
Mathematical Logic of the Jones and Homfly Polynomials of Knotted Trivalent Networks

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Abstract

Two rational functions are defined logically for special type of knotted trivalent networks as state models of planar trivalent networks. The restriction of these two rational functions reduce to the Jones and Homfly polynomials for nonoriented links. Also, these two models are used to define two invariants for this special type of knotted trivalent networks embedded in \mathbb{R}^3 . Finally, we study some congruences of these two polynomials for periodic knotted trivalent networks **this generalize the work of periodicity of the Jones and Homfly polynomials on knots to these two rational functions of knotted trivalent networks.**

Keywords: Trivalent networks, Homfly polynomial, Jones polynomial

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1 Introduction

Polynomials play an important role in the many fields of engineering, chemical, biological and mathematical sciences (**see for example** [9]). In particular, Jones and Homfly polynomials are defined as polynomial invariants of isotopy for oriented links in [4] and in [3, 13] respectively. They are denoted by $V_L(t)$ and $H_L(a, z)$ for an oriented link L and they are characterized by the following recursive mathematical relations:

1. $V_{L_1}(t) = V_{L_2}(t)$ whenever L_1 is isotopic to L_2 .
 2. If D is a simple closed curve, then $V_D(t) = 1$.
-

3.

$$t^{-1}V \begin{array}{c} \diagup \\ \diagdown \end{array} - tV \begin{array}{c} \diagdown \\ \diagup \end{array} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})V \begin{array}{c} \diagup \\ \diagdown \end{array} \left(\right),$$

for the Jones polynomial $V_L(t)$ and by

1. $H_{L_1}(a, z) = H_{L_2}(a, z)$ whenever L_1 is isotopic to L_2 .
2. If H is a simple closed curve, then $H_D(a, z) = 1$.
- 3.

$$aH \begin{array}{c} \diagup \\ \diagdown \end{array} - a^{-1}H \begin{array}{c} \diagdown \\ \diagup \end{array} = zH \begin{array}{c} \diagup \\ \diagdown \end{array} \left(\right),$$

for the Homfly polynomial $H_L(a, z)$. The diagrams in the above equations are parts of larger diagrams that are identical except as indicated.

Later, Kauffman defined the Jones polynomial as a state model in terms of disjoint union of simple closed curves in [5, 6]. Also, the authors of [8] defined a version of the Homfly polynomial as state model in terms of colored oriented trivalent planar graphs. We extend the Homfly and Jones polynomial to a special type of knotted trivalent networks as state models in terms planar trivalent networks. The restriction of these state models to links gives the Jones and the Homfly polynomials of unoriented links. Also, these state models give invariants of a special type of knotted trivalent networks embedded in \mathbb{R}^3 . At the end, we study congruences of these state models of periodic trivalent networks.

This work is inspired by the work of the authors in [1] and we follow their convention for the two types of trivalent graphs. The first type is a trivalent planar graph and the second type is knotted trivalent graph embedded in \mathbb{R}^3 . The edges of these two types of trivalent graphs are also of two types namely standard and wide edges such that there is exactly one wide edge incident to a pair of trivalent vertices. The wide edges considered as being rigid. In other words, there is a cyclic ordering of the four standard edges incident to that wide edge.

This paper is outlined as follows. In section 2, we define a rational function on the set of knotted trivalent networks that reduces to the Jones polynomial for knots. In section 3 and in a similar manner, we define a rational function on the set of knotted trivalent networks that reduces to the Homfly polynomial for knots. In section 4, we present congruences of these two rational functions for periodic knotted trivalent networks that generalize the work for periodic knots for Jones and Homfly polynomials.

2 State Model for the Jones Polynomial

Let Υ be a knotted trivalent network that is a trivalent graph in \mathbb{R}^3 of two types of edges namely standard and wide edges such that there is exactly one wide edge incident to a pair of trivalent vertices. The wide edges considered as being rigid. Here we are using the term network as a synonym for a graph. The states of Υ are planar trivalent networks obtained by replacing each crossing by one of the local diagrams $\begin{array}{c} \diagup \\ \diagdown \end{array} \left(\right)$, $\begin{array}{c} \diagdown \\ \diagup \end{array} \left(\right)$ or $\begin{array}{c} \diagup \\ \diagdown \end{array} \left(\right)$. The state networks have both standard and wide edges, such that each trivalent vertex is incident to only one wide edge.

According to [1, Theorem 1], there is a unique polynomial $P(\Gamma) \in \mathbb{Z}[t^{\pm\frac{1}{2}}]$ to each trivalent planar network Γ that takes the value 1 for any simple closed curve and satisfies the following mathematical relations:

$$P\left(\Gamma \cup \begin{array}{c} \bigcirc \end{array}\right) = \alpha P(\Gamma) \tag{2.1}$$

$$P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) = P\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \quad (2.2)$$

$$P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) = \beta P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \quad (2.3)$$

$$P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) = \alpha P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) - 2\beta P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \quad (2.4)$$

$$\begin{aligned} P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) &= \left(P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right)\right) - P\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) + 3\beta \left(P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) - P\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right)\right) \\ &+ t^{\frac{1}{2}} \left(P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) - P\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right)\right) + t^{-\frac{1}{2}} \left(P\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) - P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right)\right) + P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right), \end{aligned} \quad (2.5)$$

where $\alpha = -t^{-\frac{1}{2}} - t^{\frac{1}{2}}, \beta = t + t^{-1}$.

Proposition 2.1. *The following identities hold for the above network polynomial:*

$$\begin{aligned} P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) &= \beta P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \\ P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) &= \alpha P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) - 2\beta P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \\ P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) &= \alpha P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) - 2\beta P\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \end{aligned}$$

We associate a polynomial $\mathbb{P}_\Upsilon(t)$ to Υ by the following recursive formulas:

$$\mathbb{P}\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) = t^{\frac{3}{2}} \mathbb{P}\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) + t \mathbb{P}\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) + t \mathbb{P}\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right) \quad (2.6)$$

$$\mathbb{P}\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) = t^{\frac{-3}{2}} \mathbb{P}\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) + t^{-1} \mathbb{P}\left(\begin{array}{c} \diagup \quad \diagdown \\ \text{---} \\ \diagdown \quad \diagup \end{array}\right) + t^{-1} \mathbb{P}\left(\begin{array}{c} \diagdown \quad \diagup \\ \text{---} \\ \diagup \quad \diagdown \end{array}\right), \quad (2.7)$$

where $\mathbb{P}_\Gamma = P(\Gamma)$ is the unique polynomial assigned to the trivalent network Γ .

Throughout this paper, diagrams that appear in one equation are identical except as indicated in a small disk.

This assigns to each knotted trivalent network Υ the polynomial $\mathbb{P}_\Upsilon(t)$ obtained by summing up the network polynomials $P(\Gamma)$ weighted by monomials in the variable t . That is,

$$\mathbb{P}_\Upsilon(t) = \sum_{\Gamma} t^k P(\Gamma).$$

The first important result is as follows:

Theorem 2.1. *The polynomial $\mathbb{P}_L(t)$ is an invariant of the unoriented link L .*

$$\begin{aligned}
\mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) &= \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} \right) + t^{-\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) + t^{-\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagup \end{array} \right) + t^{\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagup \end{array} \right) + \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) + \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) \\
&+ t^{\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagup \end{array} \right) + \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) + \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) \\
&= \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagdown \end{array} \right) + t^{-\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) + t^{-\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagup \end{array} \right) + t^{\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagup \end{array} \right) + \beta \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) + \alpha \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) - 2\beta \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) \\
&+ t^{\frac{1}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagup \diagup \end{array} \right) + \alpha \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) - 2\beta \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) + \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right)
\end{aligned}$$

Now the result follows if we subtract the above two equations and apply Equation 2.5. The invariance under the third Reidemeister move follows as a result of the above argument:

$$\begin{aligned}
\mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) &= t^{\frac{3}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) + t \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) + t \mathbb{P} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagup \end{array} \right) \\
&= t^{\frac{3}{2}} \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) + t \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right) + t \mathbb{P} \left(\begin{array}{c} \diagdown \diagup \\ \diagup \diagup \end{array} \right) \\
&= \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)
\end{aligned}$$

□

Corollary 2.2. *By applying similar argument, we obtain*

1.

$$\mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right), \quad \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right)$$

2.

$$\mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagdown \end{array} \right)$$

3.

$$\mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right), \quad \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)$$

4.

$$\mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right) = \mathbb{P} \left(\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right)$$

The restriction of the above polynomial to links yields an unoriented version of the Jones polynomial of links as shown in the following proposition:

Proposition 2.2. *The link polynomial $\mathbb{P}_L(t)$ satisfies*

(a) *If D is a simple closed curve, then $\mathbb{P}_D = 1$.*

(b) *We have*

$$t^{-1}\mathbb{P} \begin{array}{c} \diagup \\ \diagdown \end{array} - t\mathbb{P} \begin{array}{c} \diagdown \\ \diagup \end{array} = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\mathbb{P} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \quad (3.1)$$

3 The State Model for the Homfly Polynomial

Let Υ be a knotted trivalent network. The states of Υ are planar trivalent networks obtained by replacing each crossing by one of the local diagrams $\begin{array}{c} \diagup \\ \diagdown \end{array}$, $\begin{array}{c} \diagdown \\ \diagup \end{array}$ or $\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array}$. The states networks have both standard and wide edges, such that each trivalent vertex is incident to only one wide edge.

According to [1, Theorem 1], there is a unique polynomial $T(\Gamma) \in \mathbb{Z}[a^{\pm 1}, A, B, (A+B)^{\pm 1}]$ to each trivalent planar network Γ that takes the value 1 for the unknot and satisfies the following network skein relations:

$$T \left(\Gamma \cup \bigcirc \right) = \alpha T(\Gamma) \quad (3.1)$$

$$T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) = T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) \quad (3.2)$$

$$T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) = \beta T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \begin{array}{c} \diagdown \\ \diagup \end{array} \quad (3.3)$$

$$T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \right) = (1 - AB)T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \right) + \gamma T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) + (A - B)T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \quad (3.4)$$

$$\begin{aligned} T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) &= T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \right) + (\beta + 2\gamma) \left(T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) - T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \right) \\ &+ AB \left(T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) - T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \right) + (B - 2A) \left(T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) - T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \right) \\ &+ (2B - A) \left(T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) - T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \right), \quad (3.5) \end{aligned}$$

where $\alpha = \frac{a+a^{-1}}{A+B}$, $\beta = \frac{aB-a^{-1}A}{A+B} - 1$, $\gamma = A - B + \alpha + 2\beta$.

Proposition 3.1. *The following identities hold for the above network polynomial:*

$$\begin{aligned} T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) &= \beta T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \begin{array}{c} \diagdown \\ \diagup \end{array} \\ T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \right) &= (1 - AB)T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) + \gamma T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \right) + (A - B)T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) \\ T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) &= (1 - AB)T \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right) + \gamma T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} \right) + (A - B)T \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} \right), \end{aligned}$$

where β and γ as before.

We associate a polynomial $\mathbb{T}_\Upsilon(a, A, B)$ to Υ by the following recursive formulas:

$$\mathbb{T} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) = a^{-1} A \mathbb{T} \left(\begin{array}{c} \diagup \\ \diagdown \end{array} \right) \left(+ a^{-1} \mathbb{T} \left(\begin{array}{c} \frown \\ \smile \end{array} \right) + a^{-1} \mathbb{T} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) \right) \quad (3.6)$$

$$\mathbb{T} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) = a B \mathbb{T} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) \left(- a \mathbb{T} \left(\begin{array}{c} \frown \\ \smile \end{array} \right) - a \mathbb{T} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) \right), \quad (3.7)$$

where $\mathbb{T}_\Gamma = T(\Gamma)$ is the unique polynomial assigned to the trivalent network Γ .

This assigns to each knotted trivalent network Υ the polynomial $\mathbb{T}_\Upsilon(a, A, B)$ obtained by summing up the network polynomials $T(\Gamma)$ weighted by monomials in the commuting variables a, A, B . That is,

$$\mathbb{T}_\Upsilon(a, A, B) = \sum_{\Gamma} a^k A^l B^m T(\Gamma).$$

The second important result is as follows:

Theorem 3.1. *The polynomial $\mathbb{T}_L(a, A, B)$ is an invariant of the unoriented link L .*

Proof. We show the invariance of this polynomial under the three Reidemeister moves. For the first Reidemeister move, we have

$$\begin{aligned} \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) &= a^{-1} A \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \left(+ a^{-1} \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \right) + a^{-1} \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) = (a^{-1} A \alpha + a^{-1} + a^{-1} \beta) \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \\ &= \left(a^{-1} A \left(\frac{a + a^{-1}}{A + B} \right) + a^{-1} + a^{-1} \left(\frac{aB - a^{-1}A}{A + B} - 1 \right) \right) \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) = \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \end{aligned}$$

Also, we have

$$\begin{aligned} \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) &= a B \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \left(- a \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \right) - a \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) = (a B \alpha - a - a \beta) \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \\ &= \left(a B \left(\frac{a + a^{-1}}{A + B} \right) - a - a \left(\frac{aB - a^{-1}A}{A + B} - 1 \right) \right) \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \\ &= \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \end{aligned}$$

Now we show the invariance under the second Reidemeister move as follows:

$$\begin{aligned} \mathbb{T} \left(\begin{array}{c} \text{crossing} \end{array} \right) &= A B \mathbb{T} \left(\begin{array}{c} \text{crossing} \end{array} \right) \left\{ - A \mathbb{T} \left(\begin{array}{c} \frown \\ \smile \end{array} \right) - A \mathbb{T} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) + B \mathbb{T} \left(\begin{array}{c} \frown \\ \smile \end{array} \right) - \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) - \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \right. \\ &\quad \left. + B \mathbb{T} \left(\begin{array}{c} \diagdown \\ \diagup \end{array} \right) - \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) - \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \right\} \\ &= A B \mathbb{T} \left(\begin{array}{c} \text{crossing} \end{array} \right) \left\{ + (B - A) \mathbb{T} \left(\begin{array}{c} \text{crossing} \end{array} \right) + (B - A - \alpha - 2\beta) \mathbb{T} \left(\begin{array}{c} \frown \\ \smile \end{array} \right) + \mathbb{T} \left(\begin{array}{c} \text{loop} \end{array} \right) \right\} \\ &= \mathbb{T} \left(\begin{array}{c} \text{crossing} \end{array} \right) \left\{ \right\} \end{aligned}$$

$$\begin{aligned}
\mathbb{T} \left(\text{crossing} \right) &= t^{\frac{3}{2}} \mathbb{T} \left(\text{link} \right) + t \mathbb{T} \left(\text{link} \right) + t \mathbb{T} \left(\text{link} \right) \\
&= t^{\frac{3}{2}} \mathbb{T} \left(\text{link} \right) + t \mathbb{T} \left(\text{link} \right) + t \mathbb{T} \left(\text{link} \right) \\
&= \mathbb{T} \left(\text{crossing} \right)
\end{aligned}$$

□

Corollary 3.2. *By applying similar argument, we obtain*

1.

$$\mathbb{T} \left(\text{link} \right) = \mathbb{T} \left(\text{link} \right), \quad \mathbb{T} \left(\text{link} \right) = \mathbb{T} \left(\text{link} \right)$$

2.

$$\mathbb{T} \left(\text{link} \right) = \mathbb{T} \left(\text{link} \right)$$

3.

$$\mathbb{T} \left(\text{crossing} \right) = \mathbb{T} \left(\text{crossing} \right), \quad \mathbb{T} \left(\text{crossing} \right) = \mathbb{T} \left(\text{crossing} \right)$$

4.

$$\mathbb{T} \left(\text{crossing} \right) = \mathbb{T} \left(\text{crossing} \right)$$

The restriction of the above polynomial to links yields an unoriented version of the Homfly polynomial of links as shown in the following proposition:

Proposition 3.2. *The link $\mathbb{T}_L(a, A, B)$ satisfies*

(a) *If D is a simple closed curve, then $\mathbb{T}_D = 1$.*

(b) *We have*

$$a \mathbb{T} \left(\text{crossing} \right) + a^{-1} \mathbb{T} \left(\text{crossing} \right) = (A + B) \mathbb{T} \left(\text{link} \right)$$

Proposition 3.3. *For any link L , we have $\mathbb{T}_L(-it, it^{\frac{1}{2}}, -it^{-\frac{1}{2}}) = \mathbb{P}_L(t)$.*

Proof. We use mathematical induction on the number of crossings of the link diagram D . It is clear that the result follows if D has no crossings since

$$\mathbb{T}_L(-it, it^{\frac{1}{2}}, -it^{-\frac{1}{2}}) = \left(\frac{-it + it^{-1}}{it^{\frac{1}{2}} - it^{-\frac{1}{2}}} \right)^{n-1} = \left(-(t^{\frac{1}{2}} + t^{-\frac{1}{2}}) \right)^{n-1} = \mathbb{P}_L(t),$$


where n is the number of components of D . Now if D has m crossings and we assume that the result holds for any crossing change. Then the result follows from Proposition 2.2 and Proposition 3.2. □

We like to point out that the result of Proposition 3.3 does not hold for the case of a knotted trivalent network with at least one rigid edge. For example, we have

$$\mathbb{T}_\Upsilon(-it, it^{\frac{1}{2}}, -it^{-\frac{1}{2}}) = \frac{(-it)(-it^{-\frac{1}{2}}) - (it^{-1})(it^{\frac{1}{2}})}{it^{\frac{1}{2}} - it^{-\frac{1}{2}}} = i,$$

while

$$\mathbb{P}_\Upsilon(t) = t + t^{-1},$$

where $\Upsilon =$  .

4 Periodic Trivalent Networks

Periodicity of these two knot polynomials has been the core of the work of in many papers like [10, 11, 12, 13, 14, 15]. In this section, we generalize the work for the periodicity of the Jones and Homfly polynomials to any knotted trivalent network.

Theorem 4.1. *For any p -periodic knotted trivalent network Υ , we have*




$$\mathbb{P}_\Upsilon(t) \equiv \mathbb{P}_{\Upsilon'}(t) \pmod{(p, t^p - 1)},$$

where Υ' is the mirror image of Υ .

Proof. We apply Equation 2.6 and Equation 2.7 to all crossings of the orbit ν in Υ and the corresponding crossings in Υ_ν . We can pair the terms in both summations in a way that the network state in both terms is identical but with possibly different coefficients. By considering the state summations modulo p , we need only to examine the contributions from states that are p -periodic.

Finally, we consider difference between \mathbb{P}_Υ and $\mathbb{P}_{\Upsilon'}$ that is

$$\begin{aligned} \mathbb{P}_\Upsilon(t) - \mathbb{P}_{\Upsilon_\nu}(t) &\equiv (t^{\frac{3p}{2}} - t^{-\frac{3p}{2}}) \mathbb{P}_{\Upsilon_0}(t) + (t^p - t^{-p}) (\mathbb{P}_{\Upsilon_1}(a) + \mathbb{P}_{\Upsilon_2}(a)) \pmod{p} \\ &\equiv 0 \pmod{(p, t^p - 1)}, \end{aligned}$$

where Υ_0, Υ_1 and Υ_2 are the knotted trivalent network Υ after replacing the crossings of the orbit ν by the local diagrams ,  and  respectively. The result follows after we apply the above argument to all orbits. \square

Lemma 4.2. *The polynomial $\alpha^{p-1} - 1 \equiv t^{-\frac{p+1}{2}} \left(\sum_{i=0}^{p-1} (-1)^i t^i - t^{\frac{p-1}{2}} \right) \pmod{p}$. Therefore, the ideal generated by $\alpha^{p-1} - 1$ and p is equal to the ideal generated by $\sum_{i=0}^{p-1} (-1)^i t^i - t^{\frac{p-1}{2}}$ and p .*

Proof. The result follows since $\binom{p-1}{j} = (-1)^j \pmod{p}$ which follows from the well known formula $\binom{n}{j} = \binom{n}{j-1} + \binom{n-1}{j-1}$ for $j \geq 1$. \square

Theorem 4.3. *For any p -periodic knotted trivalent network Υ , we have $\mathbb{P}_\Upsilon(t) \equiv (\mathbb{P}_{\Upsilon_*}(t))^p \pmod{(p, \xi(t))}$, where Υ_* is the quotient knotted trivalent network of Υ and $\xi(t) = \sum_{i=0}^{p-1} (-t)^i - t^{\frac{p-1}{2}}$.*

Proof. We use mathematical induction on the number of rigid edges in Υ_* . In the case, Υ_* has no rigid edges, then Υ represents a link diagram D of some link. Now, there is a one-to-one correspondence between the binary resolving tree of D to compute $\mathbb{P}_D(t)$ and the binary resolving tree of D_* to compute $\mathbb{P}_{D_*}(t)$ in the characterization given in Proposition 2.2. Now the result follows using mathematical induction on the number of crossings and assuming that the result holds for crossing changes.

Now if Υ_* contains $(m+1)$ orbits of rigid edges, then with the aid of the recurrence formula given in Equation 2.6, we obtain

$$\mathbb{P}_\Upsilon(t) = t^{-p}\mathbb{P}_{\Upsilon_0}(t) - t^{-\frac{p}{2}}\mathbb{P}_{\Upsilon_1}(t) - \mathbb{P}_{\Upsilon_2}(t) \pmod{p},$$

where Υ_0, Υ_1 and Υ_2 are p -periodic knotted trivalent networks obtained from Υ after replacing the rigid edges of some orbit ν by the local diagrams $\times,)$ (and \smile respectively. We also have

$$\mathbb{P}_{\Upsilon_*}(t) = t^{-1}\mathbb{P}_{\Upsilon_{0*}}(t) - t^{-\frac{1}{2}}\mathbb{P}_{\Upsilon_{1*}}(t) - \mathbb{P}_{\Upsilon_{2*}}(t),$$

where $\Upsilon_*, \Upsilon_{0*}, \Upsilon_{1*}$ and Υ_{2*} are the quotient knotted trivalent networks of $\Upsilon, \Upsilon_0, \Upsilon_1$ and Υ_2 respectively. Finally, the result follows from the mathematical induction hypothesis on each term. \square

Hereafter, we let \mathcal{R} to be the subring of $\mathbb{Z}[a^{\pm 1}, A, B, (A+B)^{\pm 1}]$ generated by $a^{\pm 1}, A, B$ and $\frac{a+a^{-1}}{A+B}$. Observe that $A+B$ is not invertible in the subring \mathcal{R} . Now we define the ideal I of the subring \mathcal{R} to be generated by p and $\left(\frac{a+a^{-1}}{A+B}\right)^{p-1} - 1$. We need the following lemma that is motivated by [12, Lemma 1.1].

Lemma 4.4. *For any knotted trivalent network Υ , we have $\mathbb{T}_\Upsilon(a, A, B) \in \mathcal{R}$.*

Proof. We use mathematical induction on the number of rigid edges in the knotted trivalent network Υ . In the case that Υ has zero pairs of trivalent vertices, then Υ simply represents a link diagram D of some link. In this case, we use mathematical induction on the number of crossings in the diagram D . The result holds for the trivial link of n components simply since $\mathbb{T}_D(a, A, B) = \left(\frac{a+a^{-1}}{A+B}\right)^{n-1} \in \mathcal{R}$. Now suppose that the result holds for any link diagram of m crossings or less. Let D be a link diagram of $m+1$ crossings and pick a crossing in this link diagram. We assume that the result holds for the link diagram D after changing a crossing. Now the results holds using the second equation of Proposition 3.2 and the mathematical induction hypothesis.

Now if the knotted trivalent network Υ contains $m+1$ pairs of trivalent vertices then with the aid of the recurrence formulas given in Equation 3.6, we obtain

$$\mathbb{T}_\Upsilon(a, A, B) = a\mathbb{T}_{\Upsilon_0}(a, A, B) - A\mathbb{T}_{\Upsilon_1}(a, A, B) - \mathbb{T}_{\Upsilon_2}(a, A, B),$$

where Υ_0, Υ_1 and Υ_2 are the knotted trivalent networks obtained by replacing the rigid edge in Υ by $\times,)$ (and \smile respectively. Finally, the result follows from the mathematical induction hypothesis on each term. \square

Lemma 4.5. *We have $a^p + a^{-p} \equiv 0 \pmod{(p, A^p + B^p)}$.*

Proof. The result follows since

$$a^p + a^{-p} \equiv (A^p + B^p) \left(\frac{a+a^{-1}}{A+B}\right)^p \equiv 0 \pmod{(p, A^p + B^p)}$$

\square

Theorem 4.6. For any p -periodic knotted trivalent network Υ , we have

$$\mathbb{T}_{\Upsilon}(a, A, B) \equiv \mathbb{T}_{\Upsilon'}(a, A, B) \pmod{(p, A^p + B^p)},$$

where Υ' is the mirror image of Υ .

Proof. We apply Equation 3.6 and Equation 3.7 to all crossings of the orbit ν in Υ and the corresponding crossings in Υ_{ν} . Now we can pair the terms in both summations in a way that the network state in both terms is identical but with possibly different coefficients. By considering the state summations modulo p , we need only to examine the contributions from states that are p -periodic.

Now we consider difference between \mathbb{T}_{Υ} and $\mathbb{T}_{\Upsilon'}$ that is

$$\begin{aligned} \mathbb{T}_{\Upsilon}(a, A, B) - \mathbb{T}_{\Upsilon'}(a, A, B) &\equiv (a^{-p}A^p - a^pB^p)\mathbb{T}_{\Upsilon_0}(a, A, B) \\ &\quad + (a^p + a^{-p})(\mathbb{T}_{\Upsilon_1}(a, A, B) + \mathbb{T}_{\Upsilon_2}(a, A, B)) \pmod{p} \\ &\equiv a^{-p}(A^p + B^p)\mathbb{T}_{\Upsilon_0}(a, A, B) \pmod{(p, A^p + B^p)} \\ &\equiv 0 \pmod{(p, A^p + B^p)} \end{aligned}$$

where Υ_0, Υ_1 and Υ_2 are the knotted trivalent network Υ after replacing the crossings of the orbit ν by the local diagrams $\begin{array}{c} \diagdown \\ \diagup \end{array}$, $\begin{array}{c} \diagup \\ \diagdown \end{array}$ and $\begin{array}{c} \diagdown \\ \diagdown \end{array}$ respectively. The result follows after we apply the above argument to all orbits. \square

Theorem 4.7. For any p -periodic knotted trivalent network Υ , we have

$$\mathbb{T}_{\Upsilon}(a, A, B) \equiv (\mathbb{T}_{\Upsilon_*}(a, A, B))^p \pmod{I},$$

where Υ_* is the quotient knotted trivalent network of Υ .

Proof. We use mathematical induction on the number of pairs of rigid edges in Υ_* . In the case, Υ_* has no rigid edges, then Υ represents a link diagram D of some link. Now, there is a one-to-one correspondence between the binary resolving tree of D to compute $\mathbb{T}_D(a, A, B)$ and the binary resolving tree of D_* to compute $\mathbb{T}_{D_*}(a, A, B)$. Now the result follows using mathematical induction on the number of crossings and assuming that the result holds for crossing changes.

Now if Υ_* contains $m+1$ rigid edges, then with the aid of the recurrence formula given in Equation 3.6, we obtain

$$\mathbb{T}_{\Upsilon}(a, A, B) = a^p\mathbb{T}_{\Upsilon_0}(a, A, B) - A^p\mathbb{T}_{\Upsilon_1}(a, A, B) - \mathbb{T}_{\Upsilon_2}(a, A, B) \pmod{p},$$

where Υ_0, Υ_1 and Υ_2 are p -periodic knotted trivalent networks obtained from Υ after replacing the rigid edges of some orbit ν by the local diagrams $\begin{array}{c} \diagdown \\ \diagdown \end{array}$, $\begin{array}{c} \diagup \\ \diagup \end{array}$ and $\begin{array}{c} \diagup \\ \diagdown \end{array}$ respectively. Also, we have

$$\mathbb{T}_{\Upsilon_*}(a, A, B) = a\mathbb{T}_{\Upsilon_{0*}}(a, A, B) - A\mathbb{T}_{\Upsilon_{1*}}(a, A, B) - \mathbb{T}_{\Upsilon_{2*}}(a, A, B),$$

where $\Upsilon_*, \Upsilon_{0*}, \Upsilon_{1*}$ and Υ_{2*} are the quotient knotted trivalent networks of $\Upsilon, \Upsilon_0, \Upsilon_1$ and Υ_2 respectively. Finally, the result follows from the mathematical induction hypothesis on each term. \square

Corollary 4.8. If $A = \omega a, B = \omega a^{-1}$ or $A = \omega a^{-1}, B = \omega a$ for some $(p-1)$ -th of unity ω , then we have

$$\mathbb{T}_{\Upsilon}(a, A, B) \equiv (\mathbb{T}_{\Upsilon_*}(a, A, B))^p \pmod{p},$$

where Υ_* is the quotient knotted trivalent network of Υ .

Proof. The result follows because the second relation in the ideal I holds for both cases since $\left(\frac{a+a^{-1}}{A+B}\right)^{p-1} = 1$. \square

5 CONCLUSIONS

We present some congruence of polynomials for periodic knotted trivalent networks. The restriction of these polynomials yield the famous Jones and Homfly knot polynomials. Therefore, this generalize the previous work of periodicity for the Jones and Homfly polynomials.

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