Power Series Generalised Power Weibull Class of Distributions

Abstract

The power series generalised power Weibull class of distributions were developed in this study by compounding the power series family of distributions and the generalised power Weibull distribution. The statistical properties of this new class were derived. Maximum likelihood parameter estimators were derived for the parameters of the power series generalised power Weibull class of distributions. Four sub-families of distributions were developed from the power series generalised power Weibull class of distribution; the generalised power Weibull geometric distribution, generalised power Weibull Poisson distribution, generalised power Weibull binomial distribution and the generalised power Weibull logarithmic distribution. The hazard rate and probability density function plots of the four sub families of distributions showed that, they can model both monotonic and non-monotonic lifetime data. Monte Carlo simulations performed on these sub-distributions showed that, their estimators were consistent estimators. Application of these sub-distributions to failure data from air conditioning system of an aircraft showed that, the generalised power Weibull geometric distribution provides a better fit to the data. Also, the generalised power Weibull Poisson distribution provides a better fit to the data on service times of 63 aircraft.

Keywords: Generalised power Weibull, power series, bathtub, compounding, estimators, monotonic.

1 Introduction

In reliability and survival modelling, probability distributions are usually used for modelling time to failure data. In probability distribution theory, significant efforts have been made in developing new classes of standard statistical distributions for many lifetime situations. Nonetheless, there are several significant situations where empirical data set do not follow these standard and traditional statistical distributions. Also in reliability and biological studies, a component or system may contain sub-systems connected in series with each of the sub-systems functioning independently and with their failure rate following independent distributions. For such system, the main component will fail if any or all of the sub-systems fails. There are however limited statistical distributions developed for modelling lifetime data from such systems in series. Also, there is a possibility that some lifetime data sets obtained from such systems might not follow any of the existing distributions. This might be due to the fact that, the time of life or failure can have different interpretations depending on the area of applications (Lai, 2013). Hence, there is the need to generate more flexible distributions for modelling the failure rate of various kinds of random variables from components connected in series.

One approach of achieving this is by compounding two or more distributions. This technique allows for greater flexibility of the tails of a distribution and can be used for engineering and biological applications. Besides, compounding families might be suitable for complementary risk problems based on the presence of latent risks. The compounding technique was pioneered by Adamidis and Loukas (1998). Compounding two or more distributions have been shown to be very useful in discovering various skewed and tailed properties of many distributions and for improving the goodness-of-fit of the traditional distributions Cordeiro et al. (2017).

This study developed the power series generalised power Weibull (PGPW) class of distributions. This class of distributions was developed on the assumptions that, the failure rate associated with the two sub-components are independent random variables.

The generalized power Weibull (GPW) model derived by Bagdonavicius and Nikulin (2002) is a modification of the Weibull distribution on the bases of accelerated failure time models. The GPW distribution was developed by Bagdonavicius and Nikulin (2002) for building accelerated failure time models to investigate the dependence of a lifetime distribution on prognostic variables. Nikulin and Haghighi, (2006) showed that, the hazard rate of the GPW model can be constant, monotonically and non-monotonically shaped. Lai (2013) described the GPW distribution as one of the generalisations of the Weibull model which is mostly essential to describe the non-monotonic nature of the observed hazard rates. On the concept of exponentiated distributions, Fernando et al. (2018) obtained the exponentiated generalised power Weibull distribution. If T follows the GPW distribution, then its cumulative distribution function (CDF), probability density function (PDF) and hazard functions are given respectively as;

$$F(t) = 1 - e^{(1-(1+\lambda t^{\gamma})^{\theta})}, \quad t > 0, \gamma > 0, \theta > 0, \lambda > 0, \tag{1}$$

$$f(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})}, \quad t > 0, \tag{2}$$

and

$$h(t) = \lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}, \quad t > 0, \tag{3}$$

where γ represents the scale parameter and λ, θ represents the shape parameters.

The power series class is a technique of deriving new distributions. Several distributions have been derived using the power series approach. Some of these are; Chahkandi and Ganjali (2009) proposed the exponential power series family. Eisa and Mitra (2012)'s exponentiated Weibull power series class of distributions was gotten by compounding the exponentiated Weibull and power series distributions. Others are Jose et al. (2013)'s complementary exponential power series distribution with increasing failure rate which was introduced as a supplement to the exponential power series distribution proposed by Chahkandi and Ganjali (2009). Baitshephi et al. (2019)'s Webull-G Power Series family of distributions among others.
Let N be the number of independent subsystems of a series system functioning at a given time. Then the zero truncated power series distribution has probability mass function

(PMF) given as;

$$P(N=n)=\frac{a_n\alpha^n}{C(\alpha)}, n=1,2,.... \quad (4)$$

$$C(\alpha)=\sum_{i=1}^{\infty}a_n\alpha^n, \quad (5)$$

where $a_n>0$, $\alpha\in(0,s)$, a_n is the coefficient of the power series, $C(\alpha)$ is the generating function and s is the parameter space. The power series family are; binomial (Bin), Poisson (Poi), geometric (Geo) and logarithmic (Log) distributions. Some useful quantities of this family are;

Table 1: **Power Series Family**

Dis	a_n	C(α)	$C'(\alpha)$	$C''(\alpha)$	$C'''(\alpha)$	s	C^{-1}	α
Geo	1	$\alpha(1-\alpha)^{-1}$	$(1-\alpha)^{-2}$	$2(1-\alpha)^{-3}$	$6(1-\alpha)^{-4}$	1	$\alpha(\alpha+1)^{-1}$	$(0,1)$
Poi	$\frac{1}{n!}$	$e^{\alpha}-1$	e^{α}	e^{α}	e^{α}	∞	$\log(\alpha+1)$	$(0,\infty)$
Log	n^{-1}	$-\log(1-\alpha)$	$(1-\alpha)^{-1}$	$(1-\alpha)^{-2}$	$2(1-\alpha)^{-3}$	1	$1-e^{-\alpha}$	$(0,1)$
Bin	$\binom{M}{n}$	$(1-\alpha)^m-1$	$\frac{m}{(1-\alpha)^{1-m}}$	$\frac{m(m-1)}{(1-\alpha)^{2-m}}$	$\frac{m(m-1)(m-2)}{(1-\alpha)^{3-m}}$	∞	$(\alpha-1)^{\frac{1}{m}}-1$	$(0,\infty)$

2 The Power Series Generalised Power Weibull Class of Distributions

Consider N to be a discrete random variable from the power series distribution (truncated at zero) and N gives the number of failures of system with independent subsystem functioning in series at a given point in time with PMF given in equation (4).
Assume also that, $T_1,T_2,...,T_N$ represents the lifetime failures associated with this system of independent and identically distributed continuous random variables following the GPW distribution with CDF in equation (1). Then $T_i, i=1,....,N$ gives the time to failure of the i^{th} series subsystem. Since the subsystems are in series, T_1 is defined by;

$$T_{(1)}=min(T_1,T_2,....,T_N). \quad (6)$$

Then the conditional CDF of $T_{(1)|N=n}$ is given as;

$$F_{T_{(1)}|N=n}(t)=1-\prod_{i=1}^{n}[1-F_i(t)]. \quad (7)$$

Hence,

$$F_{T_{(1)}|N=n}(t)=1-\left[e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]^n,\ \ t>0. \quad (8)$$

Proposition 1. The marginal CDF of $T_{(1)}$ is given by;

$$F(t;\alpha,\lambda,\gamma,\theta)=1-\frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)},\ \ t>0,\alpha>0,\gamma>0,\theta>0. \quad (9)$$

Proof. Using the concept of compounding,

$$F(t;\alpha,\lambda,\gamma,\theta)=\sum_{n=1}^{\infty}P(N=n)F_{T_{(1)}|N=n}(t). \tag{10}$$

Inputting $P(N=n)$ from the power series family and $F_{T_{(1)}|N=n}(t)$, we have;

$$\begin{aligned}F(t;\alpha,\lambda,\gamma,\theta) &= \sum_{n=1}^{\infty}\left[1-\left(e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right)^{n}\right]\times\frac{a_n\alpha^n}{C(\alpha)}\\ &= \frac{\sum_{n=1}^{\infty}a_n\alpha^n}{\sum_{n=1}^{\infty}a_n\alpha^n}-\sum_{n=1}^{\infty}\frac{a_n\alpha^n}{C(\alpha)}e^{n(1-(1+\lambda t^{\gamma})^{\theta})}\\ &= 1-\frac{\sum_{n=1}^{\infty}a_n\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]^{n}}{C(\alpha)}.\end{aligned}$$

Since $C(\alpha)=\sum_{n=1}^{\infty}a_n\alpha^n$, the PGPW class of distributions has a CDF given as;

$$F(t;\alpha,\lambda,\gamma,\theta)=1-\frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)},\ \ t>0,\alpha>0,\gamma>0,\theta>0. \tag{11}$$

The PDF of the PGPW class of distributions is given as;

$$f(t) = \alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\frac{C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}, t>0, \tag{12}$$

where $\alpha>0,\lambda>0$ are scales parameters and $\gamma>0,\theta>0$ are shape parameters.
The survival function s(t) and hazard function of the PGPW class of distributions are given respectively as;

$$s(t)=\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)},\ \ t>0, \tag{13}$$

and

$$h(t)=\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\frac{C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]},\ \ t>0. \tag{14}$$

From the PGPW class of distributions, three sub-distributions can be developed. Thus;

- The power series Weibull distribution (that is when $\theta=1$) with CDF given as;

$$F_{PW}(t)=1-\frac{C\left[\alpha e^{-\lambda t^{\gamma}}\right]}{C(\alpha)}. \tag{15}$$

- The power series exponential distribution (that is when $\theta=1$ and $\gamma=1$) with CDF given as;

$$F_{PE}(t)=1-\frac{C\left[\alpha e^{-\lambda t}\right]}{C(\alpha)}. \tag{16}$$

- The power series NH distribution (that is when $\gamma = 1$) with CDF given as;

$$F_{PNH}(t) = 1 - \frac{C\left[\alpha e^{(1-(1+\lambda t)^{\theta})}\right]}{C(\alpha)}. \tag{17}$$

Proposition 2. For $\alpha \to 0$, the GPW is a limiting distribution of the PGPW class of distributions.

Proof. Using the CDF of the PGPW class of distributions, we obtain the limits as;

$$\lim_{\alpha^{+}\to 0} (F(t)) = 1 - \lim_{\alpha^{+}\to 0} \frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}.$$

Using $C(\alpha) = \sum_{n=1}^{\infty} a_n \alpha^n$, we have;

$$\lim_{\alpha^{+}\to 0} F(t) = 1 - \lim_{\alpha^{+}\to 0} \frac{\sum_{n=1}^{\infty} a_n \alpha^n e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}}{\sum_{n=1}^{\infty} a_n \alpha^n}.$$

Using the concept of the L' Hopital rule to simplify, we obtain

$$\lim_{\alpha^{+}\to 0} F(t) = 1 - \lim_{\alpha^{+}\to 0} \frac{\sum_{n=1}^{\infty} n a_n \alpha^{n-1} e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}}{\sum_{n=1}^{\infty} n a_n \alpha^{n-1}}.$$

Hence,

$$\lim_{\alpha^{+}\to 0} F(t) = 1 - e^{[1-(1+\lambda t^{\gamma})^{\theta}]}.$$

This is the CDF of the GPW.

Proposition 3. The density function of the PGPW class of distributions has an expanded linear representation of the form;

$$f(t) = n\lambda\gamma\theta \sum_{n=1}^{\infty} P(N = n) t^{\gamma-1}(1 + \lambda t^{\gamma})^{\theta-1} e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}. \tag{18}$$

Proof. Substituting $C'(\alpha) = \sum_{n=1}^{\infty} n a_n \alpha^{n-1}$ into the PDF of the PGPW class of distributions, we have;

$$f(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1 + \lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times \sum_{n=1}^{\infty} n a_n \left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]^{n-1}}{C(\alpha)},$$

which is further simplified into;

$$f(t) = \lambda\gamma\theta \sum_{n=1}^{\infty} \frac{n a_n \alpha^n}{C(\alpha)} t^{\gamma-1}(1 + \lambda t^{\gamma})^{\theta-1} \times e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}.$$

but $P(N=n)=\frac{a_n \alpha^n}{c(\alpha)}$. Therefore,

$$f(t)=n\lambda\gamma\theta\sum_{n=1}^{\infty}P(N=n)t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{n[1-(1+\lambda t^{\gamma})^{\theta}]}.$$

2.1 Statistical Properties of the PGPW class of distribution

The properties considered are; the quantile function, ordinary (non-central) moments, moment generating function, order statistics, incomplete moment, mean deviation, median deviation, Lorenz and Bonferron curves, mean residual life and stochastic ordering property.

2.1.1 Quantile function of the PGPW class of distribution

The quantile function can be used for generating random numbers from a given distribution. It can serve as an alternative way of describing a probability distribution other than the probability density function, CDF or characteristic function.

Proposition 4: The quantile function of the PGPW class of distributions is;

$$Q_{F(p)}=\left[\frac{\left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}, \tag{19}$$

where $C^{-1}(.)$ is the inverse of $C(.)$ and $p\epsilon[0,1]$.

Proof. By definition, the quantile function is defined as; $F(X_p)=P(x\leq x_p)=p$. Thus by setting, $Q_{F(p)}=p$ in the marginal CDF of the PGPW, we have,

$$1-\frac{C\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)} = p.$$

To make t the subject, we first make the exponent function the subject, hence we have;

$$e^{[1-(1+\lambda t^{\gamma})^{\theta}]}=\frac{C^{-1}(1-p).C(\alpha)}{\alpha}.$$

Taking logarithm on both sides and making t the subject gives the quantile function as,

$$Q_{F(p)}=\left[\frac{\left[1-\log\left(\frac{C^{-1}(1-p).C(\alpha)}{\alpha}\right)\right]^{1/\theta}-1}{\lambda}\right]^{1/\gamma}.$$

Using the quantile function above, the median of the PGPW class of distributions evaluated at p=0.5 is;

$$Q_{F(0.5)} = \left[\frac{\left[1 - \log\left(\frac{C^{-1}(0.5).C(\alpha)}{\alpha}\right)\right]^{1/\theta} - 1}{\lambda} \right]^{1/\gamma}. \tag{20}$$

2.1.2 Moments of the PGPW class of Distributions

This section presents the moments of the PGPW class of distributions.

Proposition 5. The r^{th} non-central moment of the PGPW class of distributions is given as;

$$U_r^{'} = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n) e^n (-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right], \tag{21}$$

where $\Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right]$ is a complementary incomplete gamma function.

Proof. By definition, The r^{th} non-central moment of a random variable is given as;

$$\mu_r^{'} = \int_{-\infty}^{\infty} t^r f(t) dt.$$

For the PGPW class,

$$\mu_r^{'} = \int_0^{\infty} t^r \sum_{n=1}^{\infty} P(N=n) g_1(t) dt.$$

Substituting the linear expanded form of the PDF of the PGPW class of distributions, we have;

$$\begin{aligned}
\mu_r^{'} &= \int_0^{\infty} t^r \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n) t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{n[1-(1+\lambda t^{\gamma})^{\theta}]} dt \\
&= \lambda\gamma\theta n \sum_{n=1}^{\infty} P(N=n) \int_0^{\infty} t^r t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{n[1-(1+\lambda t^{\gamma})^{\theta}]} dt.
\end{aligned}$$

Further simplifying using integration by substitution, we obtain;

$$\mu_r^{'} = \sum_{n=1}^{\infty} P(N=n) e^n \lambda^{-\frac{r}{\gamma}} \int_n^{\infty} \left(\left(\frac{u}{n}\right)^{\frac{1}{\theta}} - 1\right)^{\frac{r}{\gamma}} e^{-u} du.$$

Using the binomial expanded form to further simplify we obtain the moments as;

$$\mu_r^{'} = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} \lambda^{\frac{-r}{\gamma}} P(N=n) e^n (-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)} \Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\right)\right].$$

2.1.3 Moment Generating Function of the PGPW class of distributions

The moment generating function (MGF) are distinct functions used to determine the moments of a random variable.

Proposition 6. The MGF of the PGPW class of distributions is given as;

$$M_t(z) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\sum_{r=0}^{\infty}\frac{Z^r}{r!}e^n(-1)^j\binom{\frac{r}{\gamma}}{j}n^{-\left(\frac{r-\gamma(j-\theta)}{\theta\gamma}\right)}P(N=n)\Gamma\left[\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\right)\right]. \quad (22)$$

Proof. By definition MGF is given as;

$$M_t(z) = \int_0^{\infty} e^{tz}f(t)dt.$$

Using Taylor series to expand we have

$$M_t(z) = \sum_{r=0}^{\infty}\frac{z^r}{r!}\mu_r^{'}.$$

Inputting $\mu_r^{'}$ we obtain the MGF.

2.1.4 Order Statistics of the PGPW class of distributions

Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ be a random sample of size n, then the PDF of the p^{th} order statistic is given as;

$$f_{p:n}(t) = \frac{n!}{(n-p)!(p-1)!}\left[F(t)\right]^{p-1}\left[1-F(t)\right]^{n-p}f(t). \quad (23)$$

Assuming $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ comes from the PGPW class of distributions, then;

$$f_{p:n}(t) = \frac{n!}{(n-p)!(p-1)!}f(t)\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{p-1}\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-p}. \quad (24)$$

Proposition 7. The PDF of the largest order statistics of the PGPW class of distributions is given as;

$$f_{p:n}(t) = n\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\left[\frac{C'\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}. \quad (25)$$

Proof. For the largest order statistics, $p = n$, hence;

$$f_{p=n}(t) = nf(t)\left[1-\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}.$$

Inputting the PDF of the PGPW class of distributions, we obtain the largest order PDF.

Proposition 8. The PDF of the smallest order statistic of the PGPW class of distributions is given as;

$$f_{p:1}(t) = n\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}\left[\frac{C'\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}. \tag{26}$$

Proof. For the smallest order statistic, $p = 1$, hence we have;

$$f_{p:1}(t) = nf(t)\left[\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}\right]^{n-1}.$$

Inputting the PDF of the PGPW class of distributions, the smallest order PDF is obtained.

2.1.5 Incomplete Moments, mean deviation and median deviation

Incomplete moment plays a vital role in computing the mean deviation, median deviation, inequality measures and mean residual life of the distribution of a random variable. Incomplete moments can also be used to describe the shape of a distribution of a random variable.

Proposition 9. The r^{th} incomplete moment of the PGPW class of distributions is given as;

$$M_r(y) = \sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{r}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{r}{\gamma}}{j}n^{-\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)}$$
$$\times\left[\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda y^{\gamma}\right)^{\theta}\right)\right]. \tag{27}$$

Proof. By definition, the r^{th} incomplete moment is given as;

$$M_r(y) = \int_0^y t^r f(t)dt.$$

Using the linear expanded form of the PDF of the PGPW class of distributions, the incomplete moment can be written as;

$$M_r(y) = n\lambda\gamma\theta e^{n}\sum_{n=1}^{\infty}P(N=n)\int_0^y t^r t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{-n(1+\lambda t^{\gamma})^{\theta}}dt.$$

Using integration by substitution by considering $u = n(1+\lambda t^{\gamma})^{\theta}$, then when $t \longrightarrow 0, u \rightarrow n$ and when $t \longrightarrow y, y \rightarrow n(1+\lambda y^{\gamma})^{\theta}$.
Also,

$$dt = \frac{du}{n\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}}.$$

Therefore the incomplete moment is given as;

$$M_r(y) = e^n \lambda^{-\frac{r}{\gamma}} \sum_{n=1}^{\infty} P(N=n) \int_n^{n(1+\lambda y^\gamma)^\theta} \left(\left(\frac{u}{n} \right)^{\frac{1}{\theta}} - 1 \right)^{\frac{r}{\gamma}} e^{-u} du.$$

Further simplifying using binomial expansion, we have;

$$M_r(y) = \sum_{n=1}^{\infty} \sum_{j=1}^{i} \lambda^{-\frac{r}{\gamma}} e^n P(N=n)(-1)^j \binom{\frac{r}{\gamma}}{j} n^{-\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}\right)}$$

$$\times \left[\Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n \right) - \Gamma\left(\frac{r-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda y^\gamma\right)^\theta \right) \right].$$

Proposition 10. The mean deviation of the PGPW class of distributions is given by;

$$D(\mu) = 2\mu F(\mu) - 2\sum_{n=1}^{\infty} \sum_{j=1}^{i} \lambda^{-\frac{1}{\gamma}} e^n P(N=n)(-1)^j \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}$$

$$\times \left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n \right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda \mu^\gamma\right)^\theta \right) \right]. \tag{28}$$

Proof. The mean deviation of a random variable is given as;

$$D(\mu) = 2\mu F(\mu) - 2\int_0^\mu t f(t) dt.$$

But $\int_0^\mu t f(t) dt = m_1(\mu)$ is the first incomplete moment ($r = 1$). Substituting $M_1(\mu)$, the mean deviation is obtained.

Proposition 11. The median deviation of the PGPW class of distributions is given by;

$$D(M) = -\mu + 2\sum_{n=1}^{\infty} \sum_{j=1}^{i} \lambda^{-\frac{1}{\gamma}} e^n P(N=n)(-1)^j \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}$$

$$\times \left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n \right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda m^\gamma\right)^\theta \right) \right]. \tag{29}$$

Proof. By definition the median deviation is given as;

$$D(M) = -\mu + 2[M_1(m)].$$

Inputting $M_1(m)$, the median deviation is obtained.

2.1.6 Residual and Mean Residual Life

Mean residual life (MRL) function at time y can represent the estimated added life span for a unit alive at time y. For an operating system, its residual life at time y is $T_y =$

$T-y|T>y$ which has PDF given as;

$$f(t,y)=\frac{f(t)}{1-F(y)}. \tag{30}$$

Proposition 12. The MRL of the T_y from the PGPW class of distribution is given as;

$$MRL=\frac{A\Gamma\left[\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\right)\right]-B\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda\mu^{\gamma}\right)^{\theta}\right)\right]}{\frac{C\left[\alpha e^{(1-(1+\lambda t^{\gamma})^{\theta})}\right]}{C(\alpha)}}-y, \tag{31}$$

where

$$A=\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\lambda^{\frac{-1}{\gamma}}P(N=n)e^{n}(-1)^{j}\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\theta\gamma}\right)},$$

and

$$B=\sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}.$$

Proof. The MRL $(t>0)$ is defined as;

$$\begin{aligned} MLR &= E\left(T-y|T>y\right)\\ &= \frac{\int_y^{\infty}(t-y)f(t)dt}{1-F(t)}\\ &= \frac{\mu_1^{'}-\int_0^{y}tf(t)dt}{1-F(t)}-y. \end{aligned}$$

But $\int_0^y tf(t)dt = M_1(y)$ gives the first incomplete moment and $\mu_1^{'}$ gives the first non-central moment. Substituting these, the MRL is obtained.

2.1.7 Lorenz and Bonferroni Curves

Lorenz and Bonferroni curves are used to measure the inequalities in the distribution of a random variable (for example income inequality). These curves are mostly applicable in reliability, medical, demographic, insurance and economic fields. For the PGPW class of distributions, the Lorenz curve is given as;

$$\begin{aligned} L(p) &= \frac{1}{\mu}\sum_{n=1}^{\infty}\sum_{j=1}^{i}\lambda^{-\frac{1}{\gamma}}e^{n}P(N=n)(-1)^{j}\binom{\frac{1}{\gamma}}{j}n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)}\\ &\times\left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\right)-\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta},n\left(1+\lambda y^{\gamma}\right)^{\theta}\right)\right]. \end{aligned} \tag{32}$$

Proof. By definition, the Lorenz curve is given as;

$$L(P)=\frac{1}{\mu}\int_0^y tf(t)dt.$$

But $\int_0^y tf(t)dt = M_1(y)$ is the first incomplete moment. Hence imputing $M_1(y)$ in $L(P)$, the Lorenz curve expression is obtained.

Also, the Bonferroni curve is defined as;

$$B(P) = \frac{L(P)}{F(y)}. \tag{33}$$

Therefore the Bonferroni curve for the PGPW class of distributions is given as;

$$B(p) = \frac{A \times B}{1 - \frac{C\left[\alpha e^{(1-(1+\lambda y^{\gamma})^{\theta})}\right]}{C(\alpha)}}, \tag{34}$$

where

$$A = \frac{1}{\mu} \sum_{n=1}^{\infty} \sum_{j=1}^{i} \lambda^{-\frac{1}{\gamma}} e^{n} P(N=n)(-1)^{j} \binom{\frac{1}{\gamma}}{j} n^{-\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}\right)},$$

and

$$B = \left[\Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\right) - \Gamma\left(\frac{1-\gamma(j-\theta)}{\gamma\theta}, n\left(1+\lambda y^{\gamma}\right)^{\theta}\right)\right]. \tag{35}$$

2.1.8 Stochastic Ordering

This is used to compare two random variables to know which of them is larger or smaller. Stochastic ordering is an ordering mechanism in lifetime distribution.

Proposition 13.. If $T_1 \sim PGPW(t, \alpha, \lambda, \gamma, \theta)$ and $T_2 \sim PGPW(t, \lambda, \gamma, \theta)$, then T_1 is said to be greater than T_2 in likelihood ratio order if $\frac{f_{T_1}(t)}{f_{T_2}(t)}$ is an increasing function of T.
Proof. For $T_1 \sim PGPW(t, \alpha, \lambda, \gamma, \theta)$ and $T_2 \sim PGPW(t, \lambda, \gamma, \theta)$,

$$\begin{aligned}
\frac{f_{T_1}(t)}{f_{T_2}(t)} &= \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \frac{C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}}{\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})}} \\
&= \frac{\alpha C'\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)}.
\end{aligned}$$

$$\frac{d}{dt}\left[\frac{f_{T_1}(t)}{f_{T_2}(t)}\right] = -\alpha^{2}\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} \frac{C''\left[\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right]}{C(\alpha)},$$

since $\frac{d}{dt}\left[\frac{f_{T_1}(t)}{f_{T_2}(t)}\right] < 0$ for all $t > 0$, $\frac{d}{dt}\left[\frac{f_{T_1}(t)}{f_{T_2}(t)}\right]$ is a decreasing function for $\alpha > 0$.

2.2 Sub-families of the PGPW class of distributions

From the PGPW distribution, four major sub-families of distribution are obtained. These are; the GPW geometric (GPWG) distribution, the GPW Poisson (GPWP) distribution, the GPW binomial (GPWB) distribution and the GPW logarithmic (GPWL) distribution.

2.2.1 Generalised power Weibull geometric distribution

The geometric distribution truncated at zero is a distinct case of the power series distributions with $a_n = 1$, $C(\alpha) = \alpha(1-\alpha)^{-1}$ and $C^{'}(\alpha) = (1-\alpha)^{-2}$. By inputting these functions into the PGPW class of distributions, we obtain the GPWG distribution with PDF and hazard function given respectively as;

$$f(t) \;=\; \frac{(1-\alpha)\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda)^{\theta})}}{\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{2}}, \tag{36}$$

and

$$h(t) = \frac{\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\left(1-\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)}. \tag{37}$$

The plot of the PDF of the GPWG distribution is displayed in Figures 1. The plots shows that, the PDF of this distribution can be decreasing, increasing, decreasing-constant-increasing, increasing-decreasing, right-skewed and symmetric.

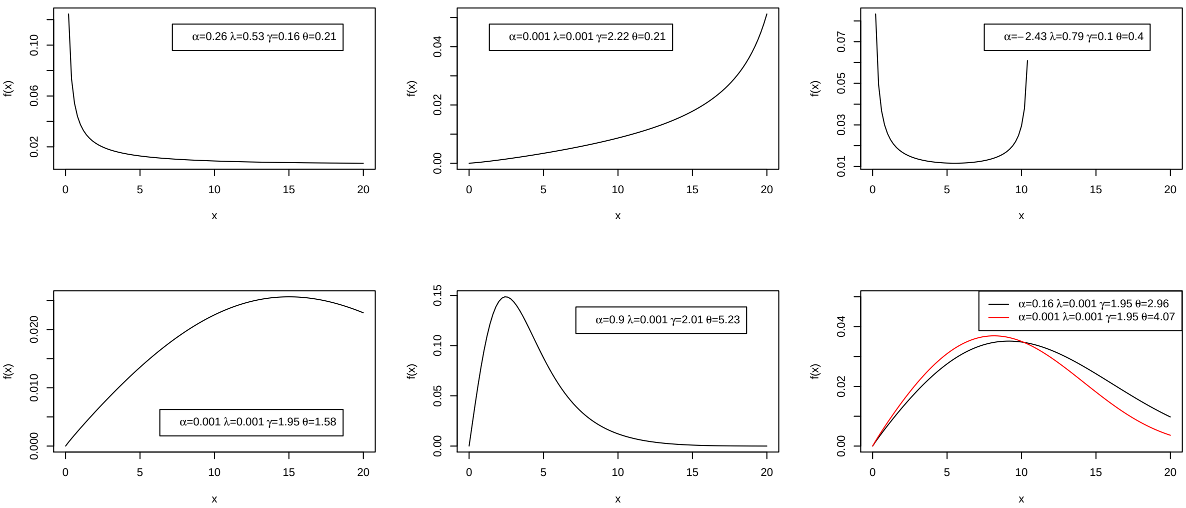

Figure 1: **PDF plot of the GPWG distribution**

Also, the hazard plots of the GPWG distribution are displayed in Figures 2. It is seen that the hazard can be increasing, decreasing, bathtub and unimodal. This shows that the GPWG distribution can model failure rate data which are both monotonically and non-monotonically shaped.

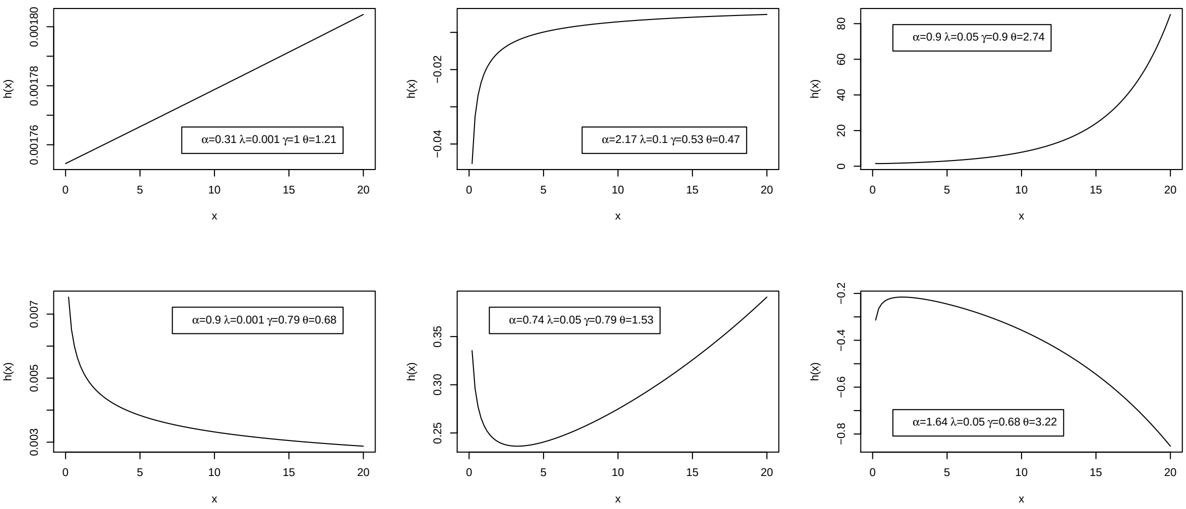

Figure 2: **Hazard plot of the GPWG distribution**

2.2.2 Generalised power Weibull Poisson distribution

The poison distribution (truncated at zero) is a special form of the power series distribution with $a_n = \frac{1}{n!}$, $C(\alpha) = e^{\alpha} - 1$ and $C^{'}(\alpha) = e^{\alpha}$. By inputting these functions into the PGPW class of distributions, we obtain the PDF and hazard function of the GPWP distribution as;

$$f(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} \times e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha} - 1}, \tag{38}$$

and

$$h(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1} e^{(1-(1+\lambda t^{\gamma})^{\theta})} e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}}}{e^{\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}} - 1}. \tag{39}$$

The plot of the PDF of the GPWP distribution displayed in Figures 3 shows that, its PDF can be decreasing, increasing, increasing-decreasing and right-skewed.

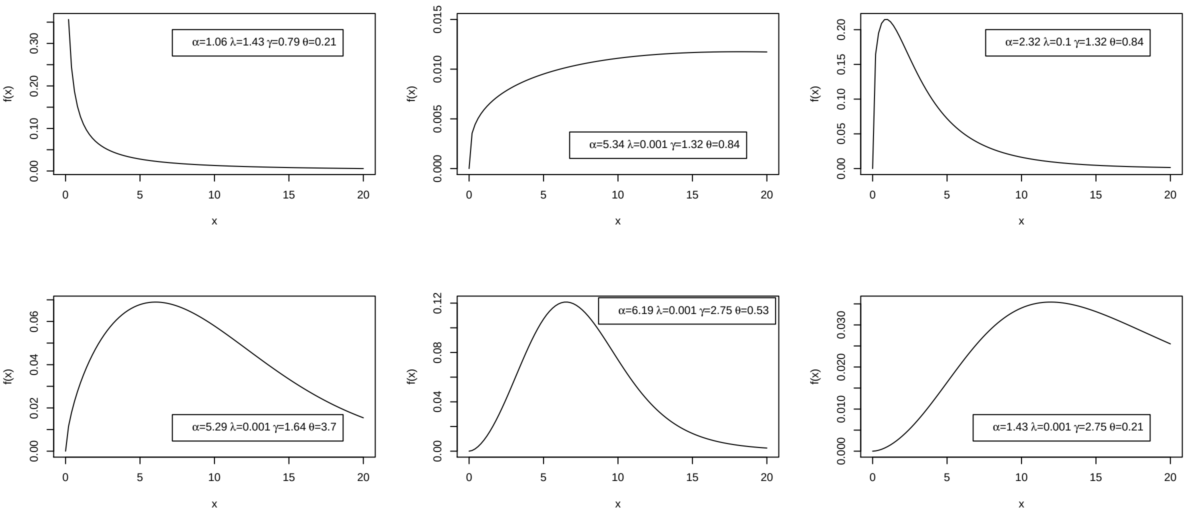

Figure 3: **PDF plot of the GPWP distribution**

Also, the GPWP distribution's hazard rate is seen to be monotonically increasing, decreasing, bathtub, unimodal, modified bathtub and modified unimodal as shown in Figure 4. This shows that the GPWP distribution can model both monotonically and non-monotonically shaped failure rate.

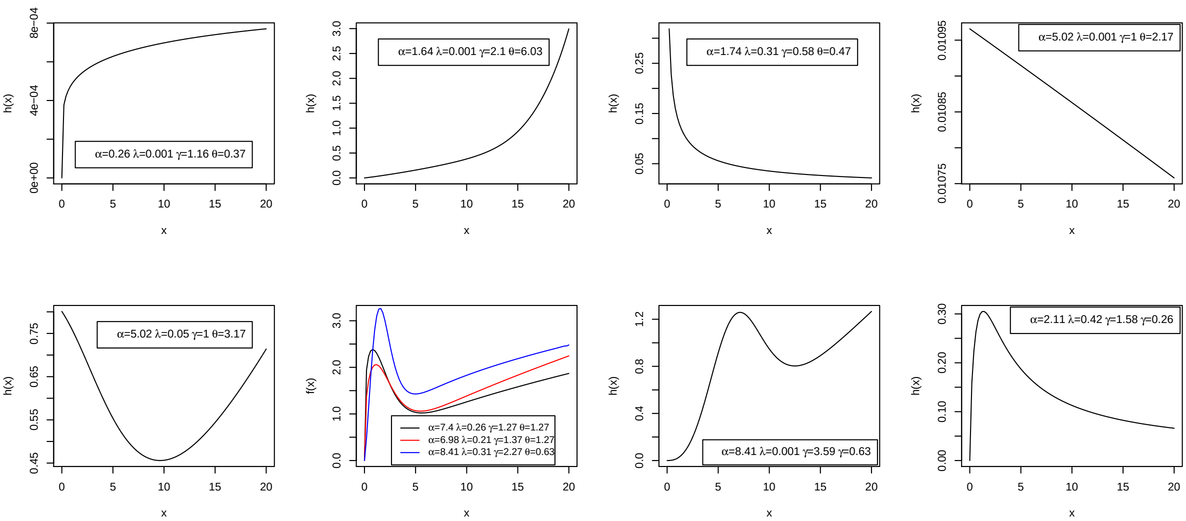

Figure 4: **Hazard plot of the GPWP distribution**

2.2.3 Generalised Power Weibull Binomial Distribution

The zero truncated binomial distribution is a special form of the power series distributions with $a_n = \binom{m}{n}$, $C(\alpha) = (1+\alpha)^m - 1$ and $C'(\alpha) = \frac{m}{(1+\alpha)^{1-m}}$. Considering these in the PGPW class of distributions, we obtain the PDF and hazard functions of the GPWB distribution respectively as;

$$f(t) = \frac{m\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}((1+\alpha)^m - 1)}, \tag{40}$$

and

$$h(t) = \frac{m\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{(1-(1+\lambda t^{\gamma})^{\theta})}}{\left(1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]}\right)^{1-m}((1+\alpha e^{[1-(1+\lambda t^{\gamma})^{\theta}]})^m - 1)}. \tag{41}$$

As displayed in Figures 5, the PDF of the GPWB distribution can be increasing, decreasing, unimodal and positively skewed.

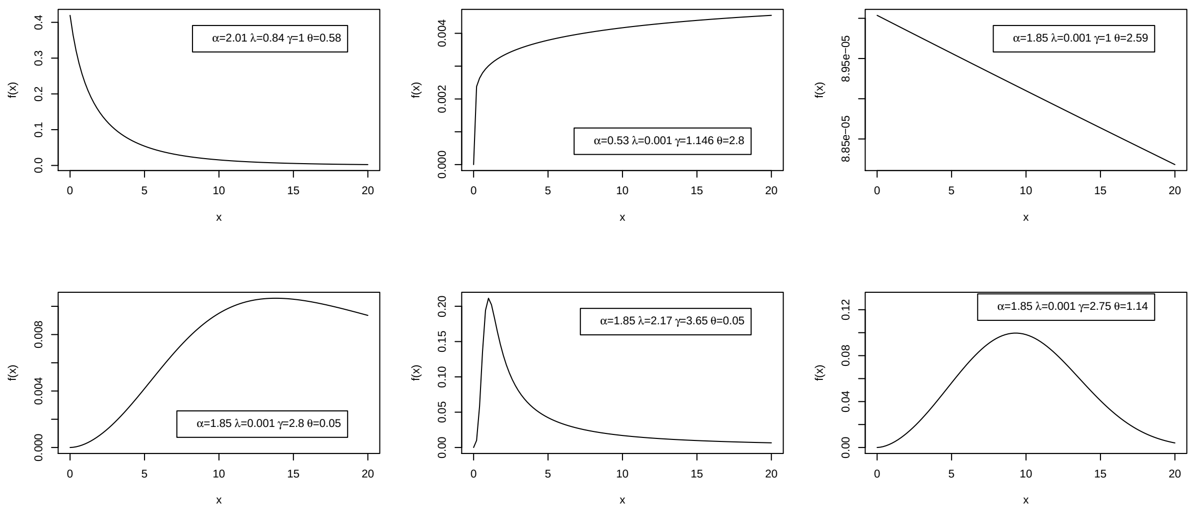

Figure 5: **PDF plot of the GPWB distribution**

Also, its hazard function, as shown in Figures 6, can be increasing, decreasing, bathtub and unimodal. This shows that the GPWG distribution can model failure rate data which are both monotonically and non-monotonically shaped.

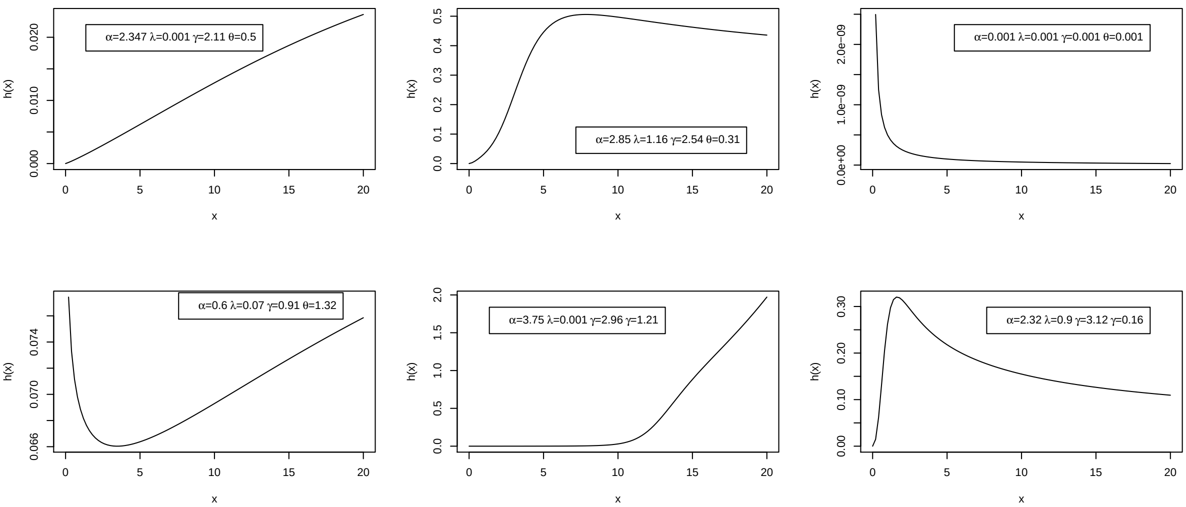

Figure 6: **Hazard plot of the GPWB distribution**

2.2.4 Generalised Power Weibull Logarithmic Family of Distributions

The zero truncated Logarithmatic distribution is also a special class of the power series family with $a_n = \frac{1}{n}$, $C(\alpha) = -\log(1-\alpha)$ and $C'(\alpha) = (1-\alpha)^{-1}$. Considering these in the PGPW class of distributions, we obtain the PDF and hazard functions of the GPWL distribution respectively as;

$$f(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}}{\left(\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}-1\right)\left(\log(1-\alpha)\right)}, \tag{42}$$

and

$$h(t) = \frac{\alpha\lambda\gamma\theta t^{\gamma-1}(1+\lambda t^{\gamma})^{\theta-1}e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}}{\left(\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}-1\right)\left(\log\left(1-\alpha e^{\left[1-(1+\lambda t^{\gamma})^{\theta}\right]}\right)\right)} \tag{43}$$

As displayed in Figures 7, the GPWL has an increasing, decreasing and unimodal PDF.

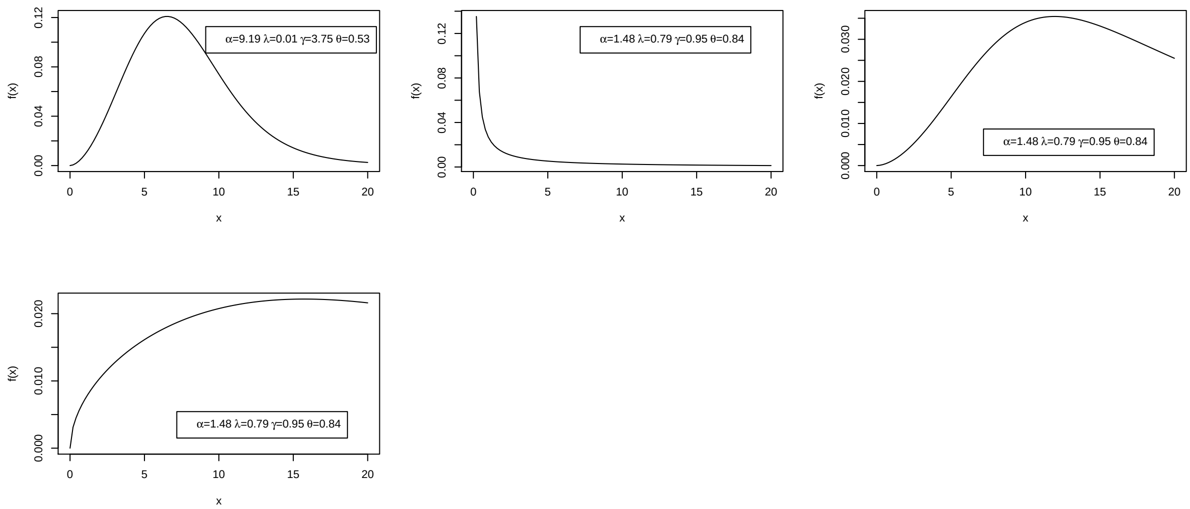

Figure 7: **PDF plot of the GPWL distribution**

The GPWL also has an increasing, decreasing, bathtub and unimodal hazard rate function. Thus can model failure rate data which are both monotonically and non-monotonically shaped.

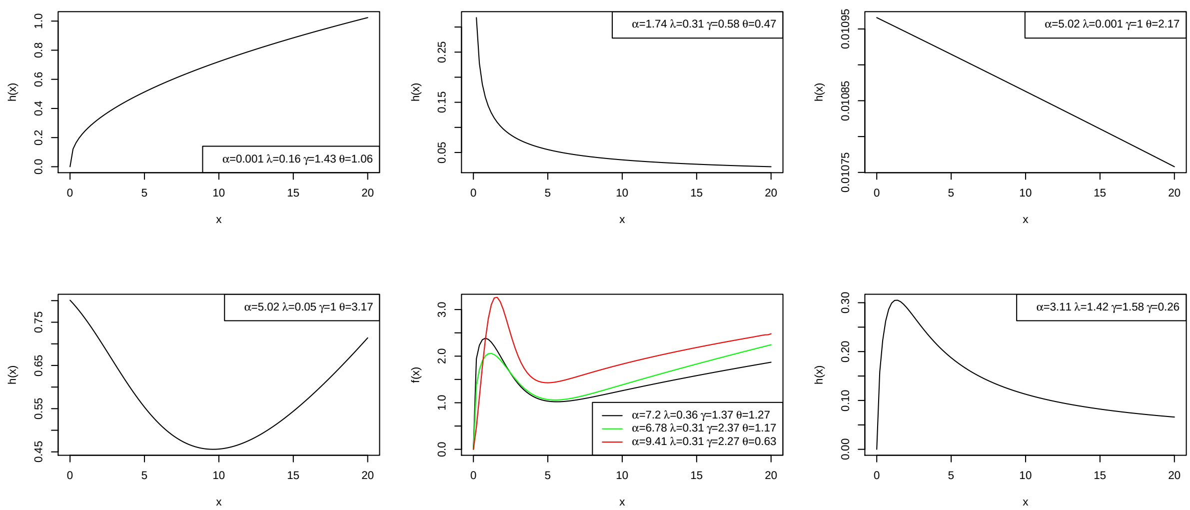

Figure 8: **Hazard plot of the GPWL distribution**

2.3 Maximum Likelihood Estimation

Maximum likelihood estimation (MLE) finds the parameter estimates by determining the values of the parameters that maximize $L(\theta; X)$. Assuming $X = (X_1, X_2, ..., X_n)$ are measurement values of a random variable with density function $f(X; \theta)$, where θ is the parameter value from the distribution, then MLE finds the value of the model parameter θ, that maximizes $L(\theta; X)$. MLE's were obtained for the four sub-families of the PGPW class of distribution.

For the GPWP distribution, the likelihood function is given as;

$$L = n\log(\alpha\lambda\gamma\theta) + (\gamma-1)\sum_{i=1}^{n}\log(t_i) + (\theta-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma}) + \sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma}))$$

$$\times\alpha\sum_{i=1}^{n}e^{1-(1+\lambda t_i^{\gamma})\theta} - n\log(e^{\alpha}-1). \tag{44}$$

For the GPWL distribution, the likelihood function is given as;

$$L = n\log(\alpha\lambda\gamma\theta)+(\gamma-1)\sum_{i=1}^{n}\log(t_i)+(\gamma-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma})+\sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta})-n\log(\alpha)$$

$$-\sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta}) - n\log(\log(1-\alpha)). \tag{45}$$

For the GPWG distribution, the likelihood function is given as;

$$L = n\log(1-\alpha)(\lambda\gamma\theta) + (\gamma-1)\sum_{i=1}^{n}\log(t_i) + (\theta-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma}) + \sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta}).$$

$$-2\sum_{i=1}^{n}(1-\alpha e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}). \tag{46}$$

The PGWB distribution has its likelihood function defined as;

$$L = n\log(m\alpha\lambda\gamma\theta) + (\gamma-1)\sum_{i=1}^{n}\log(t_i) + (\theta-1)\sum_{i=1}^{n}\log(1+\lambda t_i^{\gamma})$$

$$+\sum_{i=1}^{n}(1-(1+\lambda t_i^{\gamma})^{\theta}) - n\log((1+\alpha)^{m}-1) - (1-m)\sum_{i=1}^{n}(1+\alpha e^{(1-(1+\lambda t_i^{\gamma})^{\theta})}). \tag{47}$$

To obtain the MLE of the parameters for each class of distribution, we maximises its score function by taking the first derivative of it.

2.4 Monte Carlo Simulation

Simulation analyses were conducted to assess the performance of the maximum likelihood estimators for the parameters of the sub-families of the PGPW distribution (thus the GPWG, GPWP, GPWB and GPWL). Three parameter value combinations of each distribution were specified. The quantile function of each distribution was then used to generate five different random samples of sizes, 40,80,120,160,200. These were then used to obtain the maximum likelihood estimates of the parameters. With a replication for N=1000 times, the average bias (ABias) and mean square error (MSE) were calculated for the estimators of the parameters of each distribution. For the GPB family, m=5 was used for the simulation. The results of the simulation analyses are shown in Tables 2 to 5. The results showed that, the maximum likelihood estimates of the parameters of each distribution converges to the true parameter value since the average bias of each parameter decrease as the sample size increases and the mean square errors also approaches zero as the sample size increases. All simulations in the work are done using R-software.

Table 2: **Monte Carlo Simulation Results for the Parameters of the GPWG distribution**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.3	0.7	2.5	0.5	0.288	59.780	1.332	0.438	0.093	4.897	5.073	1.403
80	0.3	0.7	2.5	0.5	0.283	13.181	0.632	0.417	0.091	3.103	0.965	0.870
120	0.3	0.7	2.5	0.5	0.281	0.861	0.447	0.412	0.089	4.000	0.369	0.921
160	0.3	0.7	2.5	0.5	0.189	0.724	0.350	0.348	0.088	2.297	0.221	0.677
200	0.3	0.7	2.5	0.5	0.119	0.635	0.304	0.327	0.080	0.767	0.164	0.569
40	0.3	0.4	2.8	0.3	0.295	6.678	1.568	0.128	0.103	75.756	8.628	0.035
80	0.3	0.4	2.8	0.3	0.292	0.456	0.702	0.097	0.100	0.922	1.054	0.031
120	0.3	0.4	2.8	0.3	0.287	0.346	0.511	0.079	0.097	0.543	0.581	0.024
160	0.3	0.4	2.8	0.3	0.275	0.275	0.420	0.074	0.090	0.133	0.353	0.060
200	0.3	0.4	2.8	0.3	0.275	0.274	0.372	0.061	0.092	0.120	0.262	0.007

Table 3: **Monte Carlo Simulation Results for the Parameters of the GPWP distribution**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.3	0.7	2.5	0.5	0.579	0.693	0.625	0.648	0.377	2.854	0.859	5.462
80	0.3	0.7	2.5	0.5	0.554	0.525	0.436	0.330	0.354	2.502	0.400	0.454
120	0.3	0.7	2.5	0.5	0.535	0.334	0.345	0.248	0.340	0.208	0.189	0.231
160	0.3	0.7	2.5	0.5	0.329	0.145	0.129	0.072	0.329	0.145	0.129	0.072
200	0.3	0.7	2.5	0.5	0.318	0.142	0.129	0.061	0.324	0.110	0.111	0.047
40	0.4	0.4	2.7	0.3	0.482	0.281	0.913	0.273	0.265	0.706	2.756	0.604
80	0.4	0.4	2.7	0.3	0.458	0.164	0.535	0.134	0.249	0.046	0.599	0.062
120	0.4	0.4	2.7	0.3	0.454	0.146	0.413	0.089	0.245	0.033	0.282	0.020
160	0.4	0.4	2.7	0.3	0.451	0.128	0.338	0.069	0.244	0.024	0.195	0.014
200	0.4	0.4	2.7	0.3	0.446	0.118	0.294	0.055	0.238	0.022	0.143	0.006

Table 4: **Monte Carlo Simulation Results for the Parameters of the GPWL distribution**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.3	0.7	2.5	0.5	0.358	0.980	0.651	0.284	0.158	111.428	1.039	0.451
80	0.3	0.7	2.5	0.5	0.358	0.341	0.392	0.237	0.161	0.428	0.279	0.234
120	0.3	0.7	2.5	0.5	0.355	0.310	0.309	0.186	0.156	0.176	0.165	0.132
160	0.3	0.7	2.5	0.5	0.346	0.298	0.255	0.148	0.152	0.138	0.115	0.056
200	0.3	0.7	2.5	0.5	0.344	0.297	0.229	0.136	0.150	0.143	0.090	0.044
40	0.4	0.7	2.5	0.5	0.322	1.385	0.666	0.355	0.126	144.378	1.078	0.760
80	0.4	0.7	2.5	0.5	0.301	0.370	0.413	0.219	0.110	0.481	0.316	0.125
120	0.4	0.7	2.5	0.5	0.279	0.329	0.353	0.198	0.100	0.325	0.245	0.148
160	0.4	0.7	2.5	0.5	0.278	0.279	0.248	0.153	0.100	0.108	0.099	0.064
200	0.4	0.7	2.5	0.5	0.280	0.271	0.250	0.145	0.099	0.104	0.097	0.056

Table 5: **Monte Carlo Simulation Results for the Parameters of the GPWB distribution**

n	Parameter value				ABiase				MSE			
	α	λ	γ	θ	α	λ	γ	θ	α	λ	γ	θ
40	0.5	0.7	2.5	0.5	0.368	0.696	40.963	0.454	0.164	0.484	1362.307	0.207
80	0.5	0.7	2.5	0.5	0.343	0.688	40.781	0.454	0.148	0.482	1343.548	0.204
120	0.5	0.7	2.5	0.5	0.331	0.685	39.344	0.441	0.140	0.479	1109.620	0.191
160	0.5	0.7	2.5	0.5	0.310	0.680	38.716	0.438	0.128	0.475	2300.550	0.189
200	0.5	0.7	2.5	0.5	0.303	0.501	48.699	0.420	0.123	0.429	1070.184	0.182
40	0.6	0.8	2.1	0.4	0.326	0.788	32.353	0.377	0.124	0.632	2559.651	0.142
80	0.6	0.8	2.1	0.4	0.297	0.785	31.972	0.362	0.109	0.621	2100.949	0.132
120	0.6	0.8	2.1	0.4	0.282	0.777	31.235	0.360	0.100	0.619	1966.925	0.125
160	0.6	0.8	2.1	0.4	0.268	0.777	30.872	0.358	0.091	0.537	1943.259	0.123
200	0.6	0.8	2.1	0.4	0.248	0.769	30.738	0.355	0.081	0.537	1729.232	0.117

2.5 Applications of the PGPW Class of Distributions

The derived GPWG, GPWP, GPWB (with m=5) and the GPWL distributions were applied to two sets of data (failure times data of air conditioning system of aircraft and the service time of 63 aircrafts. The performance of these distributions, in terms of providing good parametric fit to the two data sets, were compared using the Kolmogorov-Smirnov (KS) statistic, Cramér-Von Mises statistic (CVM), Anderson-Darling statistic (AD), log-likelihood and model selection criteria such as the AIC, AICc and BIC.

2.5.1 Application I: Failure times of air conditioning system of an aircraft

The first application used 30 observations from the failure times of air conditioning system of an aircraft. This data is displayed in Appendix.

The TTT transformed plot of the failure times of air conditioning systems of an aircraft as shown in Figure 9 is first convex in shape followed by a concave shape which indicates that the hazard function of the data set is bathtub shaped.

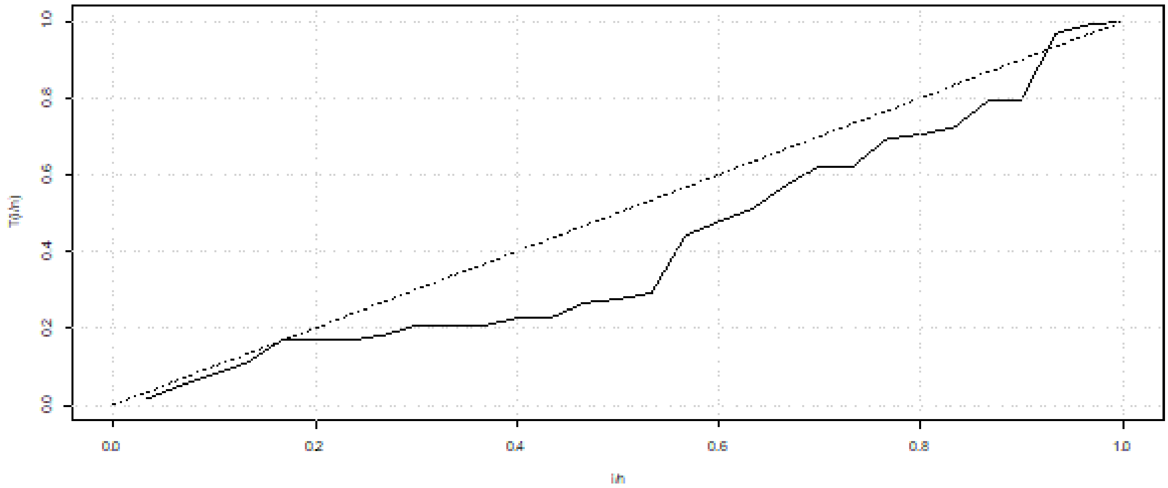

Figure 9: **TTT plot of failure times of the air conditioning system of an aircraft**

The detailed maximum likelihood parameter estimates for the four fitted families of distributions for the failure times of the air conditioning systems of an aircraft are shown in Table 6. By using the estimated standard errors and p-values for the four distributions, it is seen that all the parameters of the GPWB, GPWG and the GPWL distributions are all significant at 5 percent significance level since their standard errors are less than half of their parameter estimates and their p-values are also less than 0.05. For the GPWP family, all the parameters were significant at 0.05 significance level with the exception of the parameter θ.

Table 6: **MLE, SE and p-values of failure times of air conditioning system of an aircraft**

Distribution	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$
GPWB	1.336	11.132	17.890	0.017
	(0.413)	(0.002)	(0.003)	(0.001)
	0.001	< 0.0001	< 0.0001	< 0.0001
GPWG	-5.833	200.002	100.505	0.003
	(1.223×10^{-8})	(6.034×10^{-12})	(5.730×10^{-9})	(1.845×10^{-4})
	< 0.0001	< 0.0001	< 0.0001	< 0.0001
GPWP	2.8117	0.0071	0.9601	0.9601
	(1.0306)	(0.0079)	(0.2546)	(2.0963)
	0.006	0.042	0.001	0.559
GPWL	-39.157	99.891	105.586	0.003
	(1.202×10^{-9})	(1.066×10^{-11})	(5.285×10^{-9})	(1.857×10^{-4})
	< 0.0001	< 0.0001	< 0.0001	< 0.0001

Table 7 presents the likelihood, information criteria and goodness-of-fit measures for the fitted distributions for the failure times of the air conditioning system of an aircraft. Among the four fitted distributions, the GPWG distribution has the largest log-likelihood value with the smallest KS, AD, CVM, AIC, AICc, and BIC statistic values. This indicates that, the GPWG distribution provides a better fit to the failure times of the air conditioning system of an aircraft as compared to the other fitted family of distributions.

Table 7: **Goodness-of-fit and Information Criteria of failure times of air conditioning system of an aircraft**

Dist.	LL	$-2\log L$	AIC	AICc	BIC	CVM	AD	KS(p-value)
GPWB	−151.190	303.386	312.386	313.986	319.392	0.075	0.471	0.118(0.798)
GPWG	−151.170	302.348	310.348	311.948	315.953	0.074	0.433	0.116(0.808)
GPWP	−151.710	303.428	311.428	313.028	317.033	0.097	0.523	0.1387(0.611)
GPWL	−151.990	303.984	311.984	313.584	317.589	0.075	0.517	0.183(0.268)

Figure 10 gives the plot of the empirical CDF and the CDFs of the GPWG, GPWP, GPWB and the GPWL distributions for the failure times of the air conditioning systems of an aircraft. From the figure, the GPWG, GPWP and the GPWL distributions provides a better fit to the data than the GPWB.

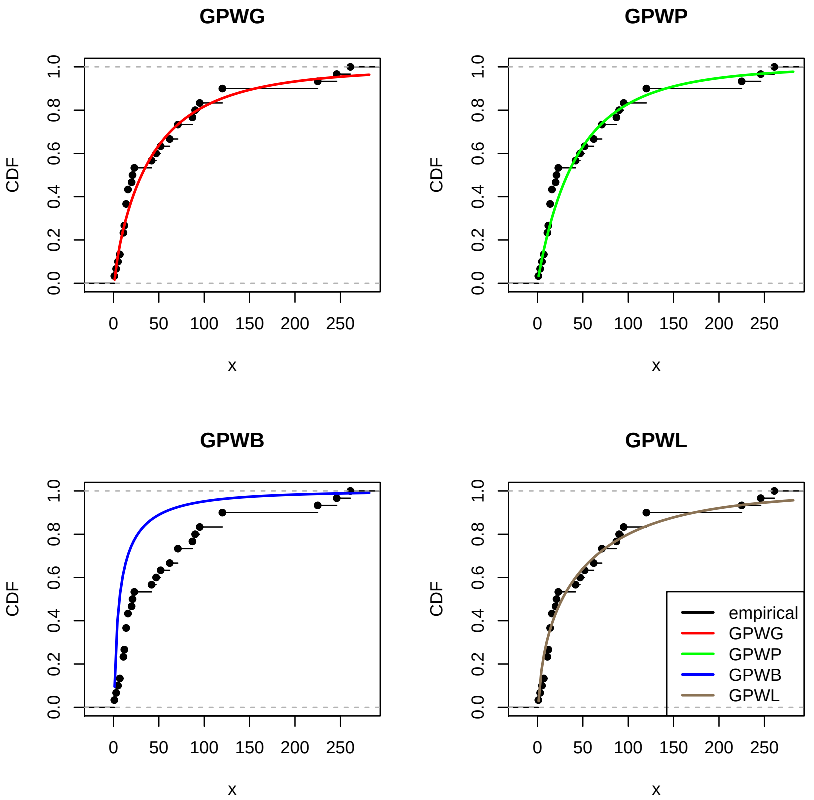

Figure 10: **Empirical CDF and CDF plots of failure times of the air conditioning systems of an aircraft**

2.5.2 Application II: Service times of 63 aircrafts Data Set

The second application of the four families of distributions used failure data on service times of 63 aircrafts given in Murthy et al. (2004) and recently studied by Tahir et al. (2015). This failure rate data is given in Appendix.
The TTT transform plot in Figure 11 indicates that, the data set has an increasing failure rate.

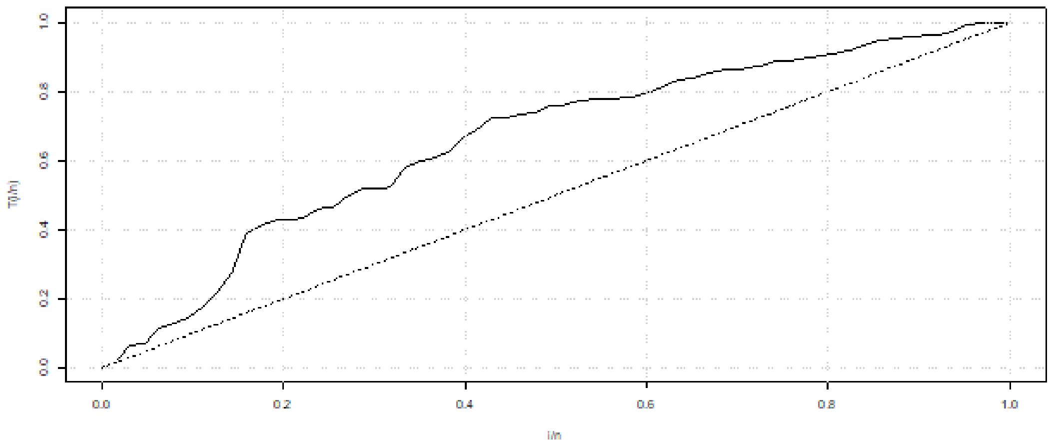

Figure 11: **TTT plot of service times of 63 aircraft**

The maximum likelihood parameter estimates, standard errors and p-values of the GPWB, GPWG, GPWP and the GPWL distributions are presented in Table 8. Using the standard errors of the parameters, all the parameters estimates of the GPWB, GPWG and the GPWL distributions are significant at 5 percent significant level. However, for the GPWP distribution, λ, θ and γ are significant at the 5 percent significance level whiles α is not.

Table 8: **MLE Parameter Estimates, SE and p-values of the service times of 63 aircrafts**

Distribution	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\gamma}$	$\hat{\theta}$
GPWB	7.209	0.009	1.595	2.871
	(0.001)	(0.002)	(0.165)	(0.001)
	0.000	0.000	0.000	0.000
GPWG	−65.228	197.833	1.910	0.253
	(0.002)	(0.001)	(0.279)	(0.016)
	0.000	0.000	0.000	0.000
GPWP	1.742	0.048	0.955	8.759
	(1.825)	(0.026)	(0.312)	(0.012)
	0.340	0.055	0.002	0.000
GPWL	−199.196	155.418	2.962	0.209
	(0.002)	(0.001)	(0.364)	(0.0126)
	0.000	0.000	0.000	0.000

The likelihood, goodness-of-fit and information criteria for the fitted distributions are presented in Table 9. The GPWP distribution provides a better fit among the four fitted distributions since it has the highest log-likelihood and the minimum AIC, AICc, BIC, KS, AD, CVM and -2log L values.

Table 9: **Goodness-of-fit and Information Criteria of service times of 63 aircrafts**

Dist.	LL	$-2\log L$	AIC	AICc	BIC	W^*	A^*	K-S(p-value)
GPWB	−100.010	200.019	210.019	210.709	220.735	0.098	0.597	0.107(0.439)
GPWG	−100.690	201.388	209.385	210.075	217.958	0.101	0.620	0.085(0.717)
GPWP	−98.020	196.039	204.039	204.729	212.611	0.033	0.225	0.065(0.940)
GPWL	−104.380	208.769	290.656	291.346	299.228	0.045	0.286	0.462(0.268)

The plot of the empirical CDF and the CDF of the GPWB, GPWP, GPWG and GPWL are shown in Figure 12. From the plots, the GPWG and the GPWP provide a better fit as compared to the other distributions considered.

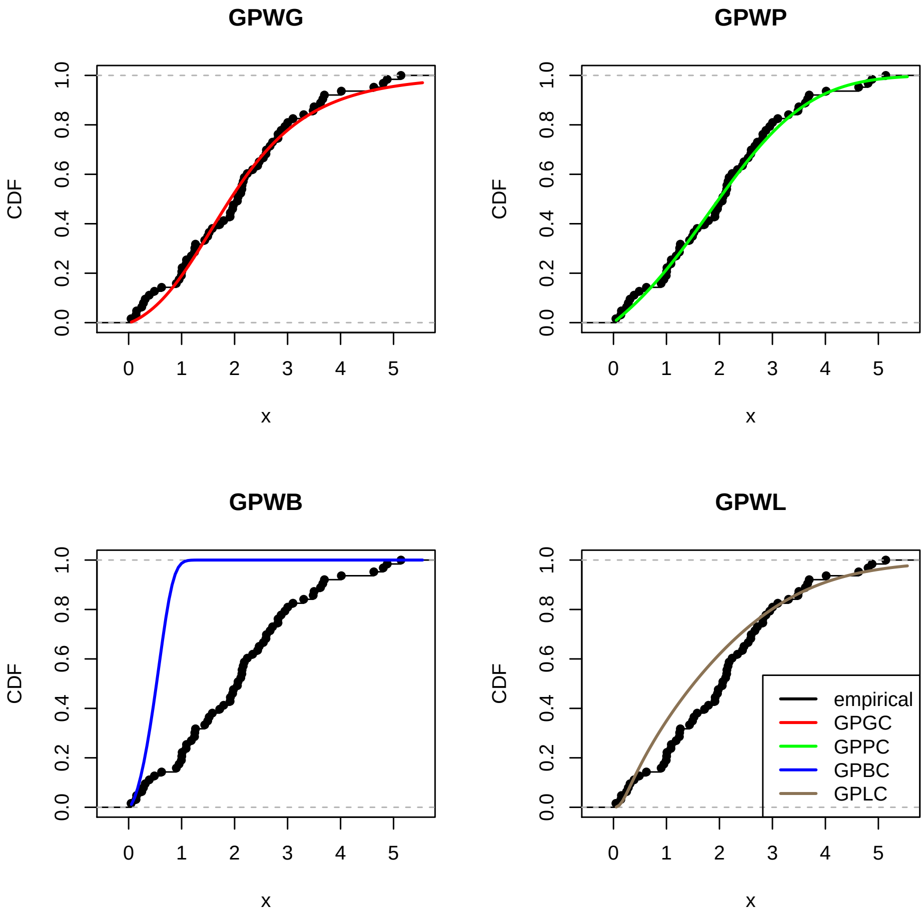

Figure 12: **Empirical CDF and CDF plots of service times of 63 aircrafts**

3 Conclusion

By discrete-continuous compounding the zero truncated power series family and the GPW distribution, the PGPW class of distributions was developed in this paper. The PGPW class of distributions contains the GPW geometric (GPWG), GPW Poisson (GPWP), GPW binomial (GPWB) and the GPW logarithmic (GPWL) as sub-distributions. As presented, the four sub-families of distributions of the PGPW class of distributions can adequately model both monotonic and non-monotonic lifetime data sets since their PDFs and hazard functions exhibit various shapes such as monotonically increasing, decreasing, bathtub, unimodal, among others. From the Monte Carlo simulation analysis, the estimators of each sub-family of distributions were consistent estimators since their mean square error and average bias approached zero as the sample size increase. Application of the four family of distributions to failure time data of air conditioning system of an aircraft showed that, the GPWG distribution provides a better fit among the other competing distributions whiles the GPWP distribution also provided a better fit for failure rate data of the service times of the aircrafts.

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APPENDIX

Table 10: **Failure Times Data of air conditioning system of an aircraft**

23	261	87	7	120	14	62	47	225	71	246	21	42
20	12	120	11	3	71	11	14	11	16	90	1	16
52	95	14	5									

Table 11: **Failure times data of service times of 63 aircraft**

0.046	1.436	1.003	2.137	2.3	3.5	1.01	2.141	3.622	1.085	2.163	2.592	0.140
1.492	2.6	0.150	1.580	2.670	0.248	1.719	2.717	2.820	0.389	1.920	0.313	1.915
1.52	2.24	4.015	1.183	2.878	0.487	1.963	2.95	0.622	1.978	3.003	0.28	1.794
2.819	2.053	3.102	0.952	2.065	3.304	0.996	0.900	1.092	2.183	3.692	2.117	3.483
3.665	2.341	4.628	1.244	2.435	4.806	4.881	1.262	2.543	5.14			