# A New Trivariate Semicopula Using Rüschendorf Method

M.M.E Abd EL-Monsef
Faculty of Science, Tanta University

M.M. Seyam
Faculty of Science, Tanta University

S.M. Elsobky
Faculty of Science, Tanta University

**Abstract**

In this paper we have introduced semicopula function by using Rüschendorf method, semicopula which is related to correlation between one or more random variables and this way is more flexible than traditional correlation approaches and dependency among variables. Every semicopula has density associated with it, which is similar to the probability density of a multivariate distribution. Our purpose is developing a new trivariate semicopula under conditions which is a trivariate cumulative distribution with uniform marginal distribution on the interval [0,1].

In order to choose a random function under specific conditions, we rely on utilizing Rüschendorf method. As a result, we will discuss that in this paper. In this theme we select an arbitrary trivariate function which adopts the Rüschendorf conditions to acquire anew function; which supposed to be a density of copula with dependence parameter. According to the evidence, we have got a semicopula function. Therefore, we can say that a semicopula is a copula function despite of missing increasing property.

**Keywords**

Copula, Semicopula, Rüschendorf Technique, Superharmonic, Subharmonic, concave.

- **Introduction**

Firstly, any random variables are a multivariate copula, which is a m-variate density function whose univariate marginals are uniformly distributed on see Kazi-Tani and Rullière (2019). Whose constituents are spread uniformly on I is distributed according to some copula.

An m-dimensional copula (or m -copula) is a function C from the unit m-cube $[0,1]^m$ to the unit interval [0,1] according to Sklar's Theorem; a function is a copula if and only if the following properties hold:

(1) $C(1,...,1, a_n,1,...,1)$ = an for every $n \leq m$ and all an in [0,1] ;

(2) $C(a_1,..., a_m) = 0$ if $a_n = 0$ for any $n \leq m$; (Property 2 says that the joint probability of all outcomes is zero if the marginal probability of any outcome is zero).

(3) C is becoming more m-increasing. (Any m-dimensional interval's C-volume is non-negative, according to Property 3).

Property 2 and 3 are multivariate cdfs' general properties.

Copula modelling has got puplicity popularity in recent years because it can model multivariate distributions without having to worry about the variables' dependence structure; where is a copula parameter called the dependence parameter, which measures reliance among the marginals.

There are numerous methods for creating multivariate distributions with uniform marginals. In the Rüschendorf technique (2014). An arbitrary bivariate function that is integrable on $[0,1]^2$ is chosen, but we employed an arbitrary trivariate function that is also integrable on $[0,1]^3$. Pauline and Mahendran are two authors who you should look into (2014). Given the use of the Rüschendorf approach and the flexibility of being able to select arbitrary functions to creat new copulas. we decided to focus on building a semicopula, utilizing the Rüschendorf method. We select an acceptable arbitrary trivariate function. This arbitrary trivariate function was chosen as:

$$f(x, y, z) = (1 - 2z)\left(\sqrt{xy}\right)^{-1} \qquad (1)$$

If $C(u,v,z)$ a trivariate function that maps $[0,1]^3$ to $[0,1]$, is a copula if it holds the following conditions:

- ***Boundary conditions:***

  $$C(u,v,0)=0 \quad ,C(u,0,z)=0, \quad C(0,v,z)=0$$

  $$C(u,1,1)=u \quad ,C(1,1,z)=z \quad C(1,v,1)=v.$$

- ***m-increasing propriety:*** m-increasing propriety is a term that refers to the process.
  Sklar (1959) invented copula functions, copulas can link multivariate marginal functions, that are uniformly distributed on $[0, 1]^m$; to form trivariate functions. Respectively $f_x$, $f_y$ ,$f_z$, and c are the density functions of $F_x$, $F_y$ ,$F_z$, and C.

- **Rüschendorf Method for Building a New Trivariate SemiCopula**

  At first we try to construct a copula and see its conditions. Assume that $f^1(x, y, z)$ has integral zero on the $[0,1]^3$ and that its three marginals are all equal to zero as :

$$\int_0^1 \int_0^1 \int_0^1 f^1(x,y,z)\, dxdydz = 0$$

and

$$\int_0^1 \int_0^1 f^1(x,y,z)\, dxdy = \int_0^1 \int_0^1 f^1(x,y,z)\, dydz = \int_0^1 \int_0^1 f^1(x,y,z)\, dxdz = 0$$

We suppose the density of a copula is $1 + f^1(x,y,z)$. There is, a need that to proof the expression be non-negative, then we can obtain a constant $\theta$, Such that :

$1 + f^1(x,y,z)$is bounded. The following steps can be used to create the function

$1 + f^1(x,y,z)$:

**the initial step to get:**

$$f(x) = \int_0^1 \int_0^1 f(x,y,z) dydz,$$

$$f(y) = \int_0^1 \int_0^1 f(x,y,z) dxdz,$$

$$f(x) = \int_0^1 \int_0^1 f(x,y,z) dxdy$$

$$A = \int_0^1 \int_0^1 \int_0^1 f(x,y,z) dxdy\, dz$$

**The second stage is to obtain:**

$$f^1(x,y,z) = f(x,y,z) - f(x) - f(y) - f(z) + A$$

Our new trivariate copula is derived from the previous phase. We choose a random function is given as:

$$f(x,y,z) = (1-2z)\left(\sqrt{xy}\right)^{-1},$$

where $x$, $y$, and $z$ are all in the range $[0,1]^3$. Following that, we compute $f(x), f(y), f(z)$and $A$ as follows:

$$f(x) = \int_0^1 \int_0^1 (1-2z)\left(\sqrt{xy}\right)^{-1} dydz = 0,$$

$$f(y) = \int_0^1 \int_0^1 (1-2z)\left(\sqrt{xy}\right)^{-1} dxdz = 0,$$

$$f(z) = \int_0^1 \int_0^1 (1-2z)\left(\sqrt{xy}\right)^{-1} dxdy = (4-8z)$$

and

$$A = \int_0^1 \int_0^1 \int_0^1 (1-2z)\left(\sqrt{xy}\right)^{-1} dxdy = 0.$$

We are going to build.

$$f^1(x,y,z) = f(x,y,z) - f(x) - f(y) - f(z) + A$$

because, in our situation,

$$f^1(x,y,z) = (1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z) \qquad (2)$$

It is critical to verify that the double integral of function (2) on the unit square equals zero, which can be demonstrated in one of two ways.

$$\int_0^1 \int_0^1 f^1(x,y,z)dxdy = \int_0^1 \int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dxdy = 0$$

$$\int_0^1 \int_0^1 f^1(x,y,z)dydz = \int_0^1 \int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dydz = 0$$

and

$$\int_0^1 \int_0^1 f^1(x,y,z)dxdz = \int_0^1 \int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dxdz = 0$$

Obviously, the integer

$$\int_0^1 \int_0^1 \int_0^1 f^1(x,y,z)dxdydz = \int_0^1 \int_0^1 \int_0^1 \left\{(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right\} dxdydz = 0$$

The density of copula is $g(x,y,z)$ as shown in the following stages.

$$g(x,y,z) = 1 + \theta\left[(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right], \theta \in [-1,1]$$

**Proposition 1.1** is a phrase that can be written in a variety of ways.

$$g(x,y,z) = 1 + \theta\left[(1-2z)\left(\sqrt{xy}\right)^{-1} - (4-8z)\right], \theta \in [-1,1] \qquad (3)$$

is the density of a copula.

**Proof**

To be $g(x,y,z)$ proven as a density of copula it should be $g(x,y,z) \geq 0$ by using the first, second, and third partial derivative of $g(x,y,z)$ , and it is given by :

$$g_x(x,y,z) = -\frac{y(1-2z)}{2(xy)^{3/2}}$$

$$g_{xx}(x,y,z) = \frac{3y^2(1-2z)}{4(xy)^{5/2}}$$

$$g_{xxx}(x,y,z) = -\frac{15y^3(1-2z)}{8(xy)^{7/2}}$$

$$g_y(x,y,z) = -\frac{x(1-2z)}{2(xy)^{3/2}}$$

$$g_{yy}(x,y,z) = \frac{3x^2(1-2z)}{4(xy)^{5/2}}$$

$$g_{yyy}(x,y,z) = -\frac{15x^3(1-2z)}{8(xy)^{7/2}}$$

$$g_z(x,y,z) = 8 - \frac{2}{\sqrt{xy}}$$

$$g_{zz}(x,y,z) = 0 \text{ and } g_{zzz}(x,y,z) = 0$$

We have found that $g(x,y,z)$ did not benefit our purpose so it is not a copula density function. To make sure $g(x,y,z)$ extensively we would suppose that it is a copula density and verify it's principles.

This paper is organized as follows :

- **The Copula of the Future**

To build the a new copula, we must first extract the triple integral of $g(x,y,z)$, which is :

$$C(u,v,w) = \int_0^u \int_0^v \int_0^w g(x,y,z)\, dxdydz$$

$$C(u,v,w) = v\left(-4(-1+v)\sqrt{uw}\theta + u\left(w + 4(-1+v)\right)w\theta\right) \quad (4)$$

**Properties of Copula**

- **1st step:** Boundary conditions

$$C(0,v,w) = 0, \quad C(u,0,w) = 0 \text{ and } C(u,v,0) = 0$$

, as well as

$$C(u,1,1) = u, \quad C(1,v,1) = v \quad \text{and } C(1,1,z) = z$$

- **Step two**: 3-increasing property

To show that $C(u,v,w)$ in the cubic $C(u,v,w)$ is increasing it should satisfy the 3-increasing property, see Ignazzi and Durante (2021). In reality, the mixed third partial derivative of $C$ may be computed, because the mixed third partial derivative of $C$ is independent of the order of the derivatives and is given by

$$\frac{\partial^3 C(u,v,w)}{\partial u \partial v \partial w} = 1 + \frac{(-1+2v)(-1+4\sqrt{uw}\theta)}{\sqrt{uw}} \quad (5)$$

We notice that C is unknown if u=0, v=0, or both of them equal zero at the same time. Our function C is called an increasing or a decreasing function, it depends on the dependence parameter $\theta$ . C will be an increasing function if , $\theta \in [-1,0)$, and it is called copula. If $\theta \in (0,1]$ ,C is a decreasing function( it is not satisfying the 3-increasing property), so it is called semicopula function S.

Semicopula function S generalises both the concept of copula and the concept of triangular norm see Klement et al., semicopula function S has all the principle properties of a copula except increasing property see F. Durante (2006), semicopula function S has all the conditions of copula except increasing property see F. Durante (2006).

- **Semicopulas**

**- Superharmonic and Subharmonic Semicopulas Functions**

Let $S$ be a semicopula with second-order partial derivatives that are continuous on$(0,1)^2$. If $S$ satisfies Laplace's equation in $I^2$:

$$\nabla^2 S(u) = \frac{\partial^2}{\partial u_1^2} S(u,v) + \frac{\partial^2}{\partial u_2^2} S(u) + \ldots + \frac{\partial^2}{\partial v_m^2} S(u) = 0$$

then $S$ is harmonic. Π It is the only harmonic semicopula.

A semicopula S is subharmonic function if $\nabla^m S(u,v) \geq 0$ and superharmonic function if $\nabla^2 S(u,v) \leq 0$. See Axler et al., for additional information on harmonic function theory (2001). It's a simple calculus problem to prove that our semicopula S function is superharmonic since, $\nabla^m S(u,v,w) \leq 0$,and

$$\frac{\partial^2 S(u,v,w)}{\partial u^2} = \frac{(v-1)vw^2\theta}{(uw)^{3/2}}$$

$$\frac{\partial^2 S(u,v,w)}{\partial v^2} = -8\left(-uw + \sqrt{uw}\right)\theta$$

and

$$\frac{\partial^2 S(u,v,w)}{\partial w^2} = uv\left(2 + \frac{(v-1)u}{(uw)^{3/2}}\right)\theta$$

then

$$\frac{\partial^2 S(u,v,w)}{\partial u^2} + \frac{\partial^2 S(u,v,w)}{\partial v^2} + \frac{\partial^2 S(u,v,w)}{\partial w^2} =$$
$$\frac{(v-1)vw^2\theta}{(uw)^{3/2}} + -8\left(-uw + \sqrt{uw}\right)\theta + uv\left(2 + \frac{(v-1)u}{(uw)^{3/2}}\right)\theta \le 0$$

It is clear that

$$\left[\frac{(v-1)v\sqrt{uw}}{u^2} + 2u(v+4w) + \frac{\sqrt{uw}\left((v-1)v - 8w^2\right)}{w^2}\right]\theta \le 0$$

$$\frac{\partial^2 S(u,v,w)}{\partial u^2} + \frac{\partial^2 S(u,v,w)}{\partial v^2} + \frac{\partial^2 S(u,v,w)}{\partial w^2} \le 0$$

so $S(u,v,w) = v\left(-4(-1+v)\sqrt{uw}\theta + u\left(w + 4(-1+v)\right)w\theta\right)$,is a superhar-monic semicopula .

**If Function S is Semicopula, The Following Sentences are Equivalent:**

1. S has a concave shape.
2. For all and any, S is super-homogeneous, $\forall u, v, w \in [0,1]$and $\theta \in [0,1]$
3. S is an idempotent variable.
4. M is concave, therefore S=M.

All semicopulas class S is a convex and log-convex subclass of all copulas class S. Furthermore, S is a complete lattice since it is closed under suprema and infima and compact with regard to the topology of uniform convergence.

**For a semicopula, the following qualities are equivalent:**

1- Semicopula S is positively homogeneous.
2- In both variables, S has linear sections.
3- Π = S.

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