Original Research Article

TWO – WAREHOUSE INVENTORY SYSTEM MODEL FOR DETERIORATING ITEMS CONSIDERING PARTIAL UPSTREAM TRADE CREDIT FINANCING

ABSTRACT

In this study, we developed an inventory system model under two – level trade credit where the supplier considers the retailer as credit risk but the retailer considers the customers as credit worthy. Therefore, the retailer is given a trade credit period on $(1-\delta)$ proportion of the goods ordered whenever he/she pays for δproportion of the goods immediately after delivery. In the same vein, the retailer passes the same grace to the customers but without attaching any condition as the customers are assumed credit worthy. This partial upstream trade credit is offered to reduce the risk of failure in payment on the business transaction especially that most retailers are involved in bulk orders. The relevant cost functions are determined and a numerical example is given. Sensitivity analysis was carried out to see the effect of changes in parameters on the optimal solution of the model.

Key words: Downstream, Partial Upstream, Deterioration, Credit –risk, Trade credit period.

1. INTRODUCTION

In inventory practice, stockists have explored various possible promotional tools to which they will stimulate demand of customers so as to enhance sales of their product. One of the promotional tools explored in the literature among others, is trade credit financing (permissible delay in payments). Trade credit is a kind of business transaction that allows delay in payment for an agreed period of time after delivery of the consignment. During such period, the beneficiary of the facility of the trade credit can make sales and then earn interest on the generated revenue without incurring any penalty. Beyond the given period, the beneficiary of the facility is charged an interest over the unsold items in stock. The literature that considered trade credit includes [1, 2, 3, 4, and 5].

In the early literature of permissible delay in payment as in Goyal [3], only retailers were given the grace of permissible delay in payment to stimulate their demand while the customers are not, but in some cases, this is not what happens. In order to reflect the reality in some market practices nowadays, researchers started looking at possibility of considering trade credit facility in the form supplier – retailer – customers known as upstream and downstream trade credit financing, Huang [6]. In short, the supplier offers the retailer a permissible delay in payment and the retailer passes the same grace to the customers. In doing so, some researchers see the need of attaching some conditions before giving in to the trade credit in order to curtail the menace of default in payments; see Shinn and Hwang [7] for instance, where the retailer is given order size dependent delay in payment.

An alternative situation is where the retailers or customers are given the choice of either cash discount or permissible delay in payment. In Chang et al [8] the retailer is given the choice between cash discount and permissible delay in payment. The cash discount is given for a shorter period say M_1 and permissible delay in payment for longer period sayM_2. In Chang [9], Economic Order Quantity (EOQ) model for deteriorating items under inflation and linked it with order size was developed. There was also consideration of situation where the retailer was offered with either cash discount or permissible delay in payment but linked it to order quantity as in [10]. Under normal circumstances, a credit – worthy retailer or credit – worthy customer obtains permissible delay in payment on the whole order placed (i.e. full upstream or full downstream), whereas a perceived to be credit – risk retailer or credit – risk customer is not given that opportunity so as to avoid the hazard of bad debt.

Thus, in order to curtail the menace of default in payment (bad debts), a facility known as partial trade credit was introduced. That is, a credit – risk retailer or a credit – risk customer is given the delay in payment only after depositing a substantial amount of money to cover a fraction of the total quantity of the items ordered. Partial downstream trade credit for a credit – risk customer was considered by Wu

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et al [11] and developed a model for deteriorating items with maximum lifetime using discounted cash flow analysis. In [14], a retailer that distinct between its bad and credit customers was considered. A model was also developed by [12] for an inventory system that considers the upstream trade credit to be linked with order quantity while offering the full downstream to customers. The upstream is linked with order quantity so as to reduce the effect of failure in payment in case of breach.

In contrast to all the papers mentioned earlier, the facility of trade credit despite it being partial as considered in this work, can led the retailer to decide to order the items in large quantity. The ordered goods can be in excess after reaching the maximum stocking capacity of the retailer's own warehouse, referred herein as OW. Thus, it becomes necessary for the retailer to rent another warehouse, referred herein as RW, of unlimited capacity where the remaining excess of goods ordered can be stored. The RW is assumed to have better preserving facilities than the OW. As such, the cost of stocking goods in RW is more than that of the OW and of course, the deterioration rate is smaller in RW than in OW. In this line, [5] developed a two - warehouse inventory system for deteriorating items under upstream permissible delay in payment. They assumed the retailer to be credit – worthy and therefore offered with full permissible delay in payment on the whole consignment ordered. There are many other works in this line such as [1, 13] and so on.

In this study, we consider two – level trade credit financing on two – warehouse inventory model but assuming the retailer as credit – risk and the customers as credit – worthy. Therefore, the retailer is offered with partial trade credit by the supplier and in turn the retailer offers customers with full trade credit in order to stimulate their demands. The partial trade credit is offered to the retailer to reduce the negative impact of failure in payment on the business transaction.

The structure of the work is as follows: In section 2, notations and assumptions are given, while in section 3, there is the model formulation. In section 4, optimization and analysis are given, while in section 5, numerical example is given. Sensitivity analysis is carried out in the same section while in section 6, conclusion and recommendations are given.

2. NOTATIONS AND ASSUMPTIONS

The following are the notations used in the model:

$I_r(t)$, $I_o(t)$ are the inventory level of the *RW* and *OW* respectively at time *t.*

D, W are the constant demand rate and the stocking capacity of OW respectively.

t_w, *T* is the time at which inventory in *RW* and that in OW drop to zero respectively.

α, β is the deterioration rate in *OW* and *RW* respectively, with $\alpha > \beta$.

h_r, h_o is the holding cost per unit per unit time of *RW* and *OW* respectively.

I_P , I_e is the interest payable and interest earned return rates respectively.

c, *p* is the purchasing cost and selling price of the item respectively.

M and N is the trade credit periods offered to the retailer by the supplier and the retailer to the customer respectively.

δ is the proportion of the quantity of goods directed to be paid by the retailer instantly whereas $1 - \delta$ is the proportion on which the trade credit period is given whereas A is the ordering cost per order.

TC is the total relevant costs per unit time (per annum) of the model to be minimized.

The following are the assumptions made in building the model:

a. Deterioration rate in *RW* is less than that in *OW*, i.e. $\alpha > \beta$ due to higher preserving facilities in *RW* and so charges higher holding cost in *RW* than in *OW*, i.e. $h_r > h_o$. Therefore it is assumed that $h_r - h_o > c(\alpha - \beta)$.
b. The demand in a warehouse is greater than the deterioration rates in the warehouses. Thus, $D > \alpha W$ and $D > \beta(Q - W)$ for OW and RW respectively. $Q - W$is the excess of the goods kept in *RW*.

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c. Due to the high holding cost constraint of *RW*, i.e.$h_r > h_o$, the goods in OW are dispatched only after the inventory in RW has dropped to zero (an economic reason).

d. In the model, the supplier offers the retailer with partial trade credit period, i.e. the retailer benefits from the trade credit only after making part payment to cover a proportion δ of the goods ordered upon delivery.

e. For economic benefit of the retailer, the proportion to be paid before giving the trade credit shall not exceed the proportion of the items given on trade credit, i.e. $\delta \ngtr (1-\delta)$, i.e. $\delta \leq (1-\delta)$.

f. Interest charged is assumed to be higher than the interest earned. This serves as penalty on the retailer whenever he/she fails to settle the account as at when agreed.

g. We restrict $M > N$ and also $N \leq t_w$.

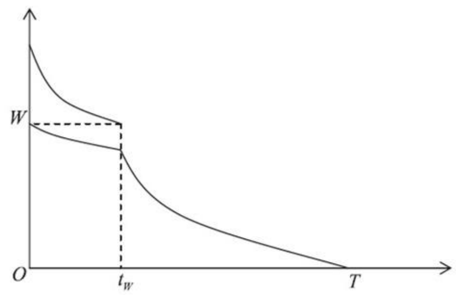

Fig 1: Pictorial presentation of the model

3. MATHEMATICAL MODEL FORMULATION

Due to economic reasons, in order to reduce the high holding cost in *RW*, goods at *RW* are first sold. At $t = t_w$ the inventory at *RW* drops to zero due to demand and deterioration during the period $[0\,, t_w]$, while goods in *OW* during that period, are depleted due to deterioration only. At $t = T$, both warehouses become empty due to depletion in *OW* by demands and deterioration during the period$[t_w\,, T]$. These phenomena are represented by the following differential equations:

$$\frac{dI_r(t)}{dt} + \beta I_r(t) = -D \quad 0 \leq t \leq t_w \tag{1}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = 0 \quad 0 \leq t \leq t_w \tag{2}$$

$$\frac{dI_o(t)}{dt} + \alpha I_o(t) = -D \quad , \quad t_w \leq t \leq T \tag{3}$$

With the boundary condition $I_r(t_w) = 0$ for *RW* in (1) and initial condition $I_o(0) = W$ and boundary condition $I_o(T) = 0$for *OW* in (2) and (3) respectively.

The solutions to equations (1) - (3) are

$$I_r(t) = \frac{D}{\beta}\left(e^{\beta(t_w - t)} - 1\right), \qquad 0 \leq t \leq t_w \tag{4}$$

$$I_o(t) = We^{-\alpha t} 0 \leq t \leq t_w \tag{5}$$

$$I_o(t) = \frac{D}{\alpha}\left(e^{\alpha(T-t)} - 1\right), \quad t_w \leq t \leq T \tag{6}$$

We have continuity in OW at $t = t_w$, using (5) and (6) we get,

$$T = t_w + \frac{1}{\alpha}\ln\left(\frac{\alpha}{D}We^{-\alpha t_w} + 1\right) \tag{7}$$

For the Total Costs (TC) per unit time, we add the following costs

a. Annual Ordering Cost

The annual ordering cost is given as $\frac{A}{T}$ (8)

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b. Annual Holding Cost

The annual holding cost of the goods in both warehouse, using equations (4), (5) and (6), is

$$\frac{Dh_r}{\beta^2 T}\left(e^{\beta t_w}-\beta t_w-1\right)+\frac{h_o}{T}\left(\frac{W}{\alpha}(1-e^{-\alpha t_w})+\frac{D}{\alpha^2}(e^{\alpha(T-t_w)}-\alpha(T-t_w)-1)\right) \quad (9)$$

c. Annual Deteriorating Cost.

The annual deterioration cost in both warehouse using (4), (5) and (6) is

$$\frac{DC}{\beta T}\left(e^{\beta t_w}-\beta t_w-1\right)+\frac{CW}{T}(1-e^{-\alpha t_w})+\frac{DC}{\alpha T}\left(e^{\alpha(T-t_w)}-\alpha(T-t_w)-1\right) \quad (10)$$

d. Annual Interest Payable And Interest Earned I_P and I_E

Based on the values of N, M, t_w and T, then, the following four cases can occur;

(1) $N < M \le t_w < T$ (2) $N \le t_w \le M < T$ (3) $T \le M < T+N$ (4) $T+N \le M$

Case 1: $N < M \le t_w < T$

Before the time *M*, the retailer will make sales and generate revenue. After the time *M*, the retailer will pay interest on all unsold items from $(1-\delta)$ proportion of the inventory. Therefore, using (4), (5) and (6), the annual interest payable by the retailer is

$$I_{P1}=\frac{(1-\delta)cI_p}{T}\left(\int_M^{t_w+N} I_r(t)dt+\int_M^{t_w} I_o(t)dt+\int_{t_w}^{T+N} I_o(t)dt\right)=\frac{(1-\delta)cI_p}{T}\left(\frac{D}{\beta^2}\left(e^{\beta(t_w-M)}-\beta(t_w+N-M)-e^{-\beta N}\right)+\frac{W}{\alpha}(e^{-\alpha M}-e^{-\alpha t_w})+\frac{D}{\alpha^2}\left(e^{\alpha(T-t_w)}-\alpha(T+N-t_w)-e^{-\alpha N}\right)\right) \quad (11)$$

For the goods from $1-\delta$ proportion of the order, the retailer will earn interest on the sales revenue recovered from the customers during the period$[N, M]$ and sales revenue made for the goods from δ proportion of the order for the period$[N, T]$. Thus, the annual interest earned by the retailer is given by

$$I_{E1}=\frac{pI_e}{T}\left((1-\delta)\int_N^M D(t-N)dt+\delta\int_N^T D(t-N)dt\right)=\frac{pDI_e}{2T}\left((1-\delta)(M-N)^2+\delta(T-N)^2\right) \quad (12)$$

Therefore, the total annual relevant costs for this case is

$$TC_1 = OC + HC + DC + I_{P1} - I_{E1}$$

Using equations (8), (9), (10), (11) and (12), we get

$$TC_1=\frac{1}{T}\left(A+\frac{D}{\beta^2}\left((h_r+c\beta)\left(e^{\beta t_w}-\beta t_w-1\right)+(1-\delta)cI_p\left(e^{\beta(t_w-M)}-\beta(t_w+N-M)-e^{-\beta N}\right)\right)+\frac{W}{\alpha}\left((h_o+c\alpha)(1-e^{-\alpha t_w})+(1-\delta)cI_p(e^{-\alpha M}-e^{-\alpha t_w})\right)+\frac{D}{\alpha^2}\left((h_o+c\alpha)\left(e^{\alpha(T-t_w)}-\alpha(T-t_w)-1\right)+(1-\delta)cI_p\left(e^{\alpha(T-t_w)}-\alpha(T+N-t_w)-e^{-\alpha N}\right)\right)-\frac{1}{2}pDI_e\left((1-\delta)(M-N)^2+\delta(T-N)^2\right)\right) \quad (13)$$

Case 2: $N \le t_w \le M < T$

In this case, goods in RW finished before the time *M*. If $M \le t_w + N$, there is outstanding payment from the last customers that bought goods from *RW* and therefore, the retailer will pay interest for the period(M, t_w+N). For the goods in OW, the retailer will pay interest for the period$(M, T+N)$. Consequently, using (4) and (6), the annual interest paid by the retailer from $(1-\delta)$ proportion of the goods ordered is given by

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$$I_{P2} = (1-\delta)\frac{cI_p}{T}\left(\int_M^{t_w+N} I_r(t)dt + \int_M^{T+N} I_o(t)dt\right) = (1-\delta)\frac{cI_p}{T}\left(\frac{D}{\beta^2}\left(e^{\beta(t_w-M)} - \beta(t_w+N-M) - e^{-\beta N}\right) + \frac{D}{\alpha^2}\left(e^{\alpha(T-M)} - \alpha(T+N-M) - e^{-\alpha N}\right)\right) \quad (14)$$

Likewise, the retailer will earn interest for the period (N,T) from the δ portion and (N,M) from the $1-\delta$ portion of the goods ordered. Therefore, the annual interest earned is given as

$$I_{E2} = \frac{pI_e}{T}\left((1-\delta)\int_N^M D(t-N)dt + \delta\int_N^T D(t-N)dt\right) = \frac{pDI_e}{2T}\left((1-\delta)(M-N)^2 + \delta(T-N)^2\right) \quad (15)$$

The total annual relevant costs of the system in this case is given by

$$TC_2 = OC + HC + DC + I_{P2} - I_{E2}$$

Using equations (8), (9), (10), (14) and (15), we obtain

$$TC_2 = \frac{1}{T}\Bigg(A + \frac{D}{\beta^2}\left((h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + (1-\delta)cI_p\left(e^{\beta(t_w-M)} - \beta(t_w+N-M) - e^{-\beta N}\right)\right) + \frac{W}{\alpha}(h_o + c\alpha)(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}\left((h_o + c\alpha)\left(e^{\alpha(T-t_w)} - \alpha(T-t_w) - 1\right) + (1-\delta)cI_p\left(e^{\alpha(T-M)} - \alpha(T+N-M) - e^{-\alpha N}\right)\right) - \frac{1}{2}pDI_e\left((1-\delta)(M-N)^2 + \delta(T-N)^2\right)\Bigg) \quad (16)$$

Case 3: $T \le M < T+N$

In this case, both warehouse are empty before the time *M*. Hence, the retailer will pay interest on only the outstanding payments from the last customers. Therefore, using (6), the annual interest payable by the retailer is

$$I_{P3} = (1-\delta)\frac{cI_p}{T}\left(\int_M^{T+N} I_o(t)dt\right) = (1-\delta)\frac{DcI_p}{\alpha^2 T}\left(e^{\alpha(T-M)} - \alpha(T+N-M) - e^{-\alpha N}\right) \quad (17)$$

The retailer will earn interest on the sales revenue recovered from the customers during$[N,M]$ from the $1-\delta$ proportion of the total inventory. The retailer will also earn interest from δ proportion of the goods sold for the period$[N\ ,M]$. Therefore, the annual interest earned is given by

$$I_{E3} = \frac{pI_e}{T}\left((1-\delta)\left(\int_N^T D(t-N)dt + DT(M-T)\right) + \delta\left(\int_N^T D(t-N)dt + DT(M-T)\right)\right) = \frac{pDI_e}{2T}\left((T-N)^2 + 2T(M-T)\right) \quad (18)$$

The total annual relevant costs for the model in this case is given by

$$TC_3 = OC + HC + DC + I_{P3} - I_{E3}$$

Using equations (8), (9), (10), (17) and (18), we obtain

$$TC_3 = \frac{1}{T}\Bigg(A + \frac{D}{\beta^2}(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + \frac{W}{\alpha}(h_o + c\alpha)(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}\left((h_o + c\alpha)\left(e^{\alpha(T-t_w)} - \alpha(T-t_w) - 1\right) + (1-\delta)cI_p\left(e^{\alpha(T-M)} - \alpha(T+N-M) - e^{-\alpha N}\right)\right) - \frac{1}{2}pDI_e\left((T-N)^2 + 2T(M-T)\right)\Bigg) \quad (19)$$

Case 4: $T + N \le M$

In this case, goods are finished at both warehouses and the retailer has no outstanding payments from customers.

Therefore, the annual interest payable by the retailer is given by

$$I_{P4} = 0 \quad (20)$$

The retailer had already sold all the items and had also received all payments from the customers, therefore, the annual interest earned by the retailer is given by

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$$I_{E4} = \frac{pI_e}{T}\left(\int_N^T D(t-N)dt + DT(T+N-T) + D(T+N)(M-T-N)\right) = \frac{pDI_e}{2T}\left(T^2 + N^2 + 2(T+N)(M-T-N)\right) \quad (21)$$

Therefore, the annual relevant costs for the model in this case is given by

$$TC_4 = OC + HC + DC + I_{P4} - I_{E4}$$

Using equations (8), (9), (10), (20) and (21), we get

$$TC_4 = \frac{1}{T}\left(A + \frac{D}{\beta^2}(h_r + c\beta)\left(e^{\beta t_w} - \beta t_w - 1\right) + \frac{W}{\alpha}(h_o + c\alpha)(1 - e^{-\alpha t_w}) + \frac{D}{\alpha^2}(h_o + c\alpha)\left(e^{\alpha(T-t_w)} - \alpha(T-t_w) - 1\right) - \frac{1}{2}pDI_e\left(T^2 + N^2 + 2(T+N)(M-T-N)\right)\right) \quad (22)$$

4 OPTIMALITY CONDITIONS

The necessary conditions for TC_1 (total cost per annum for case 1) to have minimum are $\frac{\partial TC_1}{\partial t_w} = 0$ and $\frac{\partial TC_1}{\partial T} = 0$

From equation (13) and setting the result to zero we have the following equations

$$\frac{\partial TC_1}{\partial t_w} = \frac{1}{T}\left(\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right) + (1-\delta)cI_p\left(e^{\beta(t_w - M)} - 1\right)\right) + W\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{-\alpha t_w} + \frac{D}{\alpha}\left(h_o + c\alpha + (1-\delta)cI_p\right)\left(1 - e^{\alpha(T-t_w)}\right)\right) = 0 \quad (23)$$

$$\frac{\partial TC_1}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}\left(h_o + c\alpha + (1-\delta)cI_p\right)\left(e^{\alpha(T-t_w)} - 1\right) - pDI_e\delta(T-N) - TC_1\right) = 0 \quad (24)$$

The solutions to (23) and (24) give the values of t_w and *T*. To confirm that the optimal solution exist and is unique, we show that the determinant of a Hessian matrix evaluated at (t_w^{1*}, T_1^*) is positive definite. Thus,

$$\left.\frac{\partial^2 TC_1}{\partial t_w^2}\right|_{(t_w^{1*}, T_1^*)} = \frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + (1-\delta)cI_p e^{\beta(t_w - M)}\right) - \alpha W\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{-\alpha t_w} + D\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{1*}, T_1^*)} > \frac{1}{T}\left(\left(De^{\alpha T} - \alpha W\right)\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{-\alpha t_w}\right)\Big|_{(t_w^{1*}, T_1^*)} > 0 \quad (25)$$

since $D - \alpha W > 0$ from assumption (b) and $e^{\alpha T} \geq 1$ for any value of *T*. Therefore $(De^{\alpha T} - \alpha W) > 0$ confirms the result. Also,

$$\left.\frac{\partial^2 TC_1}{\partial T^2}\right|_{(t_w^{1*}, T_1^*)} = \frac{1}{T}\left(D\left(\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{\alpha(T-t_w)} - \delta pI_e\right)\right)\Big|_{(t_w^{1*}, T_1^*)} \text{since } \left.\frac{\partial TC_1}{\partial T}\right|_{(t_w^{1*}, T_1^*)} = 0 \quad (26)$$

Thus value of $\left.\frac{\partial^2 TC_1}{\partial T^2}\right|_{(t_w^{1*}, T_1^*)} > 0$ if and only if $D\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{\alpha(T-t_w)} - D\delta pI_e > 0$.

Lemma 1: If $(1-\delta)cI_p \geq \delta pI_e$ then the quantity given by $D\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{\alpha(T-t)} - D\delta pI_e > 0$ for all values of *t*<*T*.

Proof: Let $f(t) = D\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{\alpha(T-t)} - D\delta pI_e$.

If $t < T$, then $e^{\alpha(T-t)} > 1$. From the hypothesis, $(1-\delta)cI_p \geq \delta pI_e$ and so $f(t) > 0$. Proved.

Using (13) and since $\left.\frac{\partial TC_1}{\partial t_w}\right|_{(t_w^{1*}, T_1^*)} = 0$ we find that

$$\left.\frac{\partial^2 TC_1}{\partial T \partial t_w}\right|_{(t_w^{1*}, T_1^*)} = -\frac{1}{T}\left(D\left(h_o + c\alpha + (1-\delta)cI_p\right)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{1*}, T_1^*)} = \left.\frac{\partial^2 TC_1}{\partial t_w \partial T}\right|_{(t_w^{1*}, T_1^*)} \quad (27)$$

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Note that $\left\{\frac{\partial^2 TC_1}{\partial t_w^2}\frac{\partial^2 TC_1}{\partial T^2} - \frac{\partial^2 TC_1}{\partial t_w \partial T}\frac{\partial^2 TC_1}{\partial T \partial t_w}\right\}\Big|_{(t_w^{1*},T_1^*)} > 0$ using equations (25) – (27).

Theorem 1: if $(1-\delta)cI_p \geq \delta pI_e$, then the cost function in equation (13) is a convex function.

Proof: The proof is obvious using lemma 1 and equations (25) – (27) which shows that the Hessian matrix is positive definite.

The necessary conditions for TC_2 (total cost per annum for case 2) to have minimum are $\frac{\partial TC_2}{\partial t_w} = 0$ and $\frac{\partial TC_2}{\partial T} = 0$

From equation (16) and setting the result to zero, we obtain the following equations

$$\frac{\partial TC_2}{\partial t_w} = \frac{1}{T}\left(\frac{D}{\beta}\left((h_r + c\beta)\left(e^{\beta t_w} - 1\right) + (1-\delta)cI_p\left(e^{\beta(t_w - M)} - 1\right)\right) + W(h_o + c\alpha)e^{-\alpha t_w} + \frac{D}{\alpha}(h_o + c\alpha)\left(1 - e^{\alpha(T - t_w)}\right)\right) = 0 \quad (28)$$

$$\frac{\partial TC_2}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}\left((h_o + c\alpha)\left(e^{\alpha(T - t_w)} - 1\right) + (1-\delta)cI_p\left(e^{\alpha(T-M)} - 1\right)\right) - \delta pDI_e(T - N) - TC_2\right) = 0 \quad (29)$$

The solutions to (28) and (29) give the values of t_w and T for this case. To show that the optimal solution exist and is unique, we show that the determinant of the Hessian matrix evaluated at (t_w^{2*}, T_2^*) is positive definite. Thus,

$$\frac{\partial^2 TC_2}{\partial t_w^2}\Big|_{(t_w^{2*},T_2^*)} = \frac{1}{T}\left(D\left((h_r + c\beta)e^{\beta t_w} + (1-\delta)cI_p e^{\beta(t_w - M)}\right) - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T - t_w)}\right)\Big|_{(t_w^{2*},T_2^*)} > \frac{1}{T}\left((De^{\alpha T} - \alpha W)(h_o + c\alpha)e^{-\alpha t_w}\right)\Big|_{(t_w^{2*},T_2^*)} > 0 \quad (30)$$

Also,

$$\frac{\partial^2 TC_2}{\partial T^2}\Big|_{(t_w^{2*},T_2^*)} = \frac{1}{T}\left(D\left((h_o + c\alpha)e^{\alpha(T - t_w)} + (1-\delta)cI_p e^{\alpha(T-M)}\right) - \delta pI_e D\right)\Big|_{(t_w^{2*},T_2^*)} \quad (31)$$

since $\frac{\partial TC_2}{\partial T}\Big|_{(t_w^{2*},T_2^*)} = 0$. Thus value of $\frac{\partial^2 TC_2}{\partial T^2}\Big|_{(t_w^{2*},T_2^*)} > 0$ if and only if $D\left((h_o + c\alpha)e^{\alpha(T - t_w)} + (1-\delta)cI_p e^{\alpha(T-M)}\right) > \delta pI_e D$. This has been proved in lemma 1 since t_w and M are less than T.

$$\frac{\partial^2 TC_2}{\partial T \partial t_w}\Big|_{(t_w^{2*},T_2^*)} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T - t_w)}\right)\Big|_{(t_w^{2*},T_2^*)} = \frac{\partial^2 TC_2}{\partial t_w \partial T}\Big|_{(t_w^{2*},T_2^*)} \quad (32)$$

Note that $\left\{\frac{\partial^2 TC_2}{\partial t_w^2}\frac{\partial^2 TC_2}{\partial T^2} - \frac{\partial^2 TC_2}{\partial t_w \partial T}\frac{\partial^2 TC_2}{\partial T \partial t_w}\right\}\Big|_{(t_w^{2*},T_2^*)} > 0$ from equations (30) – (32).

Theorem 2: If $(1-\delta)cI_p \geq \delta pI_e$, then the cost function in equation (16) is a convex function.

Proof: The proof is obvious using lemma 1 and equations (30) – (32) which shows that the Hessian matrix is positive definite.

The necessary conditions for TC_3 (total cost per annum for case 3) to have minimum are $\frac{\partial TC_3}{\partial t_w} = 0$ and $\frac{\partial TC_3}{\partial T} = 0$

From equation (19) and setting the result to zero, we obtain the following equations

$$\frac{\partial TC_3}{\partial t_w} = \frac{1}{T}\left(\frac{D}{\beta}(h_r + c\beta)\left(e^{\beta t_w} - 1\right) + W(h_o + c\alpha)e^{-\alpha t_w} + \frac{D}{\alpha}(h_o + c\alpha)\left(1 - e^{\alpha(T - t_w)}\right)\right) = 0 \quad (33)$$

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$$\frac{\partial TC_3}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}\left((h_o + c\alpha)\left(e^{\alpha(T-t_w)} - 1\right) + (1-\delta)cI_p\left(e^{\alpha(T-M)} - 1\right)\right) - pDI_e(M - N - T) - TC_3\right) = 0 \quad (34)$$

The solutions to Equations (33) and (34) give the values of t_w and *T* in this case. To confirm that the optimal solution exist and is unique, we show that the determinant of the Hessian matrix evaluated at (t_w^{3*}, T_3^*) is positive definite. Hence,

$$\left.\frac{\partial^2 TC_3}{\partial t_w^2}\right|_{(t_w^{3*},T_3^*)} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{3*},T_3^*)} > \frac{1}{T}\left((De^{\alpha T} - \alpha W)(h_o + c\alpha)e^{-\alpha t_w}\right)\Big|_{(t_w^{3*},T_3^*)} > 0 \quad (35)$$

also,

$$\left.\frac{\partial^2 TC_3}{\partial T^2}\right|_{(t_w^{3*},T_3^*)} = \frac{1}{T}\left(D\left((h_o + c\alpha)e^{\alpha(T-t_w)} + (1-\delta)cI_p e^{\alpha(T-M)}\right) + pDI_e\right)\Big|_{(t_w^{3*},T_3^*)} > 0 \quad (36)$$

and

$$\left.\frac{\partial^2 TC_3}{\partial T \partial t_w}\right|_{(t_w^{3*},T_3^*)} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{3*},T_3^*)} = \left.\frac{\partial^2 TC_3}{\partial t_w \partial T}\right|_{(t_w^{3*},T_3^*)} \quad (37)$$

Note that $\left\{\frac{\partial^2 TC_3}{\partial t_w^2}\frac{\partial^2 TC_3}{\partial T^2} - \frac{\partial^2 TC_3}{\partial t_w \partial T}\frac{\partial^2 TC_3}{\partial T \partial t_w}\right\}\Big|_{(t_w^{3*},T_3^*)} > 0$ from equations (35) – (37).

Theorem 3: The cost function in equation (19) is a convex function.

Proof: The proof is obvious using equations (35) – (37) which shows the Hessian matrix to be positive definite.

The necessary conditions for TC_4 (total cost per annum for case 4) to have minimum are $\frac{\partial TC_4}{\partial t_w} = 0$ and $\frac{\partial TC_4}{\partial T} = 0$

From equation (22) and setting the result to zero, we get the following equations

$$\frac{\partial TC_4}{\partial t_w} = \frac{1}{T}\left(\frac{D}{\beta}(h_r + c\beta)\left(e^{\beta t_w} - 1\right) + W(h_o + c\alpha)e^{-\alpha t_w} + \frac{D}{\alpha}(h_o + c\alpha)\left(1 - e^{\alpha(T-t_w)}\right)\right) = 0 \quad (38)$$

$$\frac{\partial TC_4}{\partial T} = \frac{1}{T}\left(\frac{D}{\alpha}(h_o + c\alpha)\left(e^{\alpha(T-t_w)} - 1\right) - pDI_e(M - 2N - T) - TC_4\right) = 0 \quad (39)$$

The solutions to Equations (38) and (39) give the values of t_w and *T* in this case. To show that the optimal solution exist and is unique, we show that the determinant of the Hessian matrix evaluated at the point (t_w^{4*}, T_4^*) is positive definite. Thus,

$$\left.\frac{\partial^2 TC_4}{\partial t_w^2}\right|_{(t_w^{4*},T_4^*)} = \frac{1}{T}\left(D(h_r + c\beta)e^{\beta t_w} - \alpha W(h_o + c\alpha)e^{-\alpha t_w} + D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{4*},T_4^*)} > \frac{1}{T}\left((De^{\alpha T} - \alpha W)(h_o + c\alpha)e^{-\alpha t_w}\right)\Big|_{(t_w^{4*},T_4^*)} > 0 \quad (40)$$

Also,

$$\left.\frac{\partial^2 TC_4}{\partial T^2}\right|_{(t_w^{4*},T_4^*)} = \frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)} + pDI_e\right)\Big|_{(t_w^{4*},T_4^*)} > 0 \quad (41)$$

$$\left.\frac{\partial^2 TC_4}{\partial T \partial t_w}\right|_{(t_w^{4*},T_4^*)} = -\frac{1}{T}\left(D(h_o + c\alpha)e^{\alpha(T-t_w)}\right)\Big|_{(t_w^{4*},T_4^*)} = \left.\frac{\partial^2 TC_4}{\partial t_w \partial T}\right|_{(t_w^{4*},T_4^*)} \quad (42)$$

Note that $\left\{\frac{\partial^2 TC_4}{\partial t_w^2}\frac{\partial^2 TC_4}{\partial T^2} - \frac{\partial^2 TC_4}{\partial t_w \partial T}\frac{\partial^2 TC_4}{\partial T \partial t_w}\right\}\Big|_{(t_w^{4*},T_4^*)} > 0$ from equations (40) – (42).

Theorem 4: The cost function in equation (22) is a convex function.

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Proof: The proof to the theorem is obvious using equations (40) – (42) which shows the Hessian matrix to be positive definite.

5. NUMERICAL EXAMPLE

Example: Given an inventory system with the following parameters; $A = 1500$, $D = 2000$, $W = 100$, $c = 10$, $p = 15$, $h_r = 3$, $h_o = 1$, $\beta = 0.06$, $\alpha = 0.1$, $\delta = 0.4$, $M = 0.5$, $N = 0.25$, $I_e = 0.12$, $I_p = 0.15$.

We find that using the model, the optimal time period (given in days for case 2) in which goods are finished at RW is $t_w^* = 205(0.5616)$ days and both warehouses becomes empty at $T^* = 222(0.6088)$ days and the associated optimal cost is $TC^* = 4113.52$. The result for all the four cases are presented in table 1 below.

Table 1: Result of the numerical example for the model

Cases	t_w^*	T^*	TC^*
Case 1	0.5315	0.6571	6222.52
Case 2	0.5616	0.6088	4113.52
Case 3	0.4630	0.5106	4335.06
Case 4	0.3890	0.4370	5289.92

Sensitivity Analysis: We now study the effect of parameter changes (Sensitivity analysis) of the inventory system W, A and D (which are presumed to be the most important parameters) on the optimal policies of example above. The values of the parameters used are $W \in (100,\ 250,\ 400)$, $A \in (1500,\ 2000,\ 2500)$ and $D \in (2000,\ 2500,\ 3000)$. The result of the sensitivity analysis is given in table 2.

Table 2: Result of the sensitivity Analysis for the model

W	*A*	*D*	t_w^*	T^*	TC^*
100	1500	2000	0.5616	0.6088	4113.52
		2500	0.5151	0.5530	4533.48
		3000	0.4822	0.5139	4906.55
	2000	2000	0.6411	0.6879	4884.53
		2500	0.5863	0.6240	5382.51
		3000	0.5452	0.5767	5823.37
	2500	2000	0.7123	0.7588	5575.68
		2500	0.6493	0.6867	6145.45
		3000	0.6027	0.6341	6649.15
		2000	0.5041	0.6222	3909.08

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250	1500	2500	0.4712	0.5661	4333.17
		3000	0.4438	0.5232	4710.60
	2000	2000	0.5836	0.7008	4664.30
		2500	0.5397	0.6340	5166.08
		3000	0.5068	0.5858	5610.97
	2500	2000	0.6520	0.7685	5344.49
		2500	0.6027	0.6965	5917.66
		3000	0.5643	0.6428	6425.06
400	1500	2000	0.4520	0.6414	3739.75
		2500	0.4301	0.5822	4163.99
		3000	0.4110	0.5381	4542.67
	2000	2000	0.5288	0.7167	4475.59
		2500	0.4959	0.6470	4977.73
		3000	0.4712	0.5976	5423.97
	2500	2000	0.5973	0.7834	5141.99
		2500	0.5562	0.7064	5715.53
		3000	0.5260	0.6517	6224.27

From table 2 above, we can deduce the following:

a. The retailer incurs highest *TC* when the capacity of *OW* is small, W=100, and A=2500 and D=3000, increases. This is obvious since the larger proportion of the goods are kept in RW with large holding cost.
b. The retailer incurs minimum *TC* when W=400 increases, and A=1500 remain as it is while D=2000 increases. This is also expected since the stocking capacity of the *OW* is increase, the *D* is at peak and the *A* is small.

6. CONCLUSION AND RECOMMENDATION

In the study, we have developed an EOQ model for a two – warehouse inventory system in a situation when only the retailer is suspected to be not credit – worthy. Partial upstream trade credit was incorporated to check the credit riskiness of the retailer. This is as a result of negative effect of failure in payment on a business transaction. The result obtained (Table 1 in particular) shows that when there is goods in RW, the retailer incurs minimum *TC* compared to other cases that considers situation when goods are finished in RW. We also realized that when *A* is increase so also the *D*, there is increase in *TC*. We recommend this work to be extended to consider the demand and the deterioration rate to be time-varying. The holding cost can also be assumed to be dependent on the price of the items.

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