Perfect i.i.d processes

This note proves a theorem about i.i.d. i.e. independent and indentically distributed processes, when the index space is a measure space. The statement of the problem corresponding to the theorem proved in this paper appears in Green [2019], in which the concept of a sample distribution limit corresponds to the concept of a perfect i.i.d process in this paper.

Theorems proved in this theme, regarding existing and non-existence, have been shown in the economics literature, when the index set is $[0,1]$, in Judd [1985], Uhlig [1996], Feldman and Gilles [1985], Aumann [1967]. The approach taken in this paper is perhaps, surprisingly elementary. We may apply standard measure extension theorems to show existence. These may be found in Tao [2011], Evans and Garzepy [2018].

1 Model

Suppose $(R, \mathcal{R}, \rho)$ is a probability space that we will call the *state space*; and $(P, \mathcal{P}, \pi)$ be a probability space called the *index space*. The following definitions convey the prime theme of the paper.

Definition 1.1. *A setting is defined as a pair* $<(R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi)>$ *consisting of a state space and an index space.*

Definition 1.2. *A measure-preserving transformation is any measurable map* $\psi : P \to R$ *such that*

$$(\forall B \in \mathcal{R})(\pi(\{p : \psi(p) \in B\}) = \rho(B)). \tag{1}$$

Definition 1.3. *A setting* $<(R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi)>$ *is said to admit a perfect i.i.d process if there exists a probability space* $<\Omega, \mathcal{F}, \mathbb{P}>$ *and measurable functions* $\{X_p\}_{p \in P}$ *where* $X_p : \Omega \to R$ *such that*

1. *For any finite* $P' \subset P$ *and collection* $\{B_p\}_{p \in P'} \subseteq \mathcal{R}$ *we have that*

$$\mathbb{P}(\cap_{p \in P'} \{X_p \in B_p\}) = \prod_{p \in P'} \rho(B_p).$$

2. *There exists* $A \in \mathcal{F}$ *such that* $\mathbb{P}(A) = 1$ *and*

$$A \subseteq \{\omega \in \Omega : X_p(\omega) \text{ is measure-preserving in } p\}.$$

Definition 1.4. *An index space* $(P, \mathcal{P}, \pi)$ *will be called fine if*

1. *For every* $p \in P$, $\{p\} \in \mathcal{P}$.

2. *For every* $p \in P$, $\pi(\{p\}) = 0$.

It follows immediately that any fine index space $(P, \mathcal{P}, \pi)$ is uncountably infinite. The following is the main theorem of the paper.

Theorem 1.1. *Let* $< (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) >$ *be a setting. Suppose that the index space* $(P, \mathcal{P}, \pi)$ *is fine. Further, suppose that there exists a measure-preserving transformation* $\psi : P \to R$. *Then, the setting* $< (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) >$ *admits a perfect i.i.d process.*

Proof. The proof proceeds in a few steps.

Step 1 : We argue that given a measure-preserving transformation $\psi : P \to R$; a countable subset $\hat{P} \subseteq P$; and any function $\hat{\psi} : \hat{P} \to R$, the map $\psi' : P \to R$ defined as

$$\psi'(p) = \begin{cases} \hat{\psi}(p) & \text{if } p \in \hat{P} \\ \psi(p) & \text{otherwise} \end{cases} \tag{2}$$

is also a measure-preserving transformation. This is true since the probability space $(P, \mathcal{P}, \pi)$ is assumed to be fine. As $\mathcal{P}$ includes all singleton sets, it follows that $\hat{P} \in \mathcal{P}$. Hence, ψ' is measurable. Further, since singletons have zero probability according to the probability measure π, implying that $\pi(\hat{P}) = 0$ (due to countable additivity), it follows that ψ' is also a measure-preserving transformation.

Step 2 : We now define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define Ω as

$$\Omega := \{\psi : P \to R : \psi \text{ is measure-preserving}\}.$$

By assumption, we have that $\Omega \neq \emptyset$. For a finite subset $\hat{P} \subseteq P$ and collection of sets $\{B_p\}_{p \in \hat{P}} \subseteq \mathcal{R}$, define the set

$$< \hat{P}, \{B_p\}_{p \in \hat{P}} > := \{\psi \in \Omega : (\forall p \in \hat{P})(\psi(p) \in B_p)\}.$$

The collection of all such sets is defined as

$$\mathcal{S} := \Big\{ < \hat{P}, \{B_p\}_{p \in \hat{P}} >: \text{ finite } \hat{P} \subseteq P \text{ and collection } \{B_p\}_{p \in \hat{P}} \subseteq \mathcal{R} \Big\}.$$

We show that $\mathcal{S}$ is a semi-ring (see Aliprantis and Border [2006]). Further, we show that the following set function $\mathbb{P}'$ defines a measure on $\mathcal{S}$

$$\mathbb{P}'(< \hat{P}, \{B_p\}_{p \in \hat{P}} >) = \prod_{p \in \hat{P}} \rho(B_p).$$

We first prove that $\mathcal{S}$ is a semi-ring. This follows simply from the following facts.

1. Suppose that we have a set of the form $< \hat{P}, \{B_p\}_{p \in \hat{P}} >$ such that $B_p = \emptyset$ for some $p \in \hat{P}$. This immediately implies that $< \hat{P}, \{B_p\}_{p \in \hat{P}} > = \emptyset \in \mathcal{S}$.

2. Suppose that $< \hat{P}, \{B_p\}_{p \in \hat{P}} >, < \hat{P}', \{B'_p\}_{p \in \hat{P}'} > \in \mathcal{S}$. Then, it is simple to prove that

$$< \hat{P}, \{B_p\}_{p \in \hat{P}} > \cap < \hat{P}', \{B'_p\}_{p \in \hat{P}'} > = < \hat{P} \cup \hat{P}', \{B_p \cap B'_p\}_{p \in \hat{P} \cap \hat{P}'} \cup \{B_p\}_{p \in \hat{P} \setminus \hat{P}'} \cup \{B'_p\}_{p \in \hat{P}' \setminus \hat{P}} > .$$

Hence, we have shown that $<\hat{P},\{B_p\}_{p\in\hat{P}}>\cap<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>\in\mathcal{S}$.

3. Suppose that $<\hat{P},\{B_p\}_{p\in\hat{P}}>,<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>\in\mathcal{S}$. We wish to show that $<\hat{P},\{B_p\}_{p\in\hat{P}}>\setminus<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>$ may be represented as a finite union of pairwise disjoint sets in $\mathcal{S}$. Note that

$$<\hat{P},\{B_p\}_{p\in\hat{P}}>\setminus<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>=<\hat{P},\{B_p\}_{p\in\hat{P}}>\cap(\Omega\setminus<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>).$$

Then, it follows that

$$\Omega\setminus<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>=\bigcup_{Q\subseteq\hat{P}';Q\neq\emptyset}<\hat{P}',\{\Omega\setminus B'_p\}_{p'\in Q}\cup\{B'_{p'}\}_{p'\in\hat{P}'\setminus Q}>,$$

which is a finite union of disjoint sets in $\mathcal{S}$. Hence, from 2., we have indeed shown that it is the case $<\hat{P},\{B_p\}_{p\in\hat{P}}>\setminus<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>$ is a finite union of disjoint sets in $\mathcal{S}$ as it may be represented as

$<\hat{P},\{B_p\}_{p\in\hat{P}}>\setminus<\hat{P}',\{B'_p\}_{p\in\hat{P}'}>$

$=\bigcup_{Q\subseteq\hat{P}';Q\neq\emptyset}<\hat{P},\{B_p\}_{p\in\hat{P}}>\cap<\hat{P}',\{\Omega\setminus B'_p\}_{p'\in Q}\cup\{B'_{p'}\}_{p'\in\hat{P}'\setminus Q}>.$

and we have proved that $\mathcal{S}$ is a semiring.

We show that $\mathbb{P}'$ defines a measure on $\mathcal{S}$. Suppose, we have a countable collection of pairwise disjoint sets $\{<\hat{P}^i,\{B^i_p\}_{p\in\hat{P}^i}>\}_{i=1}^{\infty}\subseteq\mathcal{S}$ and a set $<\hat{P},\{B_p\}_{p\in\hat{P}}>\in\mathcal{S}$ such that the following holds

$$<\hat{P},\{B_p\}_{p\in\hat{P}}>=\bigcup_{i=1}^{\infty}<\hat{P}^i,\{B^i_p\}_{p\in\hat{P}^i}>. \tag{3}$$

We prove it also holds that

$$\mathbb{P}'(<\hat{P},\{B_p\}_{p\in\hat{P}}>)=\bigcup_{i=1}^{\infty}\mathbb{P}'(<\hat{P}^i,\{B^i_p\}_{p\in\hat{P}^i}>).$$

We prove this as follows. Define the set $\hat{P}^*=\hat{P}\cup(\cup_{i=1}^{\infty}\hat{P}^i)$. Since $\hat{P}^*$ is a countable union of finite sets, it is at most countable. We denote the probability space $(\otimes_{p\in\hat{P}^*}R,\otimes_{p\in\hat{P}^*}\mathcal{R},\otimes_{p\in\hat{P}^*}\rho)$ as the product measure space where $\otimes_{p\in\hat{P}^*}R=\{\hat{\psi}:\hat{\psi}:\hat{P}^*\to R\}$ is the product space corresponding to R with index set $\hat{P}^*$; $\otimes_{p\in\hat{P}^*}\mathcal{R}$ is the product σ-field ; $\otimes_{p\in\hat{P}^*}\rho$ is denoted as the associated product measure (see Halmos [2013]).

Define the map $T:\mathcal{S}\to\otimes_{p\in\hat{P}^*}\mathcal{R}$ as

$$T(<\hat{P}',\{B_p\}_{p\in\hat{P}'}>)=\{\hat{\psi}:\hat{P}^*\to R:(\forall p\in\hat{P}'\cap\hat{P}^*)(\hat{\psi}(p)\in B_p)\}.$$

Hence, it follows that $T(<\hat{P}',\{B_p\}_{p\in\hat{P}}>)\in\otimes_{p\in\hat{P}^*}\mathcal{R}$. If we have $\hat{P}'\subseteq\hat{P}^*$, then by the definition of the product measure space $(\otimes_{p\in\hat{P}^*}R,\otimes_{p\in\hat{P}^*}\mathcal{R},\otimes_{p\in\hat{P}^*}\rho)$, we have that :

$$\otimes_{p\in\hat{P}^*}\rho(T(<\hat{P}',\{B_p\}_{p\in\hat{P}'}>))=\prod_{p\in\hat{P}'}\rho(B_p)=\mathbb{P}'(<\hat{P}',\{B_p\}_{p\in\hat{P}'}>).$$

Hence, from countable additivity of the product measure $\otimes_{p\in\hat{P}^*}$ and by applying equality 3, to prove that

$\mathbb{P}'$ defines a measure on $\mathcal{S}$, it suffices to show that for the pairwise disjoint finite collection of sets given by $\{<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>\}_{i=1}^{\infty} \subseteq \mathcal{S}$, the following holds :

$$T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) = \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>).$$

We prove that $T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) \subseteq \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. Suppose that we have that it is the case that $\hat{\psi} \in T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. By equality 3, we have that $\hat{\psi} \in T(<\hat{P}, \{B_p\}_{p\in\hat{P}}>)$. Then, from Step 1, it follows that there exists a measure-preserving transformation ψ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^*$. Hence, by equality 3 we get that $\psi \in <\hat{P}, \{B_p\}_{p\in\hat{P}}> = \cup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>$. This implies there exists a $\hat{P}^i$ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^i$. Hence, this shows that $\hat{\psi} \in \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. We have, hence established that $T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) \subseteq \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$.

This way, appropriately we may also that prove that, $\bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>) \subseteq T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. Suppose $\hat{\psi} \in \bigcup_{i=1}^{\infty} T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. Then, there exists a $\hat{P}^i$ such that $\hat{\psi} \in T(<\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$. From Step 1 again, it follows that there exists a measure-preserving transformation ψ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}$. This means that $\psi \in <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>$. By equality 3, this implies $\psi \in <\hat{P}, \{B_p\}_{p\in\hat{P}}>$. Hence, $\hat{\psi} \in T(<\hat{P}, \{B_p\}_{p\in\hat{P}}>)$, which means $\hat{\psi} \in T(\bigcup_{i=1}^{\infty} <\hat{P}^i, \{B_p^i\}_{p\in\hat{P}^i}>)$.

Hence, $\mathbb{P}'$ defines a measure on $\mathcal{S}$.

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is then completely defined as $\mathcal{F} := \sigma(\mathcal{S})$ and $\mathbb{P}$ is defined to be the extension of the measure $\mathbb{P}'$ on the defined σ-field $\mathcal{F}$ by the Caratheodory Extension Theorem.

Step 3 : We finish the proof of the theorem. We have defined the appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $p \in P$, define $X_p(\psi) := \psi(p)$ and $A := \Omega$.

This completes the proof of the theorem. □

References

Charalambos D Aliprantis and Kim C Border. *Infinite Dimensional Analysis: A Hitchhiker's Guide*. Springer Science & Business Media, 2006.

Robert J Aumann. Random measure preserving transformations. In *Proceedings of the fifth Berkeley symposium on mathematical statistics and probability*, volume 2, pages 321–326. L. M. LeCam and J. Neyman, University of California Press, 1967.

Lawrence C Evans and Ronald F Garzepy. *Measure theory and fine properties of functions*. Routledge, 2018.

Mark Feldman and Christian Gilles. An expository note on individual risk without aggregate uncertainty. *Journal of Economic Theory*, 35(1):26–32, 1985.

Edward J Green. Individual-level randomness in a nonatomic population. *arXiv preprint arXiv:1904.00849*, 2019.

Paul R Halmos. *Measure theory*, volume 18. Springer, 2013.

Kenneth L Judd. The law of large numbers with a continuum of iid random variables. *Journal of Economic theory*, 35(1):19–25, 1985.

Terence Tao. *An introduction to measure theory*, volume 126. American Mathematical Society Providence, 2011.

Harald Uhlig. A law of large numbers for large economies. *Economic Theory*, 8(1):41–50, 1996.