Modelling and Optimization of Portfolio in a DC Scheme with Return of Contributions and Tax using Weibull Force Function

Abstract

One of the major challenges faced by most pension fund managers in the defined pension (DC) scheme is how best member's contributions can be invested to yield maximum returns. To achieve this, there is need to model and developed a robust investment plan which takes into consideration the volatility of the stock market price, tax on investment on risky assets and the mortality risk of its members. Based on this, the optimal portfolio distribution of a DC pension scheme with return of premium clause is studied where the mortality force function is characterized by the Weibull model and the investment in risky asset is subject to a certain proportion of tax. A portfolio with a risk-free asset and a risky asset modeled by the geometric Brownian motion such that the remaining accumulations are equally distributed between the remaining members is considered. Furthermore, the game theoretic approach is used to establish an optimization problem from the extended Hamilton Jacobi Bellman (HJB) equation which is a non-linear partial differential equation (PDE). Using variable separation method, closed form solutions of the optimal portfolio distribution and the efficient frontier are obtained. Lastly, some numerical simulations are used to study the impact of some the parameters on the optimal portfolio distribution with observations that the optimal portfolio distribution developed by the fund manager is inversely proportional to the tax imposed on the risky asset, risk averse coefficient, initial fund size, and risk free interest rate but directly proportional to time.

Keywords: DC Pension scheme, Weibull Model, Portfolio distributions, Refund clause of contribution, Game theoretic method, Mean variance utility.

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1. INTRODUCTION

The study of optimal portfolio distribution is very crucial and has attracted so much attention by many researchers due to the volatile nature of the risky assets in the financial markets. For a defined contribution pension fund manager to make seemingly right choices in investment process involving risky assets, it is necessary to consider determining the proportion of each of the assets to be invested per time for optimal profit with minimal risk considering the mortality risk of its members during the accumulation phase. Utility maximization is very crucial in the study of optimal investment strategy and has been studied by several authors such as [1-6].

Recently, the study of optimal portfolio distribution with return of contributions clause have been studied by different researchers where the mortality risk of its members were taken into consideration while determining the optimal portfolio distribution. A number of researchers who studied optimal portfolio distributions with return clauses used the Abraham De Moivre model to describe the mortality force function; they include [7] who studied optimal portfolio distributions for a DC pension with refund of contributions under mean-variance utility. In their work, they considered investment in one risky asset and went on to determine both the optimal portfolio distributions and the efficient frontier. In [8], the same problem in [7] was studied for both accumulation and distribution phases where the risky asset was modelled by Heston volatility model. The strategic optimal portfolio management for a DC plan with refund clause was studied in [9]; in their work, they extended the work of [7] by considering investment in a risk free and two risky assets and determine the optimal portfolio distributions and also the efficient frontier of the member. In [10], investment strategies with refund clause was studied under inflation and volatility risk; they considered investment in one risk free asset, stock and inflation index bond where the stock market price was modelled by Heston's volatility model. In [11], the DC pension plan with refund clause was studied under affine interest structure; here they considered a case where the risk free interest rate was modeled by Cox Ingeroll model and also determine the optimal portfolio distributions and efficient frontier. [12], studied optimal investment for the DC plan with refund of premium under jump diffusion process; in their work, the risky asset was modeled by the jump diffusion process. The authors in [13-14] studied optimal investment strategy for DC pension plan with return of premiums clauses under (CEV) model; they considered investments in treasury, stock and bond. From the above literatures, the authors assumed that the refund contributions were without any kind of interest.

Most recently, the optimal control laws with refund of contributions with predetermined interest have been studied by some authors; in their work, they assumed the refund contributions to the death members' families are with predetermined interest from the risk free asset. They include [15], where the optimal control plan in a DC plan with a risk free and one risky asset were studied when the refund contributions were with predetermined interest and the price of the risky asset was modelled by GBM. In [16], the optimal asset allocation strategy for a DC pension system with refund of contributions with predetermined interest under Heston's volatility model was studied. Since the price process of the risk free asset is deterministic and the interest rate is predetermined, it is possible to determine the interest paid to each death member's family at each point in time during the accumulation phase. Also, [17] studied optimal control strategy when the risky asset is modelled by CEV model and the refund is with predetermined interest.

In all the above mention works, the mortality force function were characterized byAbraham De Moivre model except for [18] who used the Weibull model to describe the mortality force function by studying the optimal portfolio selection for a defined contribution plan under two administrative fees and return of premium clauses for a pension member exhibiting constant absolute risk averse (CARA) and constant relative risk averse (CRRA) utility. In their work, they considered investment in one risk free and one risky asset and assumed the risky asset is modelled by the CEV model. These form the basis of our research where we study the optimal portfolio distribution for a pension scheme with return of premium clauses under mean variance utility where the mortality force function is characterized by the Weibull function and the investment in risky asset is subject to a certain proportion of tax.

2. Pension Wealth Formulation

Assume a financial market comprising of a risk-free asset and risky asset. Assume$(\Omega, \mathcal{F}_t, \mathcal{P})$is a complete probability space over the real spaceΩ and $\mathcal{P}$ is a probability measure, $\mathcal{Z}_t(t)$ is a standard Brownian motion, $\mathcal{F}_t$ is the filtration and denotes the information generated by the Brownian motion $\mathcal{Z}_t(t)$

Let $\mathcal{U}_t(t)$and $\mathcal{V}_t(t)$ denote the prices of the risk-free asset and risky asset respectively and they are modelled as

$$\frac{d\mathcal{U}_t(t)}{\mathcal{U}_t(t)} = \mathscr{r}dt, \qquad (2.1)$$

$$\frac{d\mathcal{V}_t(t)}{\mathcal{V}_t(t)} = (\mathscr{r} + \mathscr{s}_1)dt + \mathscr{n}_1 d\mathcal{Z}_t. \qquad (2.2)$$

Where $\mathscr{r} > 0$ is the interest rate of the risk free asset, $\mathscr{r} + \mathscr{s}_1$ is an expected appreciation rate of the risky asset price and $\mathscr{n}_1$is the volatility of the stock market price and satisfies the general condition $\mathscr{n}_1 > 0$

Letπ represent the proportion of the wealth to be invested in risky assets and $\pi_1 = 1 - \pi$, the proportion to be invested in the risk free asset and $\mathscr{q}$ be the contributions received at a given time, which is predetermined, ℓ_0 represent the initial age of accumulation phase, T is the time frame in years of the accumulation phase such that $\ell_0 + T$ is the end age and ε the proportional tax on the risky. The actuarial symbol $\mathfrak{z}_{\frac{1}{e}, \ell_0+t}$ is the mortality rate from time t to $t + \frac{1}{e}$, $t\mathscr{q}$ is the premium accumulated at time t, $t\mathscr{q}\mathfrak{z}_{\frac{1}{e}, \ell_0+t}$

is the premium returned to the death membersduring the accumulation phase.

Considering the time interval $[t, t+\frac{1}{e}]$, the differential form associated with the fund size is given as:

$$\mathcal{X}\left(t+\tfrac{1}{e}\right) = \left[\mathcal{X}(t)\left(\pi\frac{\mathcal{V}_{t+\frac{1}{e}}}{\mathcal{V}_t} + \pi_1\frac{\mathcal{U}_{t+\frac{1}{e}}}{\mathcal{U}_t} - \varepsilon\pi\frac{\mathcal{V}_{t+\frac{1}{e}}}{\mathcal{V}_t}\right) + q\frac{1}{e} - tq\mathfrak{z}_{\frac{1}{e}\ell_0+t}\right]\left(\frac{1}{1-\mathfrak{z}_{\frac{1}{e},\ell_0+t}}\right) \tag{2.3}$$

$$\mathcal{X}\left(t+\tfrac{1}{e}\right) = \left[\begin{matrix}\mathcal{X}(t)\left(\pi(1-\varepsilon)\left(\frac{\mathcal{V}_{t+\frac{1}{e}}}{\mathcal{V}_t} - \frac{\mathcal{V}_t}{\mathcal{V}_t} + \frac{\mathcal{V}_t}{\mathcal{V}_t}\right) + (1-\pi)\left(\frac{\mathcal{U}_{t+\frac{1}{e}}}{\mathcal{U}_t} - \frac{\mathcal{U}_t}{\mathcal{U}_t} + \frac{\mathcal{U}_t}{\mathcal{U}_t}\right)\right) \\ +q\frac{1}{e} - tq\mathfrak{z}_{\frac{1}{e}\ell_0+t}\end{matrix}\right]\left(1 + \frac{\mathfrak{z}_{\frac{1}{e},\ell_0+t}}{1-\mathfrak{z}_{\frac{1}{e},\ell_0+t}}\right) \tag{2.4}$$

$$\mathcal{X}\left(t+\tfrac{1}{e}\right) = \left[\begin{matrix}\mathcal{X}(t)\left(\pi(1-\varepsilon)\left(\frac{\mathcal{V}_{t+\frac{1}{e}}-\mathcal{V}_t}{\mathcal{V}_t}\right) + 1 - \pi\varepsilon + (1-\pi)\left(\frac{\mathcal{U}_{t+\frac{1}{e}}-\mathcal{U}_t}{\mathcal{U}_t}\right)\right) \\ +q\frac{1}{e} - tq\mathfrak{z}_{\frac{1}{e}\ell_0+t}\end{matrix}\right]\left(1 + \frac{\mathfrak{z}_{\frac{1}{e},\ell_0+t}}{1-\mathfrak{z}_{\frac{1}{e},\ell_0+t}}\right) \tag{2.5}$$

$$\begin{cases}\mathfrak{z}_{\frac{1}{e},\ell_0+t} = 1 - \text{Exp}\left\{-\int_0^{\frac{1}{e}} w(\ell_0+t+s)ds\right\} = w(\ell_0+t)\frac{1}{e} + O(\frac{1}{e}), \\ \frac{\mathfrak{z}_{\frac{1}{e},\ell_0+t}}{1-\mathfrak{z}_{\frac{1}{e},\ell_0+t}} = w(\ell_0+t)\frac{1}{e} + O(\frac{1}{e}) \\ \frac{1}{e} \to 0, \mathfrak{z}_{\frac{1}{e},\ell_0+t} = w(\ell_0+t)dt, \frac{\mathfrak{z}_{\frac{1}{e},\ell_0+t}}{1-\mathfrak{z}_{\frac{1}{e},\ell_0+t}} = w(\ell_0+t)dt, \\ q\frac{1}{e} \to qdt, \frac{\mathcal{V}_{t+\frac{1}{e}}-\mathcal{V}_t}{\mathcal{V}_t} \to \frac{d\mathcal{V}_t(t)}{\mathcal{V}_t(t)}, \frac{\mathcal{U}_{t+\frac{1}{e}}-\mathcal{U}_t}{\mathcal{U}_t} \to \frac{d\mathcal{U}_t(t)}{\mathcal{U}_t(t)}\end{cases} \tag{2.6}$$

Substituting (2.7) into (2.6) we have

$$\mathcal{X}\left(t+\tfrac{1}{e}\right) = \left[\begin{matrix}\mathcal{X}(t)\left(\pi(1-\varepsilon)\frac{d\mathcal{V}_t(t)}{\mathcal{V}_t(t)} + 1 - \pi\varepsilon + (1-\pi)\frac{d\mathcal{U}_t(t)}{\mathcal{U}_t(t)}\right) \\ +qdt - tqw(\ell_0+t)dt\end{matrix}\right](1 + w(\ell_0+t)dt) \tag{2.8}$$

Substituting (2.1) and (2.2) into (2.8), we have

$$\mathcal{X}\left(t+\tfrac{1}{e}\right) - \mathcal{X}(t) = \left[\begin{matrix}\mathcal{X}(t)\left(\begin{matrix}\pi(1-\varepsilon)\big((r+s_1)dt + n_1 d\mathcal{Z}_t\big) - \pi\varepsilon \\ +(1-\pi)rdt + w(\ell_0+t)dt\end{matrix}\right) \\ +q\big(1 - tw(\ell_0+t)\big)dt\end{matrix}\right] \tag{2.9}$$

Since $w(t)$is the force function and ℓ is the maximal age of the life table. From Weibull [18]

The Weibull force function formula is given as

$$w(t) = ht^m \quad 0 \le t < T \quad . \tag{2.10}$$

This implies that

$$w(\ell_0+t) = h(\ell_0+t)^m \tag{2.11}$$

Substituting (2.11) into (2.9) and simplifying it, we have

$$d\mathcal{X}(t) = \left\{\begin{matrix}\left(\begin{matrix}\mathcal{X}(t)\left[\pi\big(s_1 - \varepsilon(s_1 + r + 1)\big) + h(\ell_0+t)^m + r\right] \\ +q(1 - th(\ell_0+t)^m)\end{matrix}\right)dt \\ +\pi(1-\varepsilon)\mathcal{X}(t)n_1 d\mathcal{Z}_t\end{matrix}\right\} \qquad \mathcal{X}(0) = x_0 \tag{2.12}$$

3. Methodology

In this section, we consider a fund manager whose interest is to maximize his surviving member's fund size and minimize the volatility of the wealth accumulated. Hence, there is need to develop an optimal portfolio problem using mean-variance utility as follows:

$$\mathcal{J}(t,x) = \sup_{\pi}\{E_{t,x}\mathcal{X}^{\pi}(T) - Var_{t,x}\mathcal{X}^{\pi}(T)\} \tag{3.1}$$

Next, we follow the approach in [7,8], by using the variational inequality technique. The control problem in (3.1) is equivalent to the following Markovian time inconsistent stochastic optimal control problem with value function $\mathcal{J}(t,x)$

$$\begin{cases} \mathcal{N}(t,x,\pi) = E_{t,x}[\mathcal{X}^{\pi}(T)] - \frac{\gamma}{2}Var_{t,x}[\mathcal{X}^{\pi}(T)] \\ = (E_{t,x}[\mathcal{X}^{\pi}(T)] - \frac{\gamma}{2}\left(E_{t,x}[\mathcal{X}^{\pi}(T)^2] - (E_{t,x}[\mathcal{X}^{\pi}(T)])^2\right)) \\ \mathcal{J}(t,x) = \sup_{\pi}\mathcal{N}(t,x,\pi) \end{cases} \tag{3.2}$$

Following [8] the optimal control lawπ^* satisfies:

$$\mathcal{J}(t,x) = \sup_{\pi}\mathcal{N}(t,x,\pi^*) \tag{3.3}$$

whereγrepresent the risk-averse coefficient of the members

Let $u^{\pi}(t,x) = E_{t,x}[\mathcal{X}^{\pi}(T)]$,$v^{\pi}(t,x) = E_{t,x}[\mathcal{X}^{\pi}(T)^2]$then

$\mathcal{J}(t,x) = \sup_{\pi} k\left(t,x,u^{\pi}(t,x),v^{\pi}(t,x)\right)$ where

$$a(t,x,u,v) = u - \frac{\gamma}{2}(v - u^2) \tag{3.4}$$

Theorem 3.1 (verification theorem). If there exists three real functions $\mathcal{A},\mathcal{B},\mathcal{C}$:$[0,T]\times R \to R$ satisfying the following extended Hamilton Jacobi Bellman equation equations:

$$\begin{cases} \sup_{\varphi}\left\{ \begin{array}{c} \mathcal{A}_t - a_t + \left[\begin{array}{c} x\left(\pi\left(s_1 - \varepsilon(s_1 + r + 1)\right) + h(\ell_0 + t)^m + r\right) \\ +q(1 - th(\ell_0 + t)^m) \end{array}\right](\mathcal{A}_x - a_x) \\ +\frac{1}{2}\pi^2(1-\varepsilon)^2x^2n_1^2(\mathcal{A}_{xx} - \mathcal{K}_{xx}) \end{array}\right\} = 0 \\ \mathcal{A}(T,x) = a(T,x,x,x^2) \end{cases} \tag{3.5}$$

where:

$$\mathcal{K}_{xx} = \gamma\mathcal{B}_x^2 \tag{3.6}$$

$$\left\{\begin{array}{c} \mathcal{B}_t + \left[\begin{array}{c} x\left(\pi\left(s_1 - \varepsilon(s_1 + r + 1)\right) + h(\ell_0 + t)^m + r\right) \\ +q(1 - th(\ell_0 + t)^m) \end{array}\right]\mathcal{B}_x \\ +\frac{1}{2}\pi^2(1-\varepsilon)^2x^2n_1^2\mathcal{B}_{xx} \\ \mathcal{B}(T,x) = x \end{array}\right\} = 0 \tag{3.7}$$

$$\left\{\begin{array}{c} \mathcal{C}_t + \left[\begin{array}{c} x\left(\pi\left(s_1 - \varepsilon(s_1 + r + 1)\right) + h(\ell_0 + t)^m + r\right) \\ +q(1 - th(\ell_0 + t)^m) \end{array}\right]\mathcal{C}_x \\ +\frac{1}{2}\pi^2(1-\varepsilon)^2x^2n_1^2\mathcal{C}_{xx} \\ \mathcal{C}(T,x,s) = x^2 \end{array}\right\} = 0 \tag{3.8}$$

Then $\mathcal{J}(t,x) = \mathcal{A}(t,x), u^{\pi^*} = \mathcal{B}(t,x), v^{\pi^*} = \mathcal{C}(t,x)$for the optimal investment strategy π^*.

Proof: The details of the proof can be found in [19-21]

4. The Optimal Portfolio Distributions and Efficient Frontier

In this section, we attempt to solve for the optimal portfolio distributions by solving (3.5), (3.7), (3.8).
Lemma4.1. The optimal portfolio distribution for the risky asset is given as

$$\pi^* = \frac{(s_1-\varepsilon(s_1+r+1))}{(1-\varepsilon)^2\gamma x n_1^2}\left[e^{r(t-T)+\frac{h}{m+1}((\ell_0+t)^{m+1}-(\ell_0+T)^{m+1})}\right] \tag{4.1}$$

Proof. Recall that from (3.4),

$$a_t = a_x = a_{xx} = a_{xu} = a_{xv} = a_{uv} = a_{vv} = 0,\ a_u = 1+\gamma a,\ a_{uu} = \gamma,\ a_v = -\frac{\gamma}{2} \tag{4.2}$$

Substituting (3.6), (4.2) into (3.5) and differentiating it with respect to π, we have

$$x\big(s_1-\varepsilon(s_1+r+1)\big)\mathcal{A}_x + \pi(1-\varepsilon)^2x^2n_1^2(\mathcal{A}_{xx}-\mathcal{K}_{xx}) = 0 \tag{4.3}$$

Solving equation (4.3) for π, we have

$$\pi^* = -\left[\frac{(s_1-\varepsilon(s_1+r+1))\mathcal{A}_x}{x(1-\varepsilon)^2n_1^2(\mathcal{A}_{xx}-\gamma\mathcal{B}_x^2)}\right], \tag{4.4}$$

whereπ^* is the optimal portfolio strategy.

Substituting (4.4) into (3.5) and (3.7), we have

$$\left\{\mathcal{A}_t + [(r+h(\ell_0+t)^m)x + q(1-th(\ell_0+t)^m)]\mathcal{A}_x - \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2\mathcal{A}_x^2}{2(1-\varepsilon)^2n_1^2(\mathcal{A}_{xx}-\gamma\mathcal{B}_x^2)}\right\} = 0 \tag{4.5}$$

$$\left\{\begin{aligned}&\mathcal{B}_t + [(r+h(\ell_0+t)^m)x + q(1-th(\ell_0+t)^m)]\mathcal{B}_x - \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2\mathcal{A}_x\mathcal{B}_x}{(1-\varepsilon)^2n_1^2\big(\mathcal{A}_{xx}-\gamma\mathcal{B}_x^2\big)}\\ &\qquad + \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2\mathcal{A}_x^2}{(1-\varepsilon)^2n_1^2(\mathcal{A}_{xx}-\gamma\mathcal{B}_x^2)}\mathcal{B}_{xx}\end{aligned}\right\} = 0 \tag{4.6}$$

To solve equation (4.5) and (4,6), we conjecture a solution for $\mathcal{A}(t,x)$and $\mathcal{B}(t,x)$as follows:

$$\begin{cases}\mathcal{A}(t,x) = xf(t) + g(t),\ \ f(T) = 1, g(T) = 0\\ \quad \mathcal{A}_t = f_t x + g_t, \mathcal{A}_x = f, \mathcal{A}_{xx} = 0\end{cases} \tag{4.7}$$

$$\begin{cases}\mathcal{B}(t,x) = xu(t) + v(t),\ \ u(T) = 1, v(T) = 0\\ \quad \mathcal{B}_t = u_t x + v_t, \mathcal{B}_x = u, \mathcal{B}_{xx} = 0\end{cases} \tag{4.8}$$

Substituting (4.7) and (4.8) into (4.5) and (4.6), we have:

$$\left\{f_t x + g_t + [(r+h(\ell_0+t)^m)x + q(1-th(\ell_0+t)^m)]f + \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2f^2}{2(1-\varepsilon)^2n_1^2\gamma u^2}\right\} = 0 \tag{4.9}$$

$$\left\{u_t x + v_t + [(r+h(\ell_0+t)^m)x + q(1-th(\ell_0+t)^m)]u + \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2fu}{(1-\varepsilon)^2n_1^2\gamma u^2}\right\} = 0 \tag{4.10}$$

Simplifying (4.9) and (4.10), we have

$$\left\{[f_t + (r+h(\ell_0+t)^m)f]x + \left[g_t + q(1-th(\ell_0+t)^m)f + \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2f^2}{2(1-\varepsilon)^2n_1^2\gamma u^2}\right]\right\} = 0 \tag{4.11}$$

$$\left\{[u_t + (r+h(\ell_0+t)^m)u]x + \left[v_t + q(1-th(\ell_0+t)^m)u + \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2fu}{(1-\varepsilon)^2n_1^2\gamma u^2}\right]\right\} = \tag{4.12}$$

Since$x \neq 0$, then

$$\begin{cases}f_t + (r+h(\ell_0+t)^m)f = 0\\ \qquad f(T) = 1\end{cases} \tag{4.13}$$

$$\begin{cases}u_t + (r+h(\ell_0+t)^m)u = 0\\ \qquad u(T) = 1\end{cases} \tag{4.14}$$

$$\begin{cases}g_t + q(1-th(\ell_0+t)^m)f + \frac{\big(s_1-\varepsilon(s_1+r+1)\big)^2f^2}{2(1-\varepsilon)^2n_1^2\gamma u^2} = 0\\ \qquad g(T) = 0\end{cases} \tag{4.15}$$

$$\begin{cases} \mathcal{v}_t + q(1 - th(\ell_0 + t)^m)u + \frac{\left(s_1 - \varepsilon(s_1 + r + 1)\right)^2 fu}{(1-\varepsilon)^2 n_1^2 \gamma u^2} = 0 \\ \mathcal{v}(T) = 0 \end{cases} \tag{4.16}$$

Solving (4.13) – (4.16), we have:

$$f(t) = Exp\left(r(T-t) + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\right) \tag{4.19}$$

$$u(t) = Exp\left(r(T-t) + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\right) \tag{4.20}$$

$$g(t) = \left[\begin{matrix} \frac{\left(s_1 - \varepsilon(s_1 + r + 1)\right)^2}{2(1-\varepsilon)^2 n_1^2 \gamma}(T-t) - \frac{q}{r + h(\ell_0 + T)^m} \\ + \frac{q}{r + h(\ell_0 + t)^m} Exp\left(\begin{matrix} r(T-t) \\ + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\end{matrix}\right) \\ -qhExp\left(\begin{matrix} rT \\ + \frac{h}{m+1}(\ell_0 + T)^{m+1}\end{matrix}\right)\int_t^T \left[\tau(\ell_0 + T)^m Exp\left(\begin{matrix} -r\tau \\ -\frac{h}{m+1}(\ell_0 + \tau)^{m+1}\end{matrix}\right)\right] d\tau \end{matrix}\right] \tag{4.21}$$

$$\mathcal{v}(t) = \left[\begin{matrix} \frac{\left(s_1 - \varepsilon(s_1 + r + 1)\right)^2}{(1-\varepsilon)^2 n_1^2 \gamma}(T-t) - \frac{q}{r + h(\ell_0 + T)^m} \\ + \frac{q}{r + h(\ell_0 + t)^m} Exp\left(\begin{matrix} r(T-t) \\ + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\end{matrix}\right) \\ -qhExp\left(\begin{matrix} rT \\ + \frac{h}{m+1}(\ell_0 + T)^{m+1}\end{matrix}\right)\int_t^T \left[\tau(\ell_0 + T)^m Exp\left(\begin{matrix} -r\tau \\ -\frac{h}{m+1}(\ell_0 + \tau)^{m+1}\end{matrix}\right)\right] d\tau \end{matrix}\right] \tag{4.22}$$

Substituting (4.19) and (4.21)into (4.7)and (4.20), (4.22), (4.24) into (4.8) we have:

$$\mathcal{A}(t,x) = \left[\begin{matrix} xExp\left(r(T-t) + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\right) \\ + \left[\begin{matrix} \frac{\left(s_1 - \varepsilon(s_1 + r + 1)\right)^2}{2(1-\varepsilon)^2 n_1^2 \gamma}(T-t) - \frac{q}{r + h(\ell_0 + T)^m} \\ + \frac{q}{r + h(\ell_0 + t)^m} Exp\left(\begin{matrix} r(T-t) \\ + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\end{matrix}\right) \\ -qhExp\left(\begin{matrix} rT \\ + \frac{h}{m+1}(\ell_0 + T)^{m+1}\end{matrix}\right)\int_t^T \left[\tau(\ell_0 + T)^m Exp\left(\begin{matrix} -r\tau \\ -\frac{h}{m+1}(\ell_0 + \tau)^{m+1}\end{matrix}\right)\right] d\tau \end{matrix}\right] \end{matrix}\right] \tag{4.23}$$

$$\mathcal{B}(t,x) = \left[\begin{matrix} xExp\left(r(T-t) + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\right) \\ + \left[\begin{matrix} \frac{\left(s_1 - \varepsilon(s_1 + r + 1)\right)^2}{(1-\varepsilon)^2 n_1^2 \gamma}(T-t) - \frac{q}{r + h(\ell_0 + T)^m} \\ + \frac{q}{r + h(\ell_0 + t)^m} Exp\left(\begin{matrix} r(T-t) \\ + \frac{h}{m+1}\left[(\ell_0 + T)^{m+1} - (\ell_0 + T)^{m+1}\right]\end{matrix}\right) \\ -qhExp\left(\begin{matrix} rT \\ + \frac{h}{m+1}(\ell_0 + T)^{m+1}\end{matrix}\right)\int_t^T \left[\tau(\ell_0 + T)^m Exp\left(\begin{matrix} -r\tau \\ -\frac{h}{m+1}(\ell_0 + \tau)^{m+1}\end{matrix}\right)\right] d\tau \end{matrix}\right] \end{matrix}\right] \tag{4.24}$$

Substituting $\mathcal{A}_x$,$\mathcal{A}_{xx}$, $\mathcal{B}_x$, into (4.4), we obtain (4.1) which complete the proof.

Proposition 4.2. The efficient frontier of the pension fund is given as follows

$$E_{t,x}[\mathcal{X}^{\pi}(T)] = \left[\begin{array}{c} xExp\left(r(T-t)+\frac{h}{m+1}[(\ell_0+T)^{m+1}-(\ell_0+T)^{m+1}]\right) \\ + \left[\begin{array}{c} +\frac{q}{r+h(\ell_0+t)^m} Exp\left(\begin{array}{c} r(T-t) \\ +\frac{h}{m+1}[(\ell_0+T)^{m+1}-(\ell_0+T)^{m+1}] \end{array} \right) \\ -\frac{q}{r+h(\ell_0+T)^m} - qhExp\left(\begin{array}{c} rT \\ +\frac{h}{m+1}(\ell_0+T)^{m+1} \end{array} \right) \\ \int_t^T \left[\tau(\ell_0+T)^m Exp\left(\begin{array}{c} -r\tau \\ -\frac{h}{m+1}(\ell_0+\tau)^{m+1} \end{array} \right)\right] d\tau \\ \frac{(s_1-\varepsilon(s_1+r+1))}{(1-\varepsilon)n_1}\sqrt{(T-t)Var_{t,x}[\mathcal{X}^{\pi}(T)]} \end{array} \right] \end{array} \right] \tag{4.25}$$

Proof. Recall that

$$Var_{t,x}[\mathcal{X}^{\pi}(T)] = E_{t,x}[\mathcal{X}^{\pi}(T)^2] - (E_{t,x}[\mathcal{X}^{\pi}(T)])^2$$
$$Var_{t,x}[\mathcal{X}^{\pi}(T)] = \frac{2}{\gamma}(\mathcal{B}(t,x)-\mathcal{A}(t,x)) \tag{4.26}$$

Substituting (4.23) and (4.24) for $\mathcal{A}(t,x)$ and $\mathcal{B}(t,x)$ in (4.26), we have

$$Var_{t,x}[\mathcal{X}^{\pi}(T)] = \frac{1}{\gamma^2}\left(\frac{(s_1-\varepsilon(s_1+r+1))^2}{(1-\varepsilon)^2 n_1^2}(T-t)\right) \tag{4.27}$$

$$\frac{1}{\gamma} = \frac{(1-\varepsilon)n_1}{(s_1-\varepsilon(s_1+r+1))}\sqrt{\frac{Var_{t,x}[\mathcal{X}^{\pi}(T)]}{(T-t)}} \tag{4.28}$$

Recall from theorem 3.1, the expectation is given as

$$E_{t,x}[\mathcal{X}^{\pi}(T)] = \mathcal{B}(t,x) \tag{4.29}$$

Substituting equation (4.26) into (4.29), we obtain (4.25) which complete the proof.

Remark 1. If there is no tax imposed on the investment in the risky asset, i.e $\varepsilon = 0$, the optimal portfolio strategy becomes in (4.1) becomes

$$\pi^* = \frac{s_1}{\gamma x n_1^2}\left[e^{r(t-T)+\frac{h}{m+1}((\ell_0+t)^{m+1}-(\ell_0+T)^{m+1})}\right] \tag{4.30}$$

5. Numerical Simulations and Discussion

In this section, we present numerical simulations of the optimal portfolio distribution with respect to time using the following parameters:$s_1 = 0.06$, $\ell_0 = 20$, $T = 40$, $\gamma = 0.2$, $\varepsilon = 0.1$, $m = 0.001$, $h = 0.01$, $n_1 = 20$, $r = 0.06$, $x = 0.1$; unless otherwise stated.

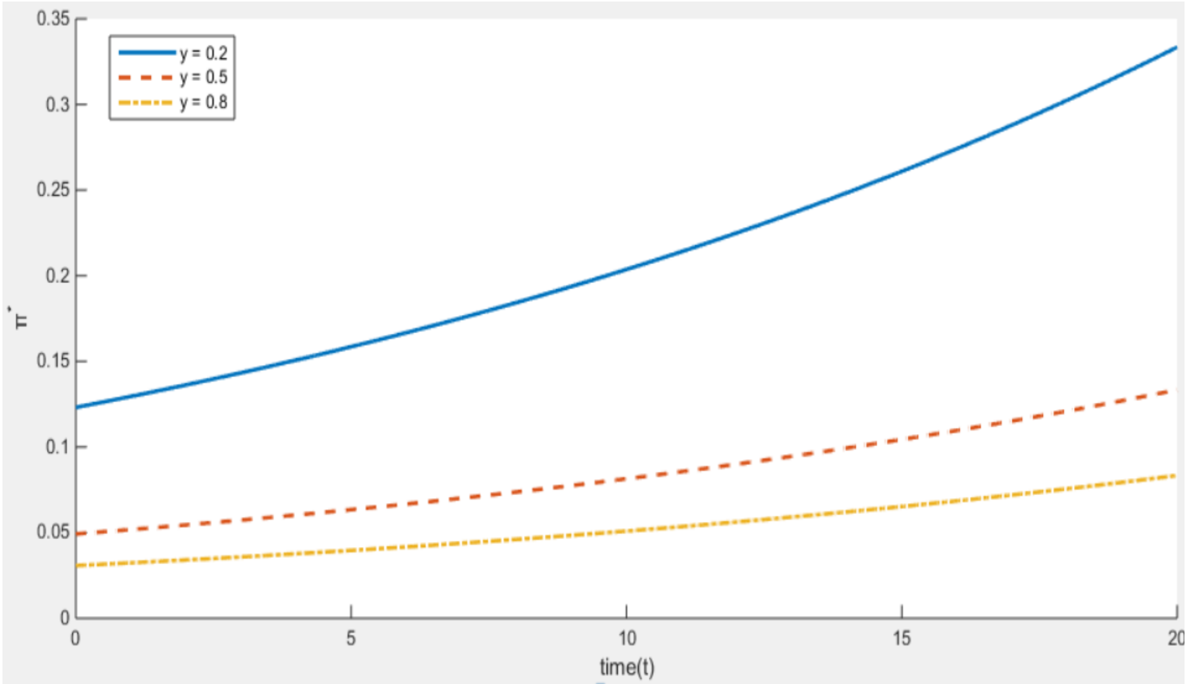

Fig. 1. Evolution of the optimal portfolio distribution $\pi^*$with different risk averse γ

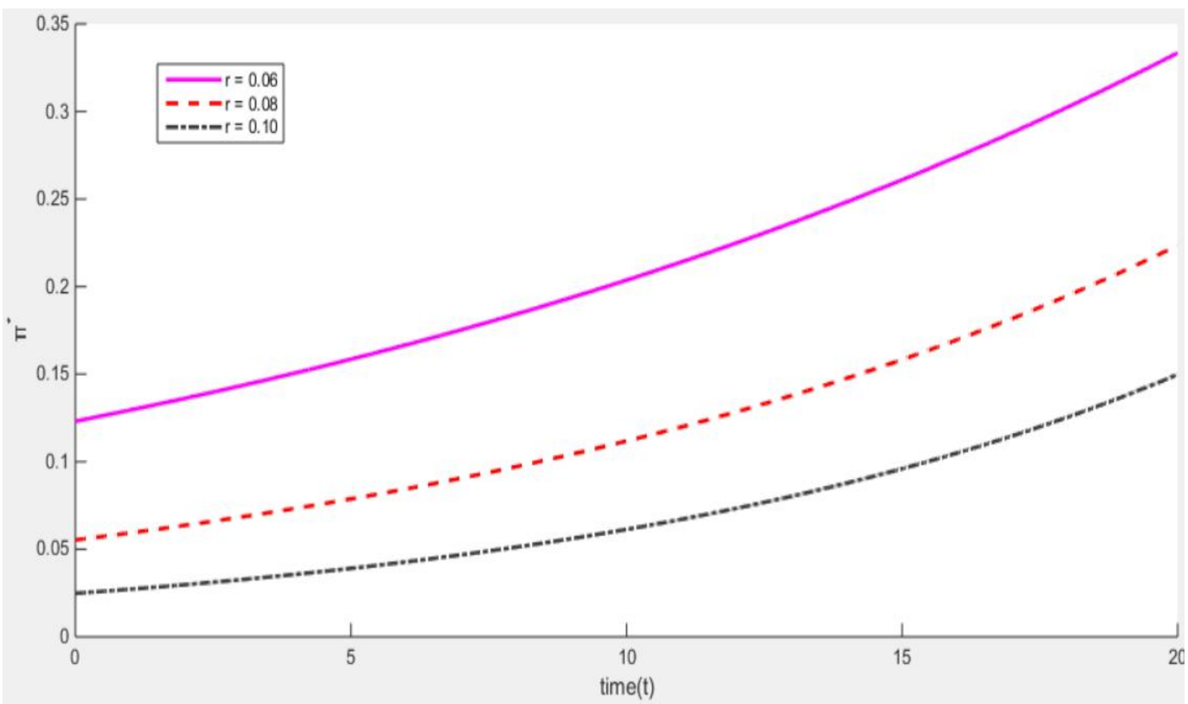

Fig. 2. Evolution of the optimal portfolio distribution $\pi^*$with different predetermined interest rate $\tilde{r}$

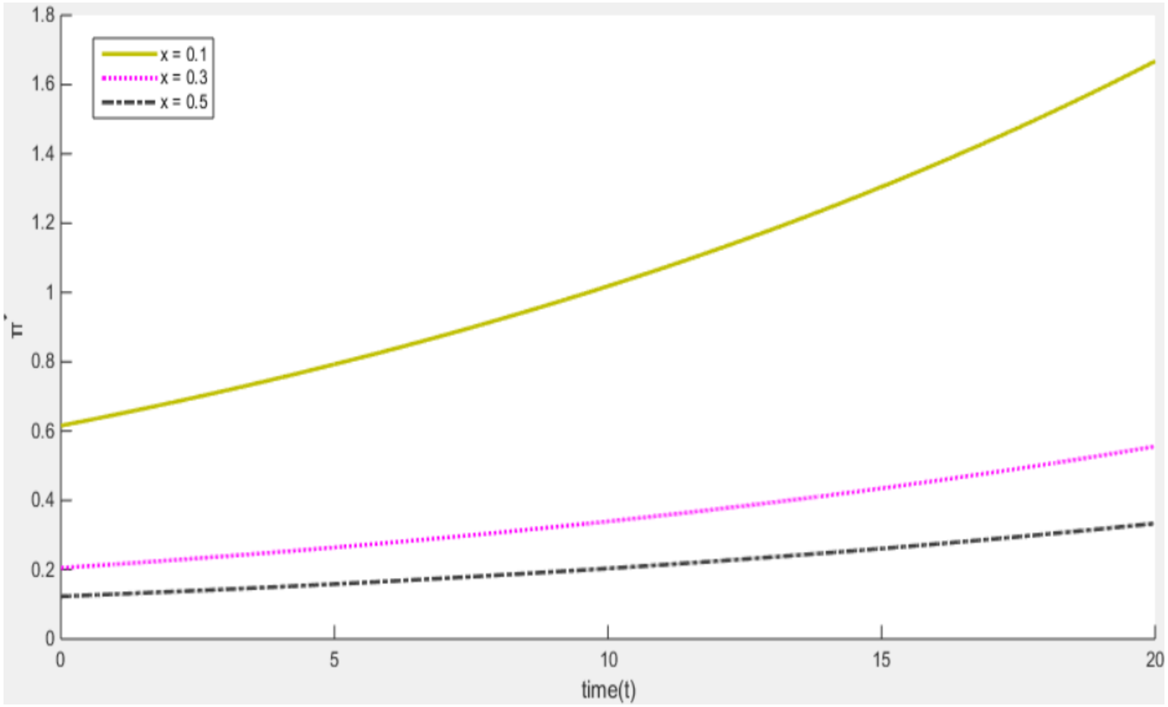

Fig. 3. Evolution of the optimal portfolio distribution $\pi^*$with different initial fund size x

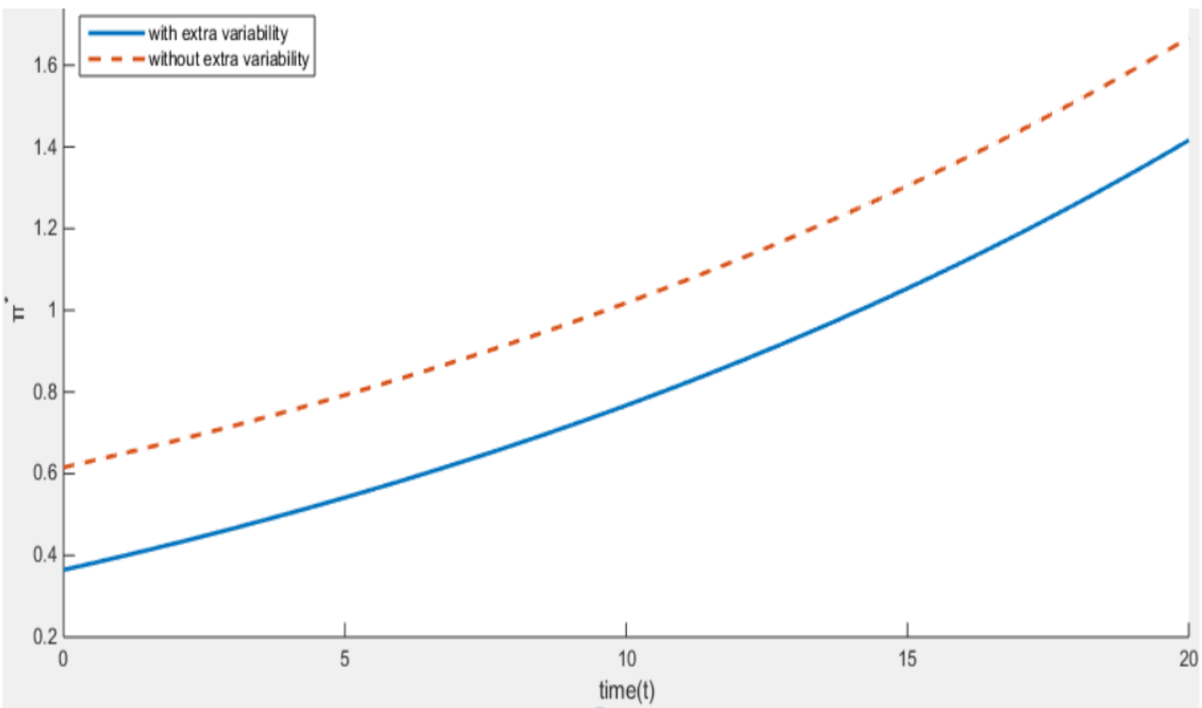

Fig. 4. Evolution of the optimal portfolio distribution $\pi^*$with and without tax

In figure 1, we observed that the optimal portfolio distribution for the risky asset is inversely proportional to the risk aversion coefficient. The consequence of the above is that members with low risk aversion coefficient will invest more in the risky asset while members with high risk aversion coefficient will invest more in the risk free asset to increase their expectations which is in accordance with proposition 4.2, where observed that the expectation is directly dependent on the variance; this means that when members take more risk, their expectation from such investment at the expiration date will be higher compared to members who invest less in risky asset. In figure 2, we observed that the optimal portfolio distribution for the risky asset is inversely proportional to the risk free interest rate offered by the risk free asset. The consequence of the above is that in the case where investment offers high interest rate, members may be attracted to such investment hence we observe that high interest rate for the risk free asset may implies more investment in the risk free asset; this is due to fact that many investors naturally do not like taking much risk but desires good returns .In figure 3, we observed that the optimal portfolio distribution for the risky asset is inversely proportional to the member's initial fund size. The consequence of the above is that if the initial fund size at the time of investment is high, members may decide to reduce the amount of risk to be taken, thereby reducing the proportion of their funds to be invested in risky asset thereby increasing the proportion to be invested in risk free asset and vice versa. Finally, in figure 4, we observed that the optimal portfolio distribution for the risky asset decreases when the invested fund is taxed and increases when it is not taxed. The consequence of this is that, member's get discouraged in investment that is highly tax and encouraged where the tax rate is slightly low.

6. Conclusion

In this work, we modelled and determined the optimal portfolio distribution for DC plan member with refund clause of contributions where the mortality force function was modelled by the Weibull function and the investment in risky asset was subject to a certain proportion of tax. We considered a portfolio consisting of a risk-free asset and a risky asset modeled by the geometric Brownian motion such that the remaining accumulations were equally distributed among the remaining members. More so, the game theoretic approach was used to establish an optimization problem from the extended Hamilton Jacobi Bellman (HJB) equation which is a non-linear partial differential equation (PDE). We used the variable separation method to find closed form solutions of the optimal portfolio distribution and the efficient frontier for the pension fund manager. Some numerical simulations were also presented and used to study how some parameters of the optimal portfolio distribution affect the investment strategy with observations that the optimal portfolio distribution developed by the pension fund manager is inversely proportional to the tax imposed on the risky asset, risk averse coefficient, initial fund size, and risk free interest rate but directly proportional to time.

In conclusion, based on the behavior of the parameters of the optimal portfolio distribution developed, this work will guide pension fund managers on how to manage the contributions of the remaining members of the pension scheme after refunds have been made to the death members' next of kin or the families for optimal profit with minimal risks when considering investments in risky assets in the presence of tax. Therefore, we recommend that government agencies should consult financial analyst before making policies in respect to tax on invested funds.

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