on f:

$\mathbf{A}_4$ a constant $\alpha > 0$ exists such that for all $x \in \mathbb{R}$, $f(x) = O(e^{-\alpha|x|})$ as $|x| \to \infty$.

The order of integration in $G_\pm$ can be exchanged, since

$$\int_{-\infty}^{-B} \frac{f(v \pm i(c-\varepsilon))\,\mathrm{d}v}{(v-u \pm i(c-\varepsilon)) \sin \frac{\pi(v \pm i(c-\varepsilon))}{h}}$$

tends to zero as $B \to \infty$, uniformly for $u \in [-C, x]$. The same holds for the integral from B to ∞. The inner integral that is left

$$\int_{-C}^{x} \frac{\sin \frac{\pi u}{h}\,\mathrm{d}u}{v-u \pm i(c-\varepsilon)} = O(h).$$

uniformly on C, v, u and ε. We can deduce from other identities already encountered that [1]

$$\begin{aligned}\left|\sin \frac{\pi(v \pm i(c-\varepsilon))}{h}\right| &\geq \sinh \frac{\pi(c-\varepsilon)}{h} \\ &\geq \frac{1}{2} e^{\frac{\pi(c-\varepsilon)}{h}},\end{aligned}$$

and then

$$|G_\pm| \leq Khe^{-\frac{\pi(c-\varepsilon)}{h}} \int_{-\infty}^{\infty} |f(v \pm i(c-\varepsilon))|\,\mathrm{d}v.$$

If C is allowed to tend to infinity and ε to zero in (2.8),

$$\int_{-\infty}^{x} f(u)\,\mathrm{d}u = h \sum_{k=-\infty}^{\infty} f(kh) \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Si}\left(\frac{\pi x - \pi kh}{h}\right)\right) + O(he^{-\frac{\pi c}{h}}) \quad (2.9)$$

as $h \to 0$. By making use of condition $\mathbf{A}_4$ and the boundedness of the sine integral,

$$\begin{aligned}\int_{-\infty}^{x} f(u)\,\mathrm{d}u = h &\sum_{k=-N}^{N} f(kh) \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Si}\left(\frac{\pi x - \pi kh}{h}\right)\right) \\ &+ O(he^{-\frac{\pi c}{h}}) + O(e^{-\alpha Nh}/h).\end{aligned} \quad (2.10)$$

To obtain the step size h, we equate the magnitude of the O terms as follows:

$$\exp(-\pi c/h) = \exp(-\alpha Nh).$$

Taking the logarithm of both sides, we have

$$h = \sqrt{\frac{\pi c}{\alpha N}}. \quad (2.11)$$

The next theorem summarises what we have done so far. The statement of this was given in [6] without a proof.

[1] If x and y are real, then $|\sin(x+iy)| = [\sinh^2 y + \sin^2 x]^{\frac{1}{2}} \geq \sinh |y|$ [32].

Theorem 2.3. *If f satisfies $\mathbf{A}_1,\ \mathbf{A}_2,\ \mathbf{A}_3$ and $\mathbf{A}_4$, and $h=\sqrt{\frac{\pi c}{\alpha N}}$, then*

$$\int_{-\infty}^{x} f(u)\,\mathrm{d}u = h\sum_{k=-\infty}^{\infty} f(kh)\left(\frac{1}{2}+\frac{1}{\pi}\,\mathrm{Si}\left(\frac{\pi x-\pi kh}{h}\right)\right)+O(\sqrt{N}e^{-\sqrt{\pi c\alpha N}}), \tag{2.12}$$

as $N\to\infty$ uniformly for $x\in\mathbb{R}$.

3 Transformation Via Conformal Mapping

The bulk of this section will be devoted to transforming the interval in (2.12) from $(-\infty, x]$ to $(-1,1)$, using conformal mapping. In addition, we shall end up with the analysis given by Haber in [6] to derive his formula A.

3.1 Approximation over $(-1,1)$

Let $\phi:(-\infty,\infty)\mapsto(-1,1)$ and $\psi:(-1,1)\mapsto(-\infty,\infty)$ such that

$$w=\phi(z)=\tanh\frac{z}{2},\quad z=\phi^{-1}(w)=\psi(w). \tag{3.1}$$

It follows from the exponential form of $\tanh z$ that

$$\tanh\frac{z}{2}=\frac{e^{\frac{z}{2}}-e^{-\frac{z}{2}}}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}}=w,$$

cross-multiplying and making z the subject of the formula gives

$$\psi(w)=z=\log\left(\frac{w+1}{1-w}\right),\quad \psi'(w)=\frac{2}{1-w^2}. \tag{3.2}$$

We define the domain as given in [24] for $\rho\in(0,\frac{\pi}{2}$ by

$$\begin{aligned} D=&[\{z:|z+i\cot\,\rho|<\mathrm{cosec}\,\rho\}\cap[\{z:x+iy,y\geq 0\}\\ &\cup[\{z:|z-i\cot\,\rho|<\mathrm{cosec}\,\rho\}\cap[\{z:x+iy,y\leq 0\}. \end{aligned} \tag{3.3}$$

Furthermore, Lunding and Stenger [10] considered

$$\Gamma=\{z\in D:\psi(z)\in(-\infty,\infty)\}=\{x:-1\leq x\leq 1\}, \tag{3.4}$$

and

$$g(x)=(1-x^2)^\beta,\quad \beta\geq 0, \tag{3.5}$$

which are very useful in understanding Haber's analysis.

The next theorem is found in [10] and gives a clear picture of g used in Haber's conditions $\mathbf{A}'_1,\ \mathbf{A}'_2,\ \mathbf{A}'_3$ and $\mathbf{A}'_5$ which we will encounter shortly.

Using the change of variables $t = \phi(u)$, and subsequently $-1 = \phi(-\infty),\ v = \phi(x),\ \ x = \phi^{-1}(v) = \log\left(\frac{1+v}{1-v}\right)$, we can write

$$\int_{-\infty}^{x} f(u)\,\mathrm{d}u = \int_{-1}^{v} g(t)\,\mathrm{d}t, \tag{3.6}$$

and using $f(u) = g(\phi(u))\phi'(u)$:

$$\int_{-\infty}^{x} f(u)\,\mathrm{d}u = \int_{-1}^{v} g(t)\,\mathrm{d}t = \int_{-\infty}^{\phi^{-1}(x)} g(\phi(u))\phi'(u)\,\mathrm{d}u,$$

so that Haber's formula A from (2.12) becomes

$$\int_{-1}^{v} g(u)\,\mathrm{d}u = h \sum_{k=-N}^{N} g(\phi(kh))\phi'(kh)\left(\frac{1}{2} + \frac{1}{\pi}\,\mathrm{Si}\left(\frac{\pi\phi^{-1}(v) - \pi kh}{h}\right)\right) + O(\sqrt{N}e^{-\sqrt{\pi c\alpha N}}). \tag{3.7}$$

Haber's conditions H_1 and H_4, which are the same as $\mathbf{A}_1$ and $\mathbf{A}_4$, can be translated into the condition

$\mathbf{A}_1'$ $\quad g$ is analytic on Φ_c.

and

$\mathbf{A}_4'$ $\quad$ there is a constant $\alpha > 0$ such that for $t \in \mathbb{R}$, $g(t) = O((1 - t^2)^{\alpha-1})$ as $t \to -1$ and $t \to +1$ from inside (-1, 1).

Furthermore, we can write the integral in condition $\mathbf{A}_3$ as

$$\int_{L_{x,\varepsilon}} |g(w)\,\mathrm{d}w|,$$

where $L_{x,\varepsilon}$ is the image of the line segment $z = x + iy,\ \ |y| < c - \varepsilon$ under ϕ. As w tends to ±1 from inside Φ_c, if we require of g that $g(w) = O(|1-w^2|^{-1})$ as $w \to \pm1$, then $\mathbf{A}_3$ would be satisfied. But we shall require the stronger condition

$\mathbf{A}_3'$ $\quad$ there is a constant $\alpha > 0$ such that $g(w) = \mathsf{O}(|1 - w^2|^{\alpha-1})$ as w tends to ±1 from inside Φ_c.

Let M_b be the image under ϕ of the line $y = b$ and $M_{b,\beta},\ \beta > 0$ be the part of M_b that lies outside circles of radius β and center ±1. With the above, we can write the integrals in $\mathbf{A}_2$ as

$$\int_{M_{\pm(c-\varepsilon)}} |g(w)\,\mathrm{d}w|. \tag{3.8}$$

In [6], it was also assumed that

$\mathbf{A}'_2$ for small β, the integrals $\int_{M_{\pm(c-\varepsilon),\beta}} |g(w)\,\mathrm{d}w|$ are bounded in ε for $\varepsilon \in (0, c)$.

To check whether $\mathbf{A}'_2$ holds for a given g in which $\mathbf{A}'_1$ and $\mathbf{A}'_3$ holds, Haber [6] considered singularities of g at points on the boundary of Φ_c except at $t = \pm 1$. In addition, if $c' > 0$ such that $c' < c$, then $\mathbf{A}'_2$ will hold with c' instead of c. This leads us to the next two theorems, which can be found in [6]. They summarise the above analysis and give us Haber's formula A (3.10).

Theorem 3.1. *If g is analytic in Φ_c for some positive $c \leq \pi$, and $g(w) = O(|1 - w^2|^{\alpha-1})$ for some $\alpha > 0$ as $w \to \pm 1$ from inside Φ_c, then [6]*

$$\int_{-1}^{v} g(t)\,\mathrm{d}t = h \sum_{k=-N}^{N} g(\phi(kh))\phi'(kh) \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Si}\left(\frac{\pi\phi^{-1}(v) - \pi kh}{h} \right) \right) + O(\sqrt{N} e^{-\sqrt{\pi c' \alpha N}}), \tag{3.9}$$

holds uniformly in $[-1, 1]$, where c' is any number in $(0, c)$ and $h = \sqrt{\frac{\pi c'}{\alpha N}}$.

Theorem 3.2. *If $0 < c \leq \pi$, $\alpha > 0$, and g satisfies conditions $\mathbf{A}'_1$, $\mathbf{A}'_2$ and $\mathbf{A}'_3$, then*

$$\int_{-1}^{v} g(t)\,\mathrm{d}t = h \sum_{k=-N}^{N} g(\phi(kh))\phi'(kh) \left(\frac{1}{2} + \frac{1}{\pi} \operatorname{Si}\left(\frac{\pi\phi^{-1}(v) - \pi kh}{h} \right) \right) + O(\sqrt{N} e^{-\sqrt{\pi c \alpha N}}), \tag{3.10}$$

holds uniformly in $[-1, 1]$.

4 Derivation of Haber's formula B

The driving force behind formula B is that, instead of calculating general values of the sine integral (Si) as in formula A, we shall only use the values of $\operatorname{Si}(k\pi), k \in \mathbb{Z}$. The latter can be calculated easily, as will be shown by the numerical experiments.

Haber's formula B involves working within the context of integrals over $\mathbb{R}$. We start by setting

$$F(x) = \int_{-\infty}^{x} f(u)\,\mathrm{d}u, \tag{4.1}$$

and F is approximated by an interpolation formula that makes use of evaluations of F at integral multiples of h, the step size.

Formula B can be derived by the change of variables $w = \phi(z) = \tan \frac{z}{2}$ and by setting $f(u) = g(\phi(u))\phi'(u)$ with $\varphi(w) = \eta(\phi^{-1}(w))$. Multiplying the numerator and denominator of η by $e^{-\alpha x}$, we will have

$$\begin{aligned}
\varphi(w) &= \eta(\psi(w)) \\
&= \eta\left(\log\left(\frac{w+1}{1-w}\right)\right) \\
&= \frac{1}{1+\exp\left(-\alpha \log\left(\frac{w+1}{1-w}\right)\right)} \\
&= \frac{1}{1+\left(\frac{1-w}{w+1}\right)^{\alpha}} \\
&= \frac{(w+1)^{\alpha}}{(w+1)^{\alpha}+(1-w)^{\alpha}}.
\end{aligned}$$

It follows from the previous analysis that conditions $\mathbf{A}'_1$ and $\mathbf{A}'_3$ on g imply $\mathbf{A}_2$, $\mathbf{A}_{3a}$ and $\mathbf{A}_4$ on f, while $\mathbf{A}'_1$ on η is equivalent to $\mathbf{A}_1$ on f. Consequently, $\mathbf{A}'_1$ on φ is the same as condition $\mathbf{A}_1$ on η. The condition $\mathbf{A}_5$ on η holds if we require that

$\mathbf{A}'_5$ **φ is continuous on $[-1, 1]$ and $\varphi(-1) = 0$, $\varphi(1) = 1$.**

In addition, φ satisfies a Hölder condition on $[-1, 1]$ of order α.

For $\mathbf{A}_{3a}$ to hold on η, Haber imposed $\mathbf{A}'_3$ on φ'; conditions $\mathbf{A}'_3$ for φ' and $\mathbf{A}'_5$ for φ imply the Hölder condition for φ. We conclude with this theorem from [6].

Theorem 4.1. *If $0 < c \le \pi$, $\alpha > 0$, g satisfies conditions $\mathbf{A}'_1, \mathbf{A}'_2$ and $\mathbf{A}'_3$, φ satisfies $\mathbf{A}'_1$ and $\mathbf{A}'_5$, and η' satisfies $\mathbf{A}_3$. Then (4.2) holds uniformly in $[-1, 1]$, and $I^\star$ is defined by (4.3).*

A close look at $\mathbf{A}'_5$ and the choice of the functions $\varphi = \frac{w+1}{2}$ satisfies the conditions for all c and for any $\alpha \le 1$, and the function $\varphi = \frac{-w^3+3w+2}{4}$ satisfies the conditions for all c for any $\alpha \le 2$, since $\varphi'(-1) = \varphi'(1) = 0$. To avoid the condition $\mathbf{A}'_2$, Haber states that we should replace c' by c in the error term. He presents this in the form of a theorem, which is Haber's formula B .

Theorem 4.2. *If $0 < c \le \pi$ and $\alpha > 0$, and*

1. *g is analytic in Φ_c and $g(w) = O(|1 - w^2|^{\alpha-1})$ as $w \to \pm 1$ from the interior or inside Φ_c;*

2. *φ is analytic in Φ_c and $\varphi(-1) = 0$, $\varphi(1) = 1$; $\varphi'(w) = O(|1 - w^2|^{\alpha-1})$ as $w \to \pm 1$ from inside Φ_c,*

then

$$\int_{-1}^{v} g(t)\,\mathrm{d}t = h \sum_{k=-N}^{N} \sum_{m=-N}^{N} g(\phi(mh))\phi'(mh)\sigma_{k-m} \operatorname{sinc}\left(\frac{\phi^{-1}(v)-kh}{h}\right)$$
$$+ I^\star \left(\varphi(v) - \sum_{k=-N}^{N} \left(\varphi(\phi(kh)) - \frac{1}{2}\right) \operatorname{sinc}\left(\frac{\phi^{-1}(v)-kh}{h}\right)\right)$$
$$+ O((\sqrt{N})^3 e^{-\sqrt{\pi c \alpha N}}), \tag{4.2}$$

holds uniformly on $[-1,1]$, $c' \in (0,c)$, $h = \sqrt{\frac{\pi c'}{\alpha N}}$ *and*

$$I^\star = h \sum_{m=-N}^{N} g(\phi(mh))\phi'(mh). \tag{4.3}$$

If $\int_{-1}^{1} g(t)\,\mathrm{d}t = 0$, *then* $I^\star$ *may be replaced by zero.*

5 Computational Considerations

We shall consider two cases for the computation of the double sum in (4.2)- on the one hand for a single value of v and on the other hand for several values.

The double sum in (4.2) involves computing $2N+1$ values of sine in the sinc function, but because k is an integer we can simplify matters.

$$\operatorname{sinc}\left(\frac{\phi^{-1}(v)-kh}{h}\right) = \frac{h\sin\left(\frac{\pi\phi^{-1}(v)}{h} - \pi k\right)}{\pi\phi^{-1}(v) - \pi kh} = \frac{h}{\pi}\frac{(-1)^k \sin\frac{\pi\phi^{-1}(v)}{h}}{\phi^{-1}(v) - kh}$$

Thus, we can write

$$h \sum_{k=-N}^{N} \sum_{m=-N}^{N} g(\phi(mh))\phi'(mh)\sigma_{k-m} \operatorname{sinc}\left(\frac{\phi^{-1}(v)-kh}{h}\right)$$
$$= \frac{h^2}{\pi} \sin\frac{\pi\phi^{-1}(v)}{h} \sum_{k=-N}^{N} \sum_{m=-N}^{N} \frac{(-1)^k \sigma_{k-m}}{\phi^{-1}(v)-kh} g(\phi(mh))\phi'(mh), \tag{5.1}$$

which uses only one sine evaluation. It follows that (4.2) involves more calculation than (3.9) because of the double sum. If we are to approximate the integral for many values of v, we write the double sum as

$$\frac{h^2}{\pi} \sin\frac{\pi\phi^{-1}(v)}{h} \sum_{k=-N}^{N} \frac{Z_k}{\phi^{-1}(v)-kh}, \tag{5.2}$$

where

$$Z_k = \sum_{m=-N}^{N} (-1)^k \sigma_{k-m} g(\phi(mh))\phi'(mh). \tag{5.3}$$

The Z_k are independent of v, which allows us to calculate Z_k first before each new value of v, which requires the calculation of a single sum. The simplification makes formula B faster than A when several values of the indefinite integral are needed. It should also be noted that each value of v requires only a sine, a logarithm and two simple sums for B, and $2N + 1$ new values of the sine integral for A.

However, developments in software has circumvented the above simplifications, since sinc values can be computed easily by typing `sinc(x)` in `Matlab` or `octave`.

One thing that is noteworthy is that the parameters α and c have to be chosen to "normalise" the functions in such a way that,

$$\int_{-1}^{1} |g(t)| \,\mathrm{d}t = 1. \tag{5.4}$$

It also becomes clear as remarked in Haber ([5], p. 148, [6]) that most of the abscissas $\phi(kh)$ are very close to ± 1 because, as $|kh| \to \infty$,

$$\tanh \frac{kh}{2} = \pm 1 + O(e^{-|kh|}). \tag{5.5}$$

It might not be possible to evaluate integrands that contain factors such as $(1 - x)$, because the integrand may be infinite at ± 1. This computational pitfall can be avoided by studying the function $\phi'(x)f(\phi(x))$.

6 Stenger's SE Formula

Before proceeding to the derivation of Stenger's SE formula, a definition is provided of single exponential transformation and the family of analytic functions $\mathbf{L}_\alpha(\mathcal{D}_c)$ and $\mathbf{M}_\alpha(\mathcal{D}_c)$ is introduced.

Definition 6.1. (Single Exponential Transformation) A function f is said to decay single exponentially [28] with respect to the conformal map ϕ, if there exist positive constants α and K such that

$$|f(\phi(x))\phi'(x)| \leq K \exp(-\alpha|x|) \quad \text{for all} \quad x \in \mathbb{R}, \tag{6.1}$$

and any ϕ that satisfies (6.1) is called a single exponential transformation.

Definition 6.2. Let α, β be positive numbers, and $\mathbf{L}_{\alpha,\beta}(\mathcal{D}_c)$ denote the family of functions f that are analytic in $\mathcal{D}_c$ [25], such that for some constants $K > 0$, and all $z \in \mathcal{D}_c$, we have

$$|f(z)| \leq K \frac{e^{\alpha z}}{(1 + |e^z|)^{\alpha+\beta}}. \tag{6.2}$$

Let $\alpha \in (0,1], \beta \in (0,1], c \in (0,\pi)$, and $\mathbf{M}_{\alpha,\beta}(\mathcal{D}_c)$ denote the family of functions in which v is holomorphic in $\mathcal{D}_c$, which has finite limits at a and b [26], such that $f \in \mathbf{L}_{\alpha,\beta}(\mathcal{D}_c)$, where f is defined by

$$f = v - \ell v, \tag{6.3}$$

and

$$\ell v(z) = \frac{v(a) + e^{\phi(z)} v(b)}{1 + e^{\phi(z)}}. \tag{6.4}$$

Stenger ([26], p. 383, 387) suggests that, in approximating integrals over various intervals, one should select a mapping ϕ that gives a one-to-one transformation of (a,b) onto the real line and that provides a conformal mapping of the region $\mathcal{D}$ on which the integrand is analytic onto $\mathcal{D}_c$. He also illustrates that for problems that involve approximating functions that decay exponentially in at least one of the points at $\pm\infty$, we take $\phi(z) = z$, so that (6.3) reduces to

$$f(x) = v(x) - \frac{[v(-\infty) + e^x v(\infty)]}{(1 + e^x)}, \tag{6.5}$$

by replacing z with x.

Remark 6.1. For the sake of notation, we shall write $\mathbf{L}_\alpha(\mathcal{D}_c)$ for $\mathbf{L}_{\alpha,\alpha}(\mathcal{D}_c)$ and $\mathbf{M}_\alpha(\mathcal{D}_c)$ for $\mathbf{M}_{\alpha,\alpha}(\mathcal{D}_c)$.

For all real x, after simplifying (6.2) we have

$$\frac{1}{2^{2\alpha}} e^{-\alpha|x|} \leq \frac{e^{\alpha x}}{(1 + e^{\alpha x})^{2\alpha}} \leq e^{-\alpha|x|}. \tag{6.6}$$

Let α', c' be positive numbers, and f a given function. Throughout this section, the assumption is that $f \in \mathbf{L}_{\alpha'}(\mathcal{D}_{c'})$.

Define v, V for $\tau > 0$ by

$$\begin{aligned} v(z) &= f(z) - \frac{\tau}{2\cosh^2(\tau z)} \int_{-\infty}^{\infty} f(u)\, \mathrm{d}u, \\ &= f(z) - \frac{\tau}{2} \operatorname{sech}^2(\tau z) \int_{-\infty}^{\infty} f(u)\, \mathrm{d}u. \end{aligned} \tag{6.7}$$

$V(z) = \int_{-\infty}^{z} v(t)\, \mathrm{d}t$. Integrating $\frac{\tau}{2} \operatorname{sech}^2(\tau t)$ with respect to t and using the exponential forms of tanh,

$$\begin{aligned} \frac{\tau}{2} \int_{-\infty}^{z} \operatorname{sech}^2(\tau t)\, \mathrm{d}t &= \frac{1}{2} \tanh(\tau t) \Big|_{-\infty}^{z} \\ &= \frac{e^{2\tau z}}{e^{2\tau z} + 1} \\ &= \frac{e^{\tau z}}{e^{\tau z} + e^{-\tau z}} \\ &= \frac{e^{\tau z}}{2\cosh(\tau z)}, \end{aligned}$$

from which we arrive at

$$V(z) = \int_{-\infty}^{z} f(u)\,\mathrm{d}u - \frac{e^{\tau z}}{2\cosh(\tau z)} \int_{-\infty}^{\infty} f(u)\,\mathrm{d}u. \tag{6.8}$$

Stenger [25] points out that we set

$$c = \min\left(c', \frac{\pi}{2\tau} - \varepsilon\right), \quad \alpha = \min(\alpha', \tau), \tag{6.9}$$

with $\varepsilon \in (0, \frac{\pi}{2\tau})$, but otherwise ε is an arbitrary positive number.

This paves the way for the following fundamental result from [27], which also helps us to find the parameters.

Theorem 6.1. *Let α_g and c_g be positive numbers. Using the variable transformation $u = \phi(t)$, the transformed function $g(t) = f(\phi(t))\phi'(t) \in \mathbf{L}_{\alpha_g}(\mathcal{D}_{\alpha_g})$, then there exists a positive number K independent of N, such that*

$$\begin{aligned} \sup_{-1<x<1} \Bigg| \int_{-1}^{x} f(u)\,\mathrm{d}u - \Bigg[\frac{\exp[\tau\phi^{-1}(x)]}{2\cosh(\tau\phi^{-1}(x))} h \sum_{k=-N}^{N} f(\phi(kh))\phi'(kh) \\ + h \sum_{k=-N}^{N} \Bigg\{ \sum_{m=-N}^{N} I(k-m) \Bigg(f(\phi(mh))\phi'(mh) \\ - \frac{\tau}{2\cosh^2(\tau m h)} h \sum_{k=-N}^{N} f(\phi(kh))\phi'(kh) \Bigg) \Bigg\} S(k, h, \phi^{-1}(x)) \Bigg] \Bigg| \\ \leq K\sqrt{N} \exp(-(\pi\alpha'_g c'_g N)^{1/2}), \end{aligned} \tag{6.10}$$

with

$$\begin{aligned} \alpha'_g &= \min(\alpha_g, 2\tau), \\ c'_g &= \min\left(c_g, \frac{\pi - 2\tau\varepsilon_c}{2\tau}\right), \\ h &= \sqrt{\frac{\pi c'_g}{\alpha'_g N}}, \end{aligned} \tag{6.11}$$

and ε_c is any number such that $c'_g > 0$.

From (6.11) one will observe that

$$\alpha'_g c'_g \leq \min(\alpha_g c_g, \pi - 2\tau\varepsilon_c), \tag{6.12}$$

$$\text{where} \quad \tau = \frac{\alpha_g}{2}, \tag{6.13}$$

because it reduces the estimated error and gives us what Tanaka *et al.* call the best SE formula .

From the above theorem, we can now deduce Stenger's formula, which Tanaka *et al.* call the SE formula,

$$
\begin{aligned}
\int_{-1}^{x} f(u)\,\mathrm{d}u = \Bigg[& \frac{\exp[\tau\phi^{-1}(x)]h}{2\cosh(\tau\phi^{-1}(x))} \sum_{k=-N}^{N} f(\phi(kh))\phi'(kh) \\
& + h \sum_{k=-N}^{N} \Bigg\{ \sum_{m=-N}^{N} I(k-m)\Bigg(f(\phi(mh))\phi'(mh) \\
& - \frac{\tau h}{2\cosh^2(\tau m h)} \sum_{k=-N}^{N} f(\phi(kh))\phi'(kh)\Bigg)\Bigg\} S(k,h,\phi^{-1}(x))\Bigg],
\end{aligned} \tag{6.14}
$$

with the step size given by (6.11).

7 Derivation of Tanaka *et al.'s* Formula

In approximating $F(x) = \int_{-1}^{x} f(u)\,\mathrm{d}u, \;\; -1 < x < 1$, we shall make a double exponential transformation, $u = \phi_1(t)$ in line with Mori $\big([33], [34]\big)$, using

$$\phi_1(t) = \tanh\left[\frac{\pi}{2}\sinh t\right], \tag{7.1}$$

giving

$$\phi_1'(t) = \frac{\pi\cosh t}{2\cosh^2(\frac{\pi}{2}\sinh t)}, \tag{7.2}$$

which maps the entire real line $(-\infty, \infty)$ to $(-1, 1)$, i.e. $\phi_1(t)$ satisfies

$$\begin{cases} -1 = \phi_1(-\infty) = \tanh\left[\frac{\pi}{2}\sinh(-\infty)\right]; \\ \;\;\; 1 = \phi_1(\infty) = \tanh\left[\frac{\pi}{2}\sinh(\infty)\right]. \end{cases} \tag{7.3}$$

To find the inverse, let $u = \tanh\left[\frac{\pi}{2}\sinh t\right]$, $\frac{\pi}{2}\sinh t = \tanh^{-1} u$, $t = \sinh^{-1}\left(\frac{2}{\pi}\tanh^{-1} u\right)$. Alternatively, using $\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = u, x = \tanh^{-1} u$, cross-multiplying and taking $\log_e$ of both sides, we have $x = \frac{1}{2}\log\left(\frac{1+u}{1-u}\right)$, thus $\frac{\pi}{2}\sinh t = \frac{1}{2}\log\left(\frac{1+u}{1-u}\right)$ and

$$t = \phi_1^{-1}(u) = \sinh^{-1}\left[\frac{1}{\pi}\log\left(\frac{1+u}{1-u}\right)\right] = \sinh^{-1}\left[\frac{2}{\pi}\tanh^{-1} u\right]. \tag{7.4}$$

With the above analysis one can now define a double exponential transformation and decay.

Definition 7.1. A function f is said to decay double exponentially [28] if there exist positive constants α and K, such that

$$|f(x)| \leq K\exp(-\alpha\exp(|x|)) \;\text{ for all }\; x \in \mathbb{R}. \tag{7.5}$$

Alternatively, a function f is said to decay double exponentially with respect to the conformal map ϕ_1, if there exist positive constants α and K, such that

$$|f(\phi_1(x))\phi_1'(x)| \leq K\exp(-\alpha\exp(|x|)) \text{ for all } x \in \mathbb{R}. \tag{7.6}$$

Thus, any ϕ_1 satisfying (7.6) is called a double exponential transformation [28].

Consequently, we have

$$\int_{-1}^{x} f(u)\,\mathrm{d}u = \int_{-\infty}^{\phi_1^{-1}(x)} f(\phi_1(t))\phi_1'(t)\,\mathrm{d}t.$$

Theorem 7.1. *Using the variable transformation $u = \phi_1(t)$, the transformed function $g(t) = f(\phi_1(t))\phi_1'(t)$ satisfies*

$$g \in \mathcal{B}(\mathcal{D}_{c_f}), \tag{7.7}$$
$$|g(x)| \leq \alpha_g \exp(-\tau_g \exp(\lambda_g|x|), \tag{7.8}$$

for positive numbers α_g, τ_g, λ_g and c_g. Then, for any $\varepsilon \in (0, c_{\hat{v}})$, there is a positive number K_ε'', independent of N, such that

$$\begin{aligned}\sup_{-1<x<1}\Bigg|\int_{-1}^{x} f(u)\,\mathrm{d}u - \Bigg[&\frac{\tanh(P\sinh(Q\phi_1^{-1}(x)))+1}{2}h\sum_{k=-N}^{N} f(\phi_1(kh))\phi_1'(kh)\\ &+ h\sum_{k=-N}^{N}\Bigg\{\sum_{m=-N}^{N}\sigma_{k-m}\Bigg(f(\phi_1(mh))\phi_1'(mh)\\ &- \frac{PQ\cosh(Qmh)}{2\cosh^2(P\sinh(Qmh))}hf(\phi_1(mh))\phi_1'(mh)\Bigg)\Bigg\}\\ &\times \operatorname{sinc}\left(\frac{\phi_1^{-1}(x)-kh}{h}\right)\Bigg]\Bigg|\\ &\leq K_\varepsilon''\exp\left[\frac{-\pi(c_{\hat{v}}-\varepsilon)\lambda_{\hat{v}}N}{\log(\pi(\lambda_{\hat{v}}-\varepsilon)\lambda_{\hat{v}}N/\tau_{\hat{v}})}\right],\end{aligned} \tag{7.9}$$

where $\hat{v}$ and h are defined as

$$\hat{v} = g - s, \quad h = \frac{\log(\pi(\lambda_{\hat{v}}-\varepsilon)\lambda_{\hat{v}}N/\tau_{\hat{v}})}{\lambda_{\hat{v}}N}, \tag{7.10}$$

and where P, Q , $\lambda_{\hat{v}}$, $\tau_{\hat{v}}$ and $c_{\hat{v}}$ are as defined in either (7.23) to (7.27) or (7.28) to (7.31), with f and v replaced by g and $\hat{v}$ respectively.

Tanaka *et al.'s* formula can now be deduced from (7.9) above as:

$$\int_{-1}^{x} f(u)\,\mathrm{d}u = \frac{\tanh(P\sinh(Q\phi_1^{-1}(x)))+1}{2} h \sum_{k=-N}^{N} f(\phi_1(kh))\phi_1'(kh)$$
$$+ h\sum_{k=-N}^{N}\Bigg\{\sum_{m=-N}^{N}\sigma_{k-m}\Bigg(f(\phi_1(mh))\phi_1'(mh)$$
$$- \frac{PQ\cosh(Qmh)}{2\cosh^2(P\sinh(Qmh))} hf(\phi_1(mh))\phi_1'(mh)\Bigg)\Bigg\} S(k,h,\phi_1^{-1}(x)). \tag{7.11}$$

The choice of the parameters P and Q is without any discretion (we are free to choose them), but with the intention of minimising the error for a given integrand f. The determination of the set of parameters (P, Q) should be such that they give the maximum value of $\lambda_v c_v$, and choosing among the maximizers [27] (P, Q) that make τ_v as large as possible. Bear in mind that τ_v, λ_v and c_v are to be determined from P and Q using the results in Proposition 7.1 and Lemma 7.2.

The following propositions and the explanations that follow help one to determine the parameters.

Proposition 7.1. *Let τ_ω, λ_ω and c_ω be determined as*

$$\tau_\omega = \begin{cases} P-\varepsilon_\tau, \lambda_\omega = Q, c_\omega = \frac{\pi - 2Q\varepsilon_c}{2Q}, & P \in (0, \pi/2); \\ P-\varepsilon_\tau, \lambda_\omega = Q, c_\omega = \frac{\sin^{-1}(\frac{\pi}{2P}) - Q\varepsilon_c}{Q}, & P \geq \pi/2; \end{cases} \tag{7.12}$$

where ε_τ and ε_c are positive numbers such that $\tau_\omega > 0$ and $c_\omega > 0$. Then we have

$$\omega \in \mathcal{B}(\mathcal{D}_{c_\omega}), \tag{7.13}$$
$$|\omega(x)| \leq \alpha_\omega \exp(-\tau_\omega \exp(\lambda_\omega |x|)),\ \forall\, x \in \mathbb{R}. \tag{7.14}$$

Lemma 7.2. *Let τ_f, λ_f and c_f be constants such that*

$$f \in \mathcal{B}(\mathcal{D}_{c_f}), \tag{7.15}$$
$$|f(x)| \leq \alpha_f \exp(-\tau_f \exp(\lambda_f |x|)),\ \forall\, x \in \mathbb{R}. \tag{7.16}$$

If τ_ω, λ_ω and c_ω are constants in (7.12), then for

$$\tau_v = \begin{cases} \tau_f, & \lambda_f < \lambda_\omega \\ \tau_\omega, & \lambda_f > \lambda_\omega \\ \min(\tau_f, \tau_\omega), & \lambda_f = \lambda_\omega \end{cases} \tag{7.17}$$

$$\lambda_v = \min(\lambda_f, \lambda_\omega) \tag{7.18}$$

$$c_v = \min(c_f, c_\omega) \tag{7.19}$$

we have

$$v \in \mathcal{B}(\mathcal{D}_{c_v}) \tag{7.20}$$

$$|v(x)| \leq \alpha_v \exp(-\tau_v \exp(\lambda_v |x|)), \quad \forall\, x \in \mathbb{R}. \tag{7.21}$$

Proposition 7.2. *If f satisfies the conditions $f \in \mathcal{B}(\mathcal{D}_{c_f})$ and $|f(x)| \leq \alpha_f \exp(-\tau_f \exp(\lambda_f |x|))$, for all $x \in \mathbb{R}$ and $f \neq 0$, then $\lambda_v c_v \leq \frac{\pi}{2}$.*

To start with, we want to emphasise that (7.18) and (7.19) imply that

$$\lambda_v c_v \leq \lambda_f c_f, \tag{7.22}$$

where $\lambda_f c_f \leq \frac{\pi}{2}$. We shall divide the argument into two cases, depending on the value of $\lambda_f c_f$.

7.0.1 First Case: $\lambda_f c_f < \frac{\pi}{2}$

$$P = \frac{\pi}{2 \sin \lambda_f c_f} - \varepsilon_P, \tag{7.23}$$

$$Q = \lambda_f, \tag{7.24}$$

where ε_P, is any positive number such that $P > \frac{\pi}{2}$. Then

$$\lambda_v = \min\{\lambda_f, Q\} = \lambda_f, \tag{7.25}$$

$$\begin{aligned} c_v &= \min\left\{c_f, \frac{\arcsin \frac{\pi}{2P} - 2\varepsilon_c Q}{Q}\right\} \\ &= \min\left\{c_f, \frac{1}{\lambda_f} \arcsin\left[\frac{\pi \sin \lambda_f c_f}{\pi - 2\varepsilon_P \sin \lambda_f c_f}\right] - \varepsilon_c\right\} \\ &= \min\left\{c_f, \frac{1}{\lambda_f} \arcsin\left[\frac{\sin \lambda_f c_f}{1 - (2\varepsilon_P/\pi) \sin \lambda_f c_f}\right] - \varepsilon_c\right\} \\ &= c_f. \end{aligned} \tag{7.26}$$

Because ε_P and ε_c are very small, $\lambda_v c_v$ in (7.22), attains the upper bound $\lambda_f c_f$, which is obtained when $\lambda_v = \lambda_f$ and $c_v = c_f$, which means that $P \in (0, \frac{\pi}{2})$ or $P \in (\frac{\pi}{2}, \frac{\pi}{2 \sin \lambda_f c_f})$.

$$\tau_v = \min\left\{\tau_f, \frac{\pi}{2 \sin \lambda_f c_f} - (\varepsilon_P + \varepsilon_\tau)\right\}, \tag{7.27}$$

$\varepsilon_\tau > 0$ such that $\tau_v > 0$.

7.0.2 Second Case: $\lambda_f c_f = \frac{\pi}{2}$

Since $\lambda_v c_v \leq \lambda_\omega c_\omega < \frac{\pi}{2}$, one cannot attain the upper bound $\lambda_f c_f$ if $\lambda_f c_f = \frac{\pi}{2}$ in (7.22). But we can make $\lambda_v c_v$ arbitrarily close to $\frac{\pi}{2}$ with

$$P = \frac{\pi}{2},\ Q = \lambda_f, \tag{7.28}$$

from which

$$\begin{aligned} \lambda_v c_v &= \min\left\{\lambda_f, Q\right\} \min\left\{c_f, \frac{\pi - 2\varepsilon_c Q}{2Q}\right\} \\ &= \lambda_f \min\left\{c_f, \frac{\pi - 2\varepsilon_c Q}{2Q}\right\} \\ &= \frac{\pi - 2\varepsilon_c Q}{2}. \qquad (7.29) \\ \lambda_v &= \lambda_f, \qquad (7.30) \\ c_v &= \frac{\pi - 2\varepsilon_c \lambda_f}{2\lambda_f}, \qquad (7.31) \end{aligned}$$

where ε_τ and ε_c are positive numbers such that $\tau_v > 0$ and $c_v > 0$.

8 Numerical Experiments

In this section, we apply the four quadrature formulas discussed in the previous section to find numerical approximations to two test problems. In doing this, we show how the parameters for finding the step size h for each of the formulas can be obtained. We also illustrate using figures, the actual form of the error by plotting the actual error against v in each case for each of the formulas, as well as plotting the error against $\phi^{-1}(v)$. This stretches out the ends of the interval $-1 < v < 1$ so that the true behaviour of v near the end-point singularities is shown more clearly. Throughout this section, error is the absolute value of the difference between the exact and the approximate values. The section concludes with a performance evaluation and an analysis of the results. The `octave` and `gnuplot` codes used for each of the computations and figures respectively can be obtained on request from the author.

8.1 Implementing Haber's Formula A

Example 8.1. *We use formula (3.10) to approximate the integral below:*

$$\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\, \mathrm{d}x. \tag{8.1}$$

Let us first find the values of the parameters α and c, bearing in mind that they are to be chosen so that the functions are "normalised". To find α, we shall use condition $\mathbf{A}_4$ *with* $\phi(x) = \tanh \dfrac{x}{2}$, $\phi'(x) = \dfrac{1}{2} \operatorname{sech}^2 \dfrac{x}{2}$*:*

$$
\begin{aligned}
f(\phi(x)) &= \frac{1}{\pi\sqrt{1-\tanh^2 \frac{x}{2}}} = \frac{1}{\pi \operatorname{sech} \frac{x}{2}} \\
f(\phi(x))\phi'(x) &= \frac{1}{2\pi \cosh \frac{x}{2}} \\
|f(\phi(x))\phi'(x)| &= \left| \frac{1}{2\pi \cosh \frac{x}{2}} \right| \\
&= O(e^{-\frac{1}{2}|x|}), \quad |x| \to \infty, \qquad (8.2)
\end{aligned}
$$

which implies that $\alpha = \frac{1}{2}$*. From Theorem 3.2, we can choose* $0 < c \leq \pi$ *and, for this example, we chose* $c = \pi$*. We used 370 values of* v*, which are*

$$
\begin{aligned}
V &= -0.999, -0.998, -0.997, \cdots, -0.9; -0.89, -0.88, -0.87, \cdots, +0.96; \\
&+ 0.911, +0.912, +0.913, \cdots, +0.999.
\end{aligned}
$$

We plugged these values of v, α, c *into* (3.10)*, we present results by Figure 1, Figure 2 and Table 1.*

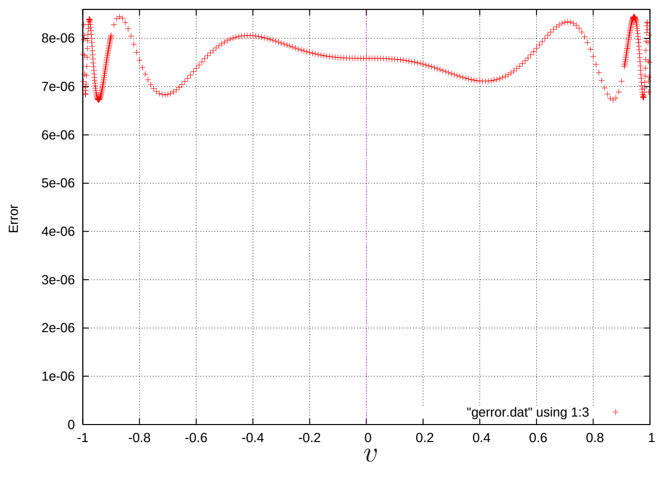

Figure 1: The error against v for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 25$, using Haber's formula A.

Figure 1 shows some oscillations that increase toward the endpoint singularities (± 1), but the behaviour towards ± 1 is shown clearly in Figure 2. From Figure 1, it is quite difficult to see the maximum absolute value of

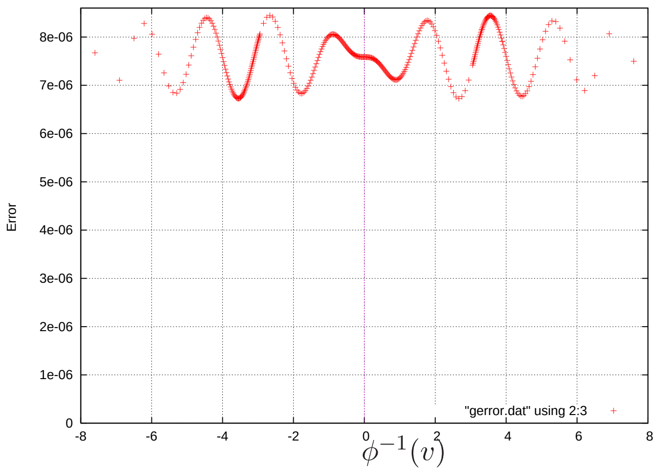

Figure 2: The error against $\phi^{-1}(v)$ for $\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 25$, using Haber's formula A.

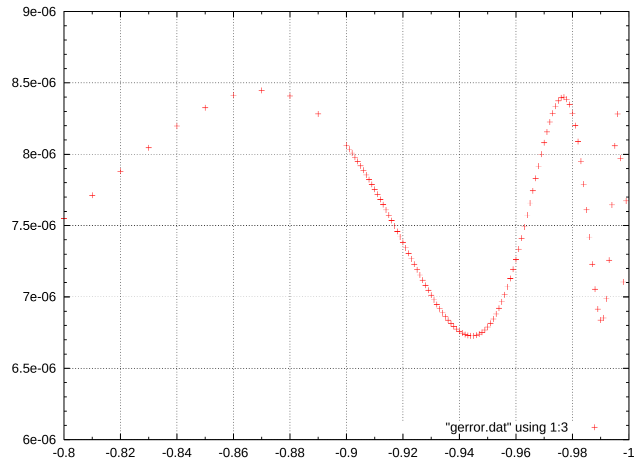

Figure 3: The error against v for $\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 25$, using Haber's formula A.

the error of applying Haber's formula A, thus we decided to plot Figure 3 ($v \in [-0.8, -1]$) to show that it occurs at $v = -0.87$.

Example 8.2. *Let us use formula (3.10) to approximate the integral below:*

$$\frac{1}{4\log 2}\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\mathrm{d}x. \tag{8.3}$$

We shall do this by finding the value of α, using condition $\mathbf{A}_4$ with $\phi(x) =$

$\tanh \frac{x}{2}$,
$\phi'(x) = \frac{1}{2} \operatorname{sech}^2 \frac{x}{2}$, *so that*

$$f(\phi(x)) = \frac{1}{4\log 2} \log \left(\frac{1 + \tanh \frac{x}{2}}{1 - \tanh \frac{x}{2}} \right).$$

After some algebra using the exponential form of $\tanh \frac{x}{2}$, *one obtains*

$$\begin{aligned} f(\phi(x)) &= \frac{1}{4\log 2} \log e^x = \frac{x}{4\log 2} \\ f(\phi(x))\phi'(x) &= \frac{x}{4\log 2} \times \frac{1}{2} \operatorname{sech}^2 \frac{x}{2} \\ &= \frac{x}{8\log(2)\cosh^2 \frac{x}{2}} \\ &= \frac{x}{2\log(2)(e^{\frac{x}{2}} + e^{-\frac{x}{2}})^2} \\ |f(\phi(x))\phi'(x)| &\leq O(e^{-|x|}), \quad |x| \to \infty. \end{aligned} \tag{8.4}$$

As in Example 8.1, 370 values of v were used with $\alpha = 1$. These values of v, α, c were substituted into (3.10)*, results are presented by Figures 4 and 5 as well as Table 2.*

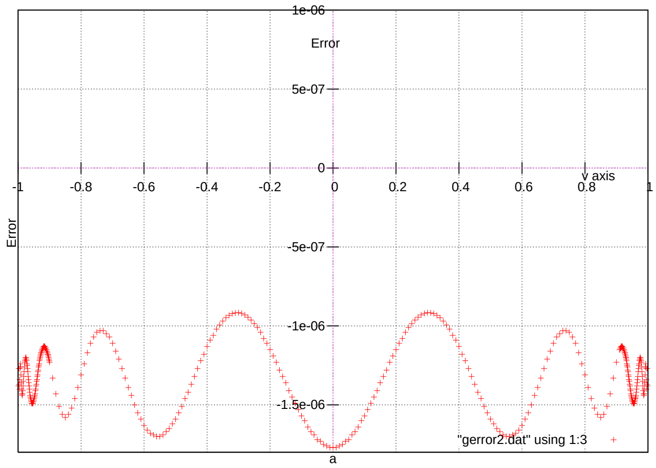

Figure 4: The error against v for $\frac{1}{4\log 2} \int_{-1}^{v} \log \left(\frac{1+x}{1-x} \right) \mathrm{d}x$, $N = 25$, using Haber's formula A.

A close look at Figure 4 shows a dying oscillation as v tends to ± 1, which means that the maximum absolute value of the error decreases towards the

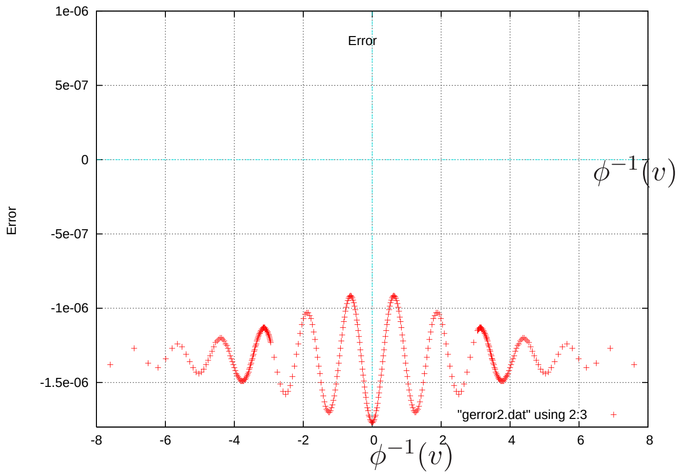

Figure 5: The error against $\phi^{-1}(v)$ for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v}\log\left(\frac{1+x}{1-x}\right)\,dx$, $N=25$, using Haber's formula A.

endpoints. The maximum absolute value of the error for Haber's formula A on the integrand $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v}\log\left(\frac{1+x}{1-x}\right)\,dx$ occurs at $v=0$, as shown in Figure 4.

8.2 Implementing Haber's Formula B

We shall try to implement Haber's Formula B on the integrals (8.1) and (8.3). From Haber's condition $\mathbf{A}_4$, they both decay single exponentially. The same analysis is applicable to the two integrals in the previous section, the only difference being that we use an auxiliary function $\varphi(x)=\dfrac{x+1}{2}$.

Example 8.3. *From the right-hand side of equation (8.2), with* $\alpha=\dfrac{1}{2}$, $c=\pi$ *and using the auxiliary function* $\varphi(x)=\dfrac{x+1}{2}$, *we substituted these values into (4.2), which is Haber's Formula B. Table 1 shows the values of* N *used and the maximum error. (See also Figures 6 and 7).*

The oscillations in Figure 6 are higher around $v=0$ and decreases towards ± 1. From Figure 8, we can see that the maximum absolute value of the error occurs at $v=-0.18$.

Example 8.4. *From the right-hand side of equation* (8.4), *with* $\alpha=1$, $c=\pi$ *and using the auxiliary function* $\varphi(x)=\dfrac{x+1}{2}$, *we then substituted these values into (4.2), which is Haber's Formula B. The results are tabulated in Table 2 and illustrated by Figure 9 and Figure 10.*

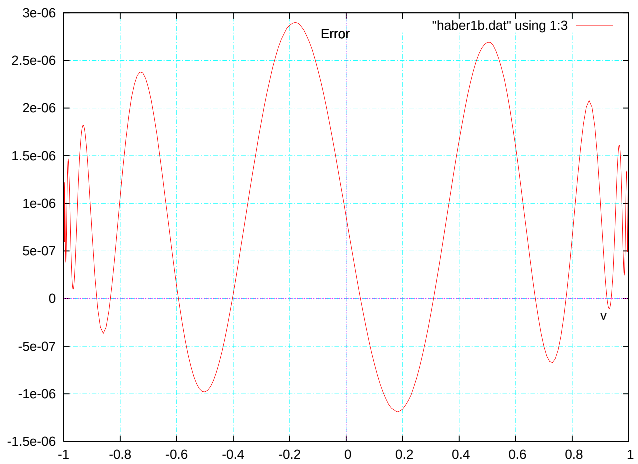

Figure 6: The error against v for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 36$, using Haber's formula B.

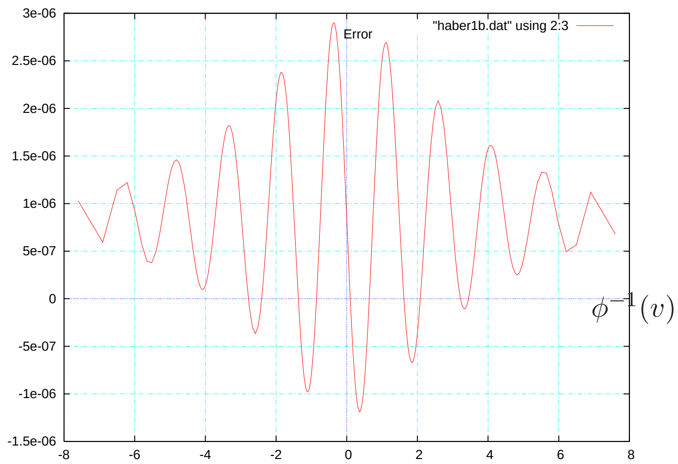

Figure 7: The error against $\phi^{-1}(v)$ for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 36$, using Haber's formula B.

As can be seen in Figure 9, the oscillations decreases towards the endpoints, but Fig 10 stretches the original figure like an "ideal spring". The maximum absolute error occurs at $v = 0$.

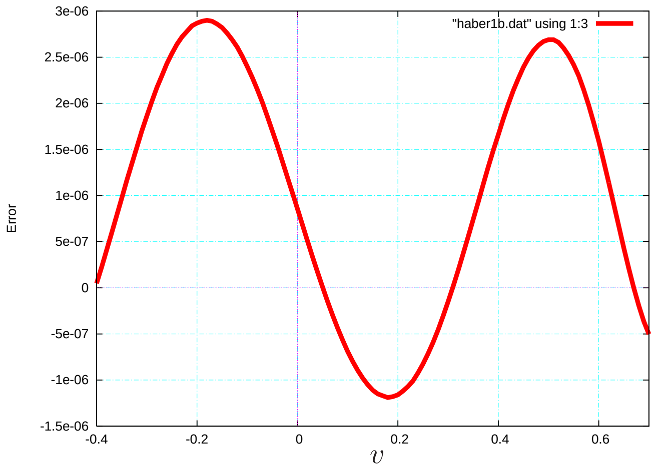

Figure 8: The error against v for $\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 36$, using Haber's formula B.

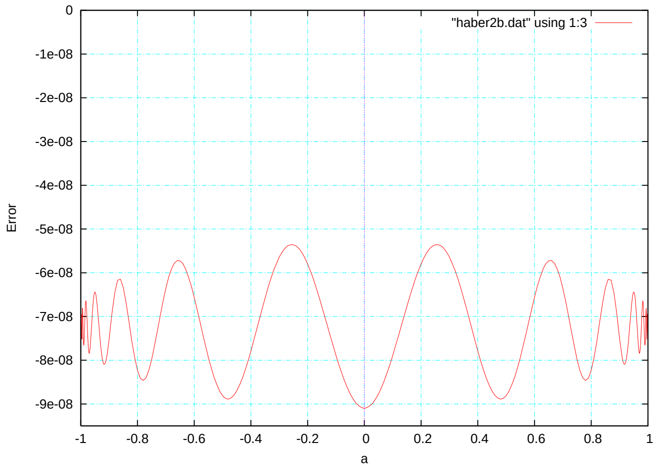

Figure 9: The error against v for $\frac{1}{4\log 2}\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\mathrm{d}x$, $N = 36$, using Haber's formula B.

8.3 Implementing the SE Formula

We shall use the integral in Examples 8.1 and 8.2 (they decay single exponentially) to implement the Single Exponential Formula. v

Example 8.5. *As already shown with Example 8.1, we can deduce that* $\alpha_f = \frac{1}{2}$, *and using (6.13), one will find that* $\tau = \frac{1}{4}$, $\alpha_f' = \min\left(\frac{1}{2}, 2(\frac{1}{4})\right) = \frac{1}{2}$,

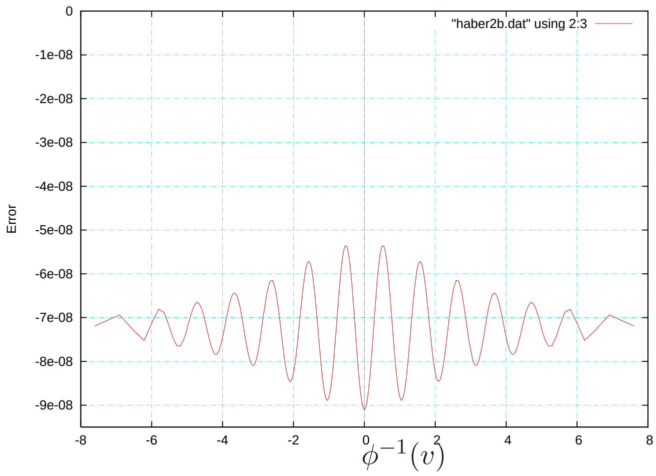

Figure 10: The error against $\phi^{-1}(v)$ for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v}\log\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$, $N = 36$, using Haber's formula B.

$c_f' = \min(c_f, 2\pi - \varepsilon_c) = \pi - \varepsilon$ *when* $\varepsilon_c = \varepsilon$. *Substituting these values into (6.11) we have that*

$$h = \sqrt{\frac{2\pi(\pi - \varepsilon)}{N}}.$$

These values are then plugged into the SE formula (6.14), as are the 370 values of v *(see Table 1, Figure 11 and Figure 12).*

By looking at Figure 11 we find that the oscillations increase toward ± 1. We plotted the figure with the gnuplot histeps option, because the other options did not join the point $v = 0$ with the other points, which is why the figure appears different to the other figures. A plot within the interval $[-1, -0.9]$, similar to that in Figure 3, shows that the maximum absolute value of the error occurs at
$v = -0.933$.

Example 8.6. *From the analysis of the integral in Example 8.2,* $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v}\log\left(\frac{1+x}{1-x}\right)\mathrm{d}x$, *we will take* $\alpha_f = 1$, *and from (6.11) and (6.13) we use* $\tau = \dfrac{1}{2}$, $c_f' = \pi - \varepsilon_c = \pi - \varepsilon$, *when* $\varepsilon_c = \varepsilon$, $\alpha_f' = \min(1, 1) = 1$*:*

$$h = \sqrt{\frac{\pi(\pi - \varepsilon)}{N}}.$$

The values obtained above, with the step size, are then substituted into (6.14) *(Refer to Table 2, Figure 13 and Figure 14).*

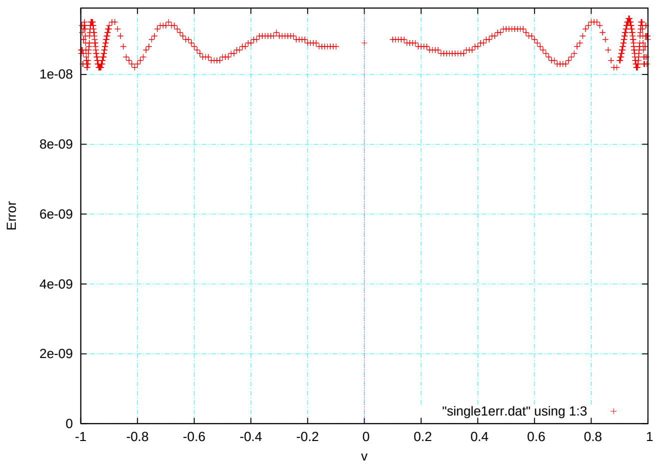

Figure 11: Error against v for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 64$, using the SE formula.

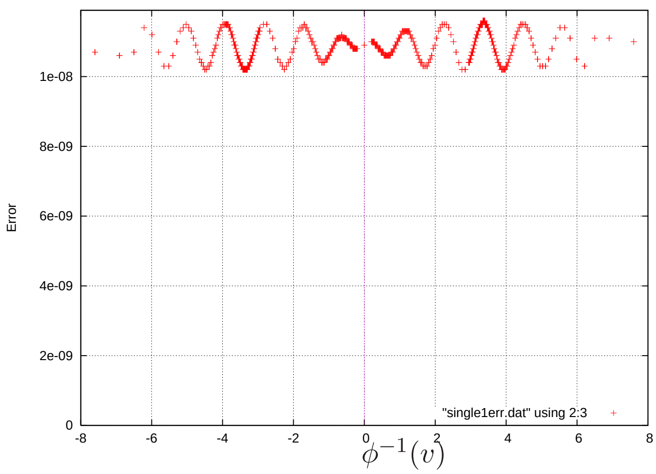

Figure 12: Error against $\phi^{-1}(v)$ for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 64$, using the SE formula.

Figure 13 shows that the maximum absolute value of the error occurs at $v = 0$. It also shows that the maximum absolute error decreases towards ± 1.

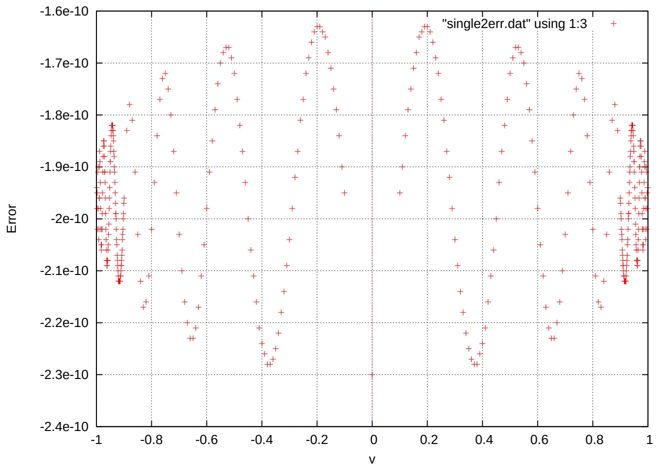

Figure 13: Error against v for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v}\log\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$, $N = 64$, using the SE formula.

8.4 Implementing Tanaka *et al.*'s Formula

Example 8.7. *We want to use formula (7.11) to approximate the integral in Example 8.1 with* $\phi_1(x) = \tanh\left(\frac{\pi}{2}\sinh x\right)$.

$$f(x) = \frac{1}{\pi\sqrt{1-x^2}} = O(|1-x^2|)^{-\frac{1}{2}}; \quad (x \to \pm 1)$$

$$\begin{aligned} f(\phi_1(x))\phi_1'(x) &= \frac{1}{\pi\sqrt{1-\tanh^2(\frac{\pi}{2}\sinh x)}}\frac{\pi\cosh x}{2\cosh^2(\frac{\pi}{2}\sinh x)} \\ &= \frac{\cosh x}{2\,\mathrm{sech}(\frac{\pi}{2}\sinh x)\cosh^2(\frac{\pi}{2}\sinh x)} \\ &= \frac{\cosh x}{2\cosh(\frac{\pi}{2}\sinh x)} \\ |f(\phi_1(x))\phi_1'(x)| &= O(\exp(-\frac{\pi}{4}\exp|x|)); \quad (x \to \pm\infty). \end{aligned}$$

Comparing this with the right-hand side of the expression $|f(x)| \le \alpha_f \exp(-\tau_f \exp(\lambda_f|x|))$, *one finds that*

$$\tau_f = \frac{\pi}{4}, \ \lambda_f = 1; \ \ c_f = \frac{\pi}{2Q} - \varepsilon_c = \frac{\pi}{2} - \varepsilon_c;$$

and using (7.23) *to* (7.27), $Q = \lambda_f = \lambda_v = 1, \ \ \tau_v = \frac{\pi}{4}, \ \ c_v = \frac{\pi}{2} - \varepsilon_c;$

$$P = \frac{\pi}{2\sin(\frac{\pi}{2} - \varepsilon_c)} - \varepsilon_P; \quad \varepsilon_P > 0 \ni P > 0. \tag{8.5}$$

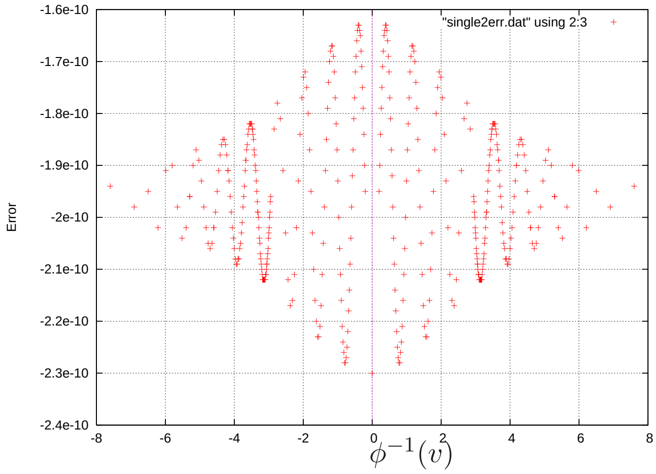

Figure 14: Error against $\phi^{-1}(v)$ for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right) \mathrm{d}x$, $N = 64$, using the SE formula.

For any small ε_c, $\sin\left(\frac{\pi}{2} - \varepsilon_c\right) \equiv 1$ *and* P *may be close to* $\frac{\pi}{2}$, *thus* $P = \frac{\pi}{2} - \varepsilon$ *must be chosen when* $\varepsilon_P = \varepsilon$ *and, making the appropriate substitutions,*

$$h = \frac{\log(\pi(c_v - \varepsilon)\lambda_v N/\tau_v)}{\lambda_v N} = \frac{\log((2\pi - 8\varepsilon)N)}{N}.$$

(Refer to Table 1, Figure 15 and Figure 16). Figure 15 displays no oscillations and, in contrast to the other figures, we see a clustering around zero, except at $v = 0.911$, which is the maximum absolute value of the error and at $v = -0.25$.

Example 8.8. *We want to use formula (7.11) to approximate the integral*

$$\frac{1}{4\log 2}\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right) \mathrm{d}x. \tag{8.6}$$

We illustrate how to obtain the parameters below.

$$\begin{aligned}
f(\phi_1(x))\phi_1'(x) &= \frac{1}{4\log 2}\log\left(\frac{1+\tanh(\frac{\pi}{2}\sinh x)}{1-\tanh(\frac{\pi}{2}\sinh x)}\right)\frac{\pi\cosh x}{2\cosh^2(\frac{\pi}{2}\sinh x)} \\
&= \frac{\pi\log(\exp(\pi\sinh x))\cosh x}{8\log 2\cosh^2(\frac{\pi}{2}\sinh x)} \\
&= \frac{\pi^2\sinh x\cosh x}{8\log 2\cosh^2(\frac{\pi}{2}\sinh x)} \\
|f(\phi_1(x))\phi_1'(x)| &= O(\exp(-\frac{\pi}{2}\exp(|x|))), \quad x \to \pm\infty.
\end{aligned}$$

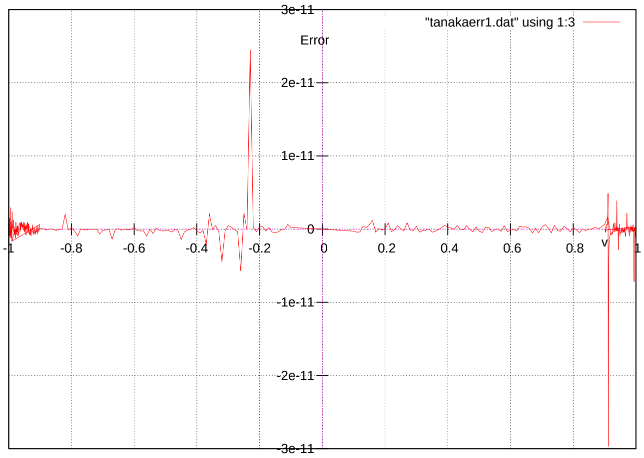

Figure 15: The error against v for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 64$, using Tanaka *et al.*'s formula.

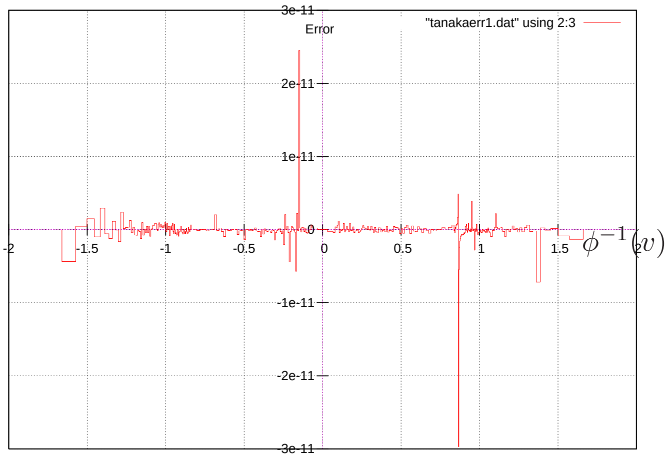

Figure 16: The error against $\phi^{-1}(v)$ for $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$, $N = 64$, using Tanaka *et al.*'s formula.

Thus $|f(\phi_1(x))\phi_1'(x)|$ *decays double exponentially with* $\tau_f = \frac{\pi}{2}$, $\lambda_f = 1$. *Using* (7.23) *to* (7.27), $P = \frac{\pi}{2}$, $\lambda_v = 1$, $\tau_v = \min\left\{\frac{\pi}{2}, \frac{\pi}{2} - (\varepsilon_P - \varepsilon_\tau)\right\} =$

$$\min\left\{\frac{\pi}{2}, \frac{\pi}{2} - 2\varepsilon\right\} = \frac{\pi}{2} - 2\varepsilon,\ c_v = \frac{\pi}{2} - \varepsilon_c = \frac{\pi}{2} - \varepsilon \textit{ and}$$

$$h = \frac{\log(\pi(c_v - \varepsilon)\lambda_v N/\tau_v)}{\lambda_v N} = \frac{\log \pi N}{N}.$$

A closer look at Figure 17 shows a clustering or increase in the error towards $+1$. Figure 18, which is supposed to stretch the plot so that we can see the behaviour towards the end-points, does not really help. So for us to trully see the maximum value of the error $v = 0.994$, and the behaviour towards $+1$, we plotted Figure 19, on the interval $[0.9, 1]$.

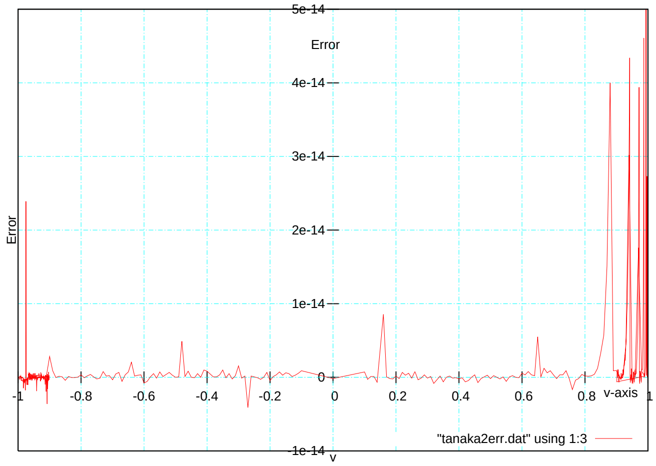

Figure 17: The error against v for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$, $N = 49$, using Tanaka *et al.'s* formula.

From Table 1, we discover that, for $N \geq 81$, there was a blow up for Haber's formulas A and B for the reason given in the final paragraph of section 5. This blow up is illustrated in Figure 20 by the vertical line going downwards at $N = 64$. In addition, one can see that Tanaka *et al.'s* formula gives the most accurate results as N becomes larger. However, Stenger's SE formula perform better than the Haber's formulas and did not blow up at $N = 64$.

A close look at Table 2 shows that Tanaka *et al.'s* formula gives more accurate results than the other formulas as well as a faster convergence to the exact solution. Figure 21 shows that the maximum absolute errors for the other three formulas coincide, indicating that they give almost the same results to a certain degree of accuracy on the integral $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$.

After deriving and showing how to implement the four quadrature formulas for numerical approximation to indefinite integrals based on the sinc

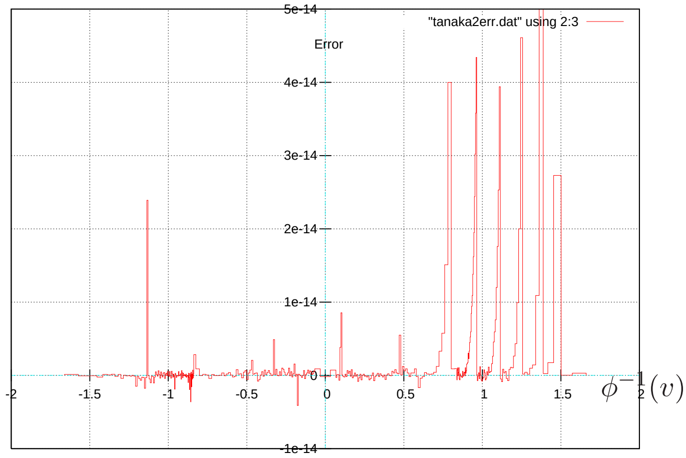

Figure 18: The error against $\phi^{-1}(v)$ for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v} \log\left(\dfrac{1+x}{1-x}\right)\,\mathrm{d}x$, $N = 49$, using Tanaka *et al.'s* formula.

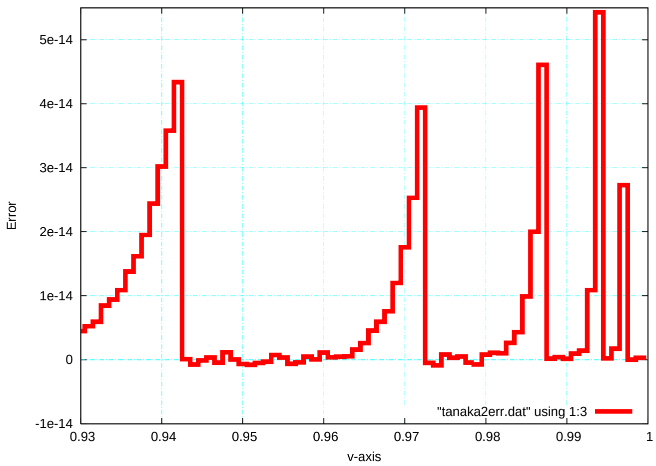

Figure 19: The error against v for $\dfrac{1}{4\log 2}\displaystyle\int_{-1}^{v} \log\left(\dfrac{1+x}{1-x}\right)\,\mathrm{d}x$, $N = 49$, using Tanaka *et al.'s* formula.

method, and in the earlier part of this section. Next, we want to ascertain the performances of the methods on the computational examples. This will be done by comparing the maximum absolute errors of the four formulas and the CPU times. In each table, "Max. Error" represents the maximum absolute value of the error of the approximation for the values of v for which the integral was evaluated, and N represents the number of function

N	Max. Error			
	Haber's A	**Haber's B**	**SE**	**Tanaka**
1	5.92e-02	1.08e-01	7.15e-02	2.88e-01
4	5.80e-03	1.70e-02	5.37e-03	1.08e-02
9	6.67e-04	2.09e-03	4.37e-04	1.07e-04
16	7.58e-05	2.36e-04	4.90e-05	2.84e-07
25	8.45e-06	2.63e-05	5.25e-06	1.78e-10
36	9.34e-07	2.90e-06	5.78e-07	2.97e-11
49	1.03e-07	3.18e-07	6.29e-08	2.97e-11
64	1.13e-08	3.48e-08	6.89e-09	2.97e-11
81			7.51e-10	2.97e-11
100			1.08e-10	2.97e-11

Table 1: The maximum error of the formulas on $\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$.

N	Max. Error			
	Haber's A	**Haber's B**	**SE**	**Tanaka**
1	1.67e-01	1.67e-01	1.66e-01	2.24e-01
4	1.06e-02	1.06e-02	1.06e-02	9.83e-03
9	6.01e-04	6.01e-04	6.03e-04	6.18e-05
16	3.35e-05	3.35e-05	3.38e-05	8.13e-08
25	1.77e-06	1.77e-06	1.79e-06	3.54e-11
36	9.10e-08	9.10e-08	9.25e-08	5.39e-14
49	4.55e-09	4.55e-09	4.65e-09	5.43e-14
64	2.24e-10	2.24e-10	2.30e-10	5.43e-14
81	1.08e-11	1.08e-11	1.12e-11	5.41e-14
100	5.20e-13	5.20e-13	5.40e-13	5.42e-14

Table 2: The maximum error of the formulas on $\frac{1}{4\log 2}\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$.

evaluations.

In order to ascertain which of the four quadrature formulas is the fastest, we computed the minimum CPU time it took for the numerical approximation of the two integrals under discuss. The results are as tabulated in Tables 3 and 4. For example, a close look at Table 3 shows that for $N = 64$, while the minimum CPU time for Haber's formula A is 1.92, it took 303, 295 and 354 seconds for Haber's formula B, Steger's Single Exponential formula and Tanaka *et al.'s* formula respectively. A similar performance was observed in Table 4. Therefore, this shows that Haber's formula A which involves a

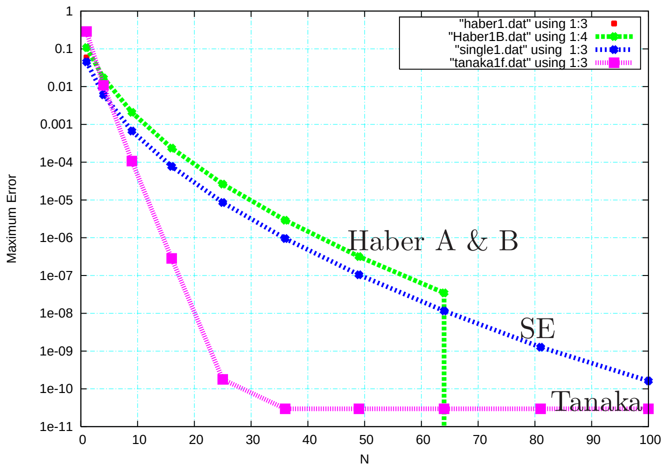

Figure 20: The logarithm of the maximum absolute error against N on $\displaystyle\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$.

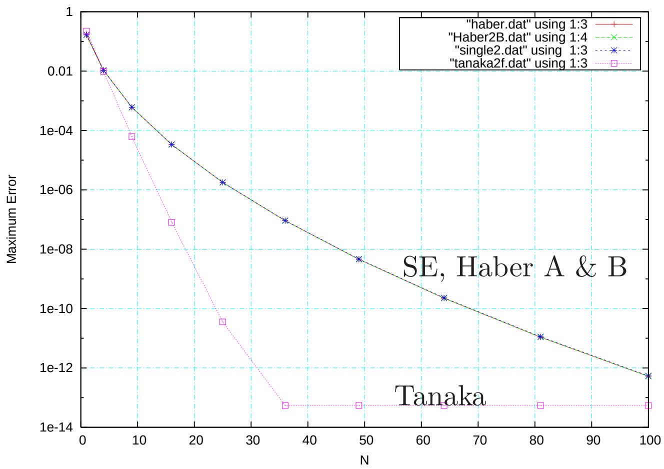

Figure 21: The logarithm of the maximum absolute error against N on $\displaystyle\frac{1}{4\log 2}\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\,\mathrm{d}x$.

single sum evaluation converges the fastest.

N	Minimum CPU Time			
	Haber's A	**Haber's B**	**SE**	**Tanaka**
1	3.60e-02	3.60e-02	3.20e-02	3.20e-02
4	6.80e-02	3.16e-01	2.96e-01	2.52e-01
9	1.24e-01	1.82e+00	1.65e+00	1.79e+00
16	2.48e-01	7.24e+00	6.84e+00	6.92e+00
25	4.12e-01	2.25e+01	2.15e+01	2.17e+01
36	7.64e-01	6.04e+01	5.84e+01	5.85e+01
49	1.24e+00	1.42e+02	1.38e+02	1.38e+02
64	1.92e+00	3.03e+02	2.95e+02	3.54e+02
81			5.63e+02	5.69e+02
100			1.05e+03	1.06e+03

Table 3: A comparison of the minimum CPU time of the four quadrature formulas on $\int_{-1}^{v} \frac{1}{\pi\sqrt{1-x^2}}\,\mathrm{d}x$.

N	Minimum CPU Time			
	Haber's A	**Haber's B**	**SE**	**Tanaka**
1	4.80e-02	2.00e-02	2.00e-02	2.80e-02
4	4.80e-02	1.68e-01	1.56e-01	3.44e-01
9	1.00e-01	9.88e-01	9.32e-01	1.62e+00
16	2.20e-01	4.08e+00	3.92e+00	6.58e+00
25	4.08e-01	1.30e+01	1.27e+01	2.10e+01
36	7.92e-01	3.51e+01	3.41e+01	6.14e+01
49	1.20e+00	8.23e+01	8.11e+01	1.34e+02
64	1.84e+00	1.78e+02	1.75e+02	2.87e+02
81	2.98e+00	3.42e+02	3.37e+02	5.76e+02
100	4.08e+00	6.35e+02	6.27e+02	1.06e+03

Table 4: A comparison of the minimum CPU time of the four quadrature formulas on $\frac{1}{4\log 2}\int_{-1}^{v} \log\left(\frac{1+x}{1-x}\right)\mathrm{d}x$.

9 Conclusions

As shown by the tables and figures above, the Double Exponential sinc method proposed by Tanaka, Sugihara and Murota [27] is the most accurate for the numerical approximation of indefinite integrals of functions with or without singularities. However, Haber's formula A, which involves one single sum evaluation had the smallest minimum CPU time as shown in Tables 5 and 6; in agreement with the unsubstantiated claims in [23].

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