Original Research Article

Gerber-Shiu function in a discrete-time risk model with dividend strategy

ABSTRACT

In this paper, a discrete-time risk model with dividend strategy and a general premium rate is considered. Under such a strategy, once the insurer's surplus hits a constant dividend barrier b, dividends are paid off to shareholders at α instantly. Using the roots of a generalization of Lundberg's fundamental equation and the general theory on difference equations, two difference equations for the Gerber-Shiu discounted penalty function are derived and solved. The analytic results obtained are utilized to derive the probability of ultimate ruin when the claim sizes is a mixture of two geometric distributions. Numerical examples are also given to illustrate the applicability of the results obtained.

Keywords: Compound binomial model; two-step premium; defective renewal equation; Gerber-Shiu discounted penalty function; dividend strategy.

1. INTRODUCTION

Risk theory has a long development time, Lundberg [1] and Gramer [2] established the connection of risk theory. The compound binomial model that was first proposed by Gerber [3] have received considerable attention. For instance, Shiu [4], Willmot [5] and Dickson [6] have analyzed the compound binomial model. Markov chain is understood to be a stochastic process in discrete time possessing a certain conditional independence property. The state space may be finite, countably infinite or even more general. Cossette et al. [7] consider the so-called compound Markov binomial model which introduces dependency between claim occurrences. For an generalization of the classical risk model see Landriault [8].Furthermore, in the discrete time risk model, the issue related to dividend is also widely considered.

Dividend strategies for insurance risk models were first proposed by DeFinetti [9] to reflect more realistically the surplus cash flows in an insurance portfolio. Because of the certainty of ruin for a risk model with a constant dividend barrier, the calculation of the Gerber-Shiu discounted penalty function is a major problem of interest in the context. Among the class of discrete-time risk models, Tan and Yang [10] derived a recursive algorithm to compute a particular class of Gerber-Shiu penalty functions in the framework of the compound binomial model with randomized dividend payments. Landriault [11] then generalized Tan and Yang's model to consider the compound binomial model with a multi-threshold dividend structure and randomized dividend payments. In the discrete time risk model, He and Yang [12] considered that dividends are paid randomly to shareholders and policyholders in the framework of the compound binomial model. In the framework of a discrete semi-Markov risk model, a randomized dividend policy is studied by Yuen et al. [13]. Zhang and Liu [14] consider a discrete-time risk model with a mathematically tractable dependence structure between interclaim times and claim sizes in the presence of an impulsive dividend strategy.

The paper is structured as follows: a brief description of the discrete-time model and the introduction of the Gerber-Shiu discounted penalty function are considered in Section 2. In section 3, we obtain and solve a non-homogeneous difference equation satisfied by the the Gerber-Shiu discounted penalty function $m(u;b)$. Closed-form solutions fr $m_b(u)$ are obtained when the claim sizes is a mixture of two geometric distributions and corresponding numerical examples are also provided in Section 4.

2. THE MODEL

Throughout, denote by N the set of natural numbers and $N^+ = N/\{0\}$. In the compound binomial model, the claim number process $\{N_k, k \in N\}$ is assumed to be a renewal process with independent and identically distributed (i.i.d.) interclaim times $\{W_j, j \in N^+\}$ having probability mass function (p.m.f.) $f_W(j) = q(1-q)^{j-1}$ for $l \in N^+$. Equivalently, the probability of having a claim is $p(0<p<1)$ and the probability of no claim is $q = 1-p$. The individual claim amount r.v.'s(random variables) $\{X_j, j \in N^+\}$ form a sequence of strictly positive, integer-valued and i.i.d. r.v.'s. We suppose that the r.v.'s $\{X_j, j \in N^+\}$ are distributed as a generic r.v. X with p.m.f. $f(x)$, probability generating function (p.g.f.) $\tilde{f}(x)$. Moreover, it is assumed that the r.v.'s $W_1, W_2, \ldots$ and $X_1, X_2, \ldots$ are mutually independent. Let $S_k = \sum_{i=1}^{N_k} X_i$ be the total amount of settled claims up the end of the kth time period with $S_0 = 0$.

Suppose that premiums are received at the beginning of each time period, and claims are paid out at the end of each time period. Denote $u \geq 0$ to be the initial surplus, $b > 0$ the constant barrier level, and $c_1 > 0$ the annual premium . Under such a strategy, let $\alpha(0 < \alpha \leq c_1)$ be the annual dividend rate, once the insurer's surplus at time k hits or exceeds a constant dividend barrier b, dividends are paid off to shareholders at α instantly. In this case, the net premium after dividend payments is $c_2 = c_1 - \alpha \geq 0$. The corresponding surplus of the insurer at the end of the kth time period is $U_b(k)$ for $k = 1, 2, \cdots$ can be described as

$$U_b(k) = \begin{cases} U_b(k-1) + c_1 - \eta_k X_k, & U_b(k-1) \leq b \\ U_b(k-1) + c_2 - \eta_k X_k, & U_b(k-1) > b \end{cases}, \tag{1}$$

where $U_b(0) = u$, $\{\eta_k, k \in N\}$ is an independent and identically distributed Bernoulli sequence, we denote by $\eta_k = 1$ the event of having a claim at the time k and denote by $\eta_k = 0$ the event that no claim at the time k. We assume that
$P(\eta_i = 1) = p$ and $P(\eta_i = 0) = 1 - p = q$ and surplus process $U_b(k)$ has a positive drift by letting $c_2 > pE[X]$ (known as the positive security loading condition in ruin theory).

Define $\tau_b = \min\{k : U_b(k) < 0\}$ to be the time of ultimate ruin. Let v be a constant annual discount rate for each period. When ruin occurs, $U_b(\tau - 1)$ is the surplus one period prior to ruin and $|U_b(\tau)|$ is the deficit at ruin. For $v \in (0,1]$, the well-known Gerber-Shiu discounted penalty function is then defined as

$$m(u;b) = E\left\{v^{\tau_b} \omega(U_{\tau_b - 1}, |U_{\tau_b}|) I_{\{\tau_b < \infty\}} \middle| U_0 = u\right\}, \tag{2}$$

where $\omega : N \times N^+ \to R$ is a penalty function and $I_{\{Q\}}$ is the indicator function of an event Q. Also, we consider some special cases of (2) with successively simplified the penalty functions. If $\omega(n_1,n_2)=1$ for $(n_1,n_2)\in N\times N^+$, we get the generating function of the time to ruin, i.e.

$$m_b(u)=E\left\{v^{\tau_b} I_{\{\tau_b<\infty\}} \middle| U_0=u\right\}.$$

3. THE GERBER-SHIU DISCOUNTED PENALTY FUNCTION

In this section, we derive two difference equations for the Gerber-Shiu discounted penalty function: one for the initial surplus below the barrier level b and the other for the initial surplus above the barrier level b. Clearly, the Gerber-Shiu discounted penalty function $m(u;b)$ behaves differently, depending on whether its initial surplus u is below or above the barrier level b. Hence, we write

$$m(u;b)=\begin{cases} m_1(u), & 0\le u<b \\ m_2(u), & u\ge b \end{cases}$$

In order to identify the structural form of the solution for the Gerber-Shiu discounted penalty function, three cases will be considered separately.

3.1 For initial surpluses less than the barrier b

In the first scenario, the initial surplus below the barrier b, for $u=0,1,\dots,b-c_1-1$, we have

$$\begin{aligned} m_1(u) &= vqm_1(u+c_1)+vp\sum_{j=1}^{u+c_1} m_1(u+c_1-j)f(j)+vp\sum_{j=u+c_1+1}^{\infty}\omega(u+c_1;j-u-c_1)f(j) \\ &= vqm_1(u+c_1)+vp(m_1*f)(u+c_1)+\gamma_1(u), \end{aligned} \tag{3}$$

where

$$\gamma_1(u)=vp\sum_{j=u+c_1+1}^{\infty}\omega(u+c_1;j-u-c_1)f(j).$$

and m_1*f holds for the convolution product of m_1 and f.

To state that (3) is a non-homogeneous difference equation of order c_1, we re-express (3) according to the forward difference operator Δ and its property (see Chapter 2 of Kelly & Peterson [15]),

$$m(u+c)=\sum_{j=0}^{c}\binom{c}{j}\Delta^j m(u), \tag{4}$$

substituting (4) into (3) shows

$$m_1(u)=vq\sum_{j=0}^{c_1}\binom{c_1}{j}\Delta^j m_1(u)+vp\sum_{j=0}^{c_1}\binom{c_1}{j}\Delta^j (m_1*f)(u)+\gamma_1(u), \tag{5}$$

for $u=0,1,\dots,b-c_1-1$, (5) can be simplified to

$$\sum_{j=0}^{c_1} a_{1,j}\Delta^j m_1(u)=\sum_{j=0}^{c_1} b_{1,j}\Delta^j (m_1*f)(u)+\gamma_1(u), \tag{6}$$

where

$$a_{1,j}=I_{\{j=0\}}-vq\binom{c_1}{j},\ b_{1,j}=vp\binom{c_1}{j}.$$

and $A_1(z)$, $B_1(z)$ are polynomials (in z) defined as

$$A_1(z)=\sum_{j=0}^{c_1} a_{1,j} z^j, B_1(z)=\sum_{j=0}^{c_1} b_{1,j} z^j .$$

becomes

$$A_1(\Delta)m_1(u)=B_1(\Delta)(m_1 * f)(u)+\gamma_1(u), \qquad u=0,1,2,...,b-c_1-1 , \tag{7}$$

We know from (7) that $m_1(u)$ satisfies a non-homogeneous difference equation of order c_1. From the general theory on difference equations, every solution to a c_1-th order difference equation can be expressed as a particular solution to this difference equation plus a linear combination of c_1 linearly independent solutions to the associated homogeneous difference equation (cf. Elaydi [16], Theorem 2.30). Therefore, for $u=0,1,\ldots,b-1$, the Gerber-Shiu discounted penalty function can be expressed as

$$m_1(u)=\varphi_1(u)+\sum_{j=0}^{c_1-1}\alpha_{1,j} y_{1,j}(u), \quad u=0,1,2,...,b-1 . \tag{8}$$

where $\{y_{1,j}(u)\}_{u=0}^{\infty}$ $(j=0,1,...,c_1-1)$ are c_1 fundamental solutions to the following homogeneous difference equation

$$A_1(\Delta)y_1(u)=B_1(\Delta)(y_1 * f)(u) \qquad u\ge 0 . \tag{9}$$

$\{\varphi_1(u)\}_{u=0}^{\infty}$ is a particular solution to

$$A_1(\Delta)\varphi_1(u)=B_1(\Delta)(\varphi_1 * f)(u)+\gamma_1(u) \qquad u\ge 0 . \tag{10}$$

Combining (3) and (9), we get

$$y_1(u)=vqy_1(u+c_1)+vp(y_1 * f)(u+c_1) . \tag{11}$$

Multiplying (11) by z^{u+c_1} and then summing over u from 0 to ∞ lead to

$$\sum_{u=0}^{\infty} z^{u+c_1} y_1(u)=vq\sum_{u=0}^{\infty} z^{u+c_1} y_1(u+c_1)+vp\sum_{u=0}^{\infty} z^{u+c_1}(y_1 * f)(u+c_1) , \tag{12}$$

routine calculations lead to

$$z^{c_1}\tilde{y}_1(z)=vq\left[\tilde{y}_1(z)-\sum_{u=0}^{c_1-1} z^u y_1(u)\right]+vp\left\{\tilde{y}_1(z)\tilde{f}(z)-\sum_{u=0}^{c_1-1} z^u (y_1 * f)(u)\right\}.$$

After some algebra, one could see that (12) can be written as

$$\tilde{y}_1(z)=\frac{-v\left\{q\sum_{u=0}^{c_1-1} z^u y_1(u)+p\sum_{u=0}^{c_1-1} z^u (y_1 * f)(u)\right\}}{z^{c_1}-vq-vp\tilde{f}(z)} . \tag{13}$$

By choosing $y_{1,j}(u)=I_{\{j=u\}}$ for $j,u\in\{0,1,...,c_1-1\}$. According to (13) , the generating function associated to the fundamental solution $\{y_{1,j}(u)\}_{u=o}^{\infty}$ is

$$\tilde{y}_{1,j}(z)=\frac{-v\left\{qz^j+p\sum_{u=j+1}^{c_1-1} z^u f(u-j)\right\}}{z^{c_1}-vq-vp\tilde{f}(z)}=\frac{-R_{1,j}(z)}{\tilde{h}_{1,1}(z)-\tilde{h}_{1,2}(z)} \qquad u\ge 0, \tag{14}$$

where

$$\tilde{h}_{1,1}(z)=z^{c_1},\ \tilde{h}_{1,2}(z)=vq+vp\tilde{f}(z), R_{1,j}(z)=v\left\{qz^j+p\sum_{u=j+1}^{c_1-1} z^u f(u-j)\right\}.$$

Lemma 3.1: When $v\in(0,1)$, the denominator in (14) has exactly c_1 zeros, say $\{z_i\}_{i=1}^{c_1}$ inside the unit circle $C=\{z:|z|=1\}$.

Lemma 3.2: When $v=1$, the denominator in (14) has exactly c_1-1 zeros, say $\{z_i\}_{i=1}^{c_1-1}$ inside the unit circle $C=\{z:|z|=1\}$ and another trivial root $z_{c_1}=1$.

For the rest of the paper, we assume that all $\{z_i\}_{i=1}^{c_1}$ are distinct, since the analysis of the multiple roots of Lundberg's generalized equation leads to tedious derivations.

Let $\pi_i(z)=\prod_{j=1}^{c_i}(z-z_j)$ and $\pi_i^{'}(z_k)=\prod_{l=1,l\neq k}^{c_i}(z_k-z_l)$, from Liu and Bao [17], we have

$$\frac{\widetilde{h}_{1,1}(z)-\widetilde{h}_{1,2}(z)}{\pi_1(z)}=1-vpT_zT_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1)\,, \tag{15}$$

where T_z is an operator (see Li [18]) defined as

$$T_z y(c)=\sum_{u=0}^{\infty}z^u y(u+c)=\sum_{u=c}^{\infty}z^{u-c}y(u).$$

(14) can be rewrote as

$$\widetilde{y}_{1,j}(z)=\frac{\dfrac{-R_{1,j}(z)}{\pi_1(z)}}{\dfrac{\widetilde{h}_{1,1}(z)-\widetilde{h}_{1,2}(z)}{\pi_1(z)}}. \tag{16}$$

Regarding the numerator in (16), partial fractions yield the equivalent representation

$$\frac{-R_{1,j}(z)}{\pi_1(z)}=\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\cdot\frac{1}{z_k-z}. \tag{17}$$

By inserting (15) and (17) into (16), we obtain

$$\widetilde{y}_{1,j}(z)=v\,p\widetilde{y}_{1,j}(z)T_zT_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1)+\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi^{'}(z_k)}\frac{1}{z_k-z}. \tag{18}$$

Theorem 3.1: For $j=0,1,\cdots,c_1-1$, $y_{1,\ j}(u)$ satisfies the following defective renewal equation

$$y_{1,j}(u)=\varsigma_1\sum_{n=0}^{u}y_{1,j}(u-n)\chi_1(n)+\zeta_1(u), \tag{19}$$

where

$$\varsigma_1=vpT_1T_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1),\ \ \chi_1(n)=\frac{T_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1+n)}{T_1T_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1)},\ \ \zeta_1(u)=\sum_{k=1}^{c_1}\frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)}\left(\frac{1}{z_k}\right)^{u+1}.$$

Now, we turn our attention to the calculation of the particular solutions $\{\varphi_1(u)\}_{u=0}^{\infty}$, combining (3) and (10), $\{\varphi_1(u)\}_{u=0}^{\infty}$ satisfies

$$\varphi_1(u)=vq\varphi_1(u+c_1)+vp(\varphi_1*f)(u+c_1)+\gamma_1(u). \tag{20}$$

We use a solution procedure analogous to the fundamental solutions, we get

$$\begin{aligned}\widetilde{\varphi}_1(z)&=\frac{z^{c_1}\widetilde{\gamma}_1(z)-v\left\{q\sum_{u=0}^{c_1-1}z^u\varphi_1(u)+p\sum_{u=0}^{c_1-1}z^u(\varphi_1*f)(u)\right\}}{z^{c_1}-vq-vp\widetilde{f}(z)}\\&=\frac{z^{c_1}T_z\gamma_1(0)-Q_{1,j}(z)}{z^{c_1}-vq-vp\widetilde{f}(z)}.\end{aligned} \tag{21}$$

where $Q_{1,j}(z)=v\left\{q\sum_{u=0}^{c_1-1}z^u\varphi_1(u)+p\sum_{u=0}^{c_1-1}z^u(\varphi_1*f)(u)\right\}$ is a polynomial of degree c_1-1 (or less) in z. It is known from (35) in Liu and Zhang [14] that

$$\frac{z^{c_1}T_z\gamma_1(0)-Q_{1,j}(z)}{\pi_1(z)}=T_zT_{z_{c_1}}\dots T_{z_2}T_{z_1}\gamma_1(0). \qquad (22)$$

By substituting (15) and (22) into (21), we get

$$\widetilde{\varphi}_1(z)=\frac{T_zT_{z_{c_1}}\dots T_{z_2}T_{z_1}\gamma_1(0)}{1-vpT_zT_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1)}=\frac{\widetilde{\ }_1(z)}{1-vpT_zT_{z_{c_1}}\dots T_{z_2}T_{z_1}f(c_1)}. \qquad (23)$$

Theorem 3.2: For $u\in N$, it holds that

$$\varphi_1(u)=\varsigma_1\sum_{n=0}^{u}\varphi_1(u-n)\chi_1(n)+\ _1(u), \qquad (24)$$

where $\ _1(u)=T_{z_{c_1}}\dots T_{z_2}T_{z_1}\gamma_1(u)$.

In the second scenario, for $u=b-c_1,\cdots,b-1$,

$$m_1(u)=vqm_2(u+c_1)+vp(m*f)(u+c_1)+\gamma_1(u). \qquad (25)$$

3.2 For initial surpluses equal to or more than the barrier b

The last scenario, for $u\geq b$,

$$m_2(u)=vqm_2(u+c_2)+vp(m*f)(u+c_2)+\gamma_2(u), \qquad (26)$$

where $\gamma_2(u)=vp\sum_{j=u+c_2+1}^{\infty}\omega(u+c_2;j-u-c_2)f(j)$.

The structural form (8) for $m_1(u)$ is expressed in terms of the $\alpha_{1,j}$, and also depends on $m_2(u)$ in (26). In order to drive the solutions of $m(u)$, shifting the argument u in (26) by b units, for $u\geq 0$ (26) can be rewritten as

$$m_2(u+b)=vqm_2(u+b+c_2)+vp\sum_{j=b}^{u+b+c_2}m_2(j)f(u+b+c_2-j)$$

$$+vp\sum_{j=0}^{b-1}m_1(j)f(u+b+c_2-j)+\gamma_2(u+b),$$

Let $\xi_2(u)\equiv m_2(u+b)$, (26) becomes

$$\xi_2(u)=vq\xi_2(u+c_2)+vp(\xi_2*f)(u+c_2)+\eta(u), \qquad (27)$$

where $\eta(u)=vp\sum_{j=0}^{b-1}m_1(j)f(u+b+c_2-j)+\gamma_2(u+b)$.

We use a solution procedure analogous to that of Section 3.1, $\xi_2(u)$ satisfies

$$A_2(\Delta)\xi_2(u)=B_2(\Delta)(\xi_2*f)(u)+\eta(u) \qquad u\geq 0, \qquad (28)$$

where

$$A_2(z)=\sum_{j=0}^{c_2}a_{2,j}z^j, B_2(z)=\sum_{j=0}^{c_2}b_{2,j}z^j, a_{2,j}=I_{\{j=0\}}-vq\binom{c_2}{j}, b_{2,j}=vp\binom{c_2}{j}.$$

From the general theory on difference equations, can be expressed as

$$m_2(u+b)\equiv\xi_2(u)=\varphi_2(u) \quad u=0,1,\dots$$

where $\{\varphi_2(u)\}_{u=0}^{\infty}$ satisfies

$$A_2(\Delta)\varphi_2(u)=B_2(\Delta)(\varphi_2*f)(u)+\eta(u) \qquad u\geq 0. \qquad (29)$$

Some solution procedures are omitted, similar discussions can be find in Section 3.1.
Generating function of the particular solution $\varphi_2(u)$ is

$$\tilde{\varphi}_2(z) = \frac{z^{c_2}\tilde{\eta}(z) - v\left\{q\sum_{u=0}^{c_2-1} z^u\varphi_2(u) + p\sum_{u=0}^{c_2-1} z^u\left(\varphi_2 * f\right)(u)\right\}}{z^{c_2} - vq - vp\tilde{f}(z)} \tag{30}$$
$$= \frac{z^{c_2}T_z\eta(0) - R_{2,j}(z)}{\tilde{h}_{2,1}(z) - \tilde{h}_{2,2}(z)},$$

where

$$\tilde{h}_{2,1}(z) = z^{c_2},\ \tilde{h}_{2,2}(z) = vq + vp\tilde{f}(z), R_{2,j}(z) = v\left\{q\sum_{u=0}^{c_2-1} z^u\varphi_2(u) + p\sum_{u=0}^{c_2-1} z^u\left(\varphi_2 * f\right)(u)\right\}.$$

Theorem 3.3: For $u \in N$, it holds that

$$\varphi_2(u) = \varsigma_2\sum_{n=0}^{u}\varphi_2(u-n)\chi_2(n) + \delta_2(u), \tag{31}$$

where

$$\varsigma_2 = vpT_1T_{z_{c_2}}\ldots T_{z_2}T_{z_1}f(c_2),\ \ \chi_2(n) = \frac{T_{z_{c_2}}\ldots T_{z_2}T_{z_1}f(c_2+n)}{T_1T_{z_{c_2}}\ldots T_{z_2}T_{z_1}f(c_2)},\ \ \delta_2(u) = T_{z_{c_2}}\ldots T_{z_2}T_{z_1}\eta(u).$$

So for $u \geq b$,

$$m_2(u) \equiv \xi_2(u-b) = \varphi_2(u-b), \tag{32}$$

4. NUMERICAL RESULTS

It is well-known that $f(x) = (1-\rho)\rho^{x-1}$ is a geometric distribution. In this section, it is further assumed that $f(x)$ is a mixture of two geometric distributions with

$$f(x) = \theta(1-\rho_1)\rho_1^{x-1} + (1-\theta)(1-\rho_2)\rho_2^{x-1},$$

Obviously, probability generating function is $\tilde{f}(x) = \frac{z\left[(1-\rho_1)(1-\rho_2) + \beta(1-z)\right]}{(1-\rho_1 z)(1-\rho_2 z)}$ where $\beta = \theta\rho_2(1-\rho_1) + (1-\theta)\rho_1(1-\rho_2)$, and mean is $\mu = \frac{\theta}{1-\rho_1} + \frac{1-\theta}{1-\rho_2}$, we rewrite (16) as

$$\tilde{y}_{1,j}(z) = \frac{-R_{1,j}(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)}, \tag{33}$$

where

$$\Lambda_i(z) = z^{c_i}(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp\left[z(1-\rho_1)(1-\rho_2) + \beta(1-z)\right]$$

Since $\Lambda_i(z)$ is a polynomial of degree $c_i + 1$, with leading coefficient $\rho_1\rho_2$, it can be expressed as

$$\Lambda_i(z) = \rho_1\rho_2\pi_i(z)\prod_{j=1}^{\hbar}(z-\xi_j),$$

where ξ_i are solutions of $\Lambda_i(z)$ on the complex plane. It is notable that ξ_i have a module larger than 1,from performing partial fraction, we have

$$\frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_i(z)} = \frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\rho_1\rho_2\pi_i(z)\prod_{j=1}^{\hbar}(z-\xi_j)} = 1 + \sum_{i=1}^{\hbar}\frac{\omega_i}{\xi_i - z}, \tag{34}$$

where

$$\omega_i = \frac{\prod_{k=1}^{\hbar}\left(\rho_k^{-1} - \xi_i\right)}{\prod_{k=1,k\neq i}^{\hbar}\left(\xi_k - \xi_i\right)}.$$

For $i = 1$, substituting (34) into (16) shows

$$\tilde{y}_{1,j}(z) = \frac{(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_1(z)} \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)} \frac{\pi_1(z)}{z_k - z} = \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)} \left(1 + \sum_{i=1}^{\hbar} \frac{\omega_i}{\xi_i - z}\right) \frac{1}{z_k - z}. \quad (35)$$

Upon inversion, we obtain from (35) that

$$\begin{aligned} y_{1,j}(u) &= \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)} \left[z_k^{-(u+1)} + \sum_{i=1}^{\hbar} \omega_i \sum_{l=0}^{u} \xi_i^{-(u+1-l)} z_k^{-(l+1)} \right] \\ &= \sum_{k=1}^{c_1} \frac{R_{1,j}(z_k)}{\pi_1^{'}(z_k)} \left(1 - \sum_{i=1}^{\hbar} \frac{\omega_i}{z_k - \xi_i}\right) z_k^{-(u+1)} + \sum_{k=1}^{c_1} \frac{R_{1.j}(z_k)}{\pi_1^{'}(z_k)} \sum_{i=1}^{\hbar} \frac{\omega_i}{z_k - \xi_i} \xi_i^{-(u+1)}, \end{aligned} \quad (36)$$

Use the same method, we obtain from (30) and (34) that,

$$\tilde{\varphi}_i(z) = \frac{\pi_i(z)(1-\rho_1 z)(1-\rho_2 z)}{\Lambda_i(z)} \tilde{\delta}_i(z) = \left(1 + \sum_{j=1}^{\hbar} \frac{\omega_j}{\xi_j - z}\right) \tilde{\delta}_i(z), \quad (37)$$

Upon the inversion of the generating functions, one obtains from (37) that

$$\varphi_i(u) = \;{}_i(u) + \sum_{j=1}^{\hbar} \omega_j \sum_{l=0}^{u} \xi_j^{-(u+1-l)} \delta_i(l). \quad (38)$$

Example: Suppose $c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, v = 0.95, \rho_1 = 0.3, \rho_2 = 0.6$, from (33)

$\Lambda_1(z) = z^2(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp\left[z(1-\rho_1)(1-\rho_2) + \beta(1-z)\right]$,

$\Lambda_2(z) = z(1-\rho_1 z)(1-\rho_2 z) - vq(1-\rho_1 z)(1-\rho_2 z) - vp\left[z(1-\rho_1)(1-\rho_2) + \beta(1-z)\right]$.

and the relatively safety loading condition $c_2 - p\mu > 0$ holds for all $\theta \in (0,1)$. Hence, θ is chosen to be $0.1, 0.3, 0.5, 0.7, 0.9$, respectively. By solving Lundberg's equation $\Lambda_1(z) = 0$, we obtain the values of z_i's and ξ_j's, see Table 1. By solving Lundberg's equation $\Lambda_2(z) = 0$, we obtain the values of z_1's and ξ_j's, see Table 2.

Table 1: Numerical results of z_i's and ξ_j's, for $c_1 = 2$

θ	z_1	z_2	ξ_1	ξ_2
0.1	-0.8738391540	0.9674521058	1.5385860914	3.3678009567
0.3	-0.8801119231	0.9682497475	1.5521641647	3.5969801084
0.5	-0.8863334780	0.9690110686	1.5658315556	3.3514908538
0.7	-0.8925049154	0.9697383777	1.5795907333	3.3431758044
0.9	-0.8986272938	0.9704338015	1.5934445298	3.3347489625

Table 2: Numerical results of z_*'s and ξ_2's, for $c_2 = 1$.

θ	z_1	ξ_1	ξ_2
0.1	0.916804131469313	1.37945341276706	3.46374245576363
0.3	0.921092162877287	1.40497341947108	3.43393441765163
0.5	0.925035115678722	1.43171692276687	3.4032479615544
0.7	0.928664417778473	1.45973526774272	3.37160031447881
0.9	0.932009119109698	1.48909606873408	3.33889481215622

Explicit expressions for $y_{1,j}(u)$ is determined by (36), so we obtain the values of $y_{1,j}(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$.

For instance, one has for $\theta = 0.5$,

$$y_0(u) = -0.42464 \times (-0.88633)^{-u} + 0.47519 \times 0.96901^{-u} - 0.05183 \times 1.56583^{-u} + 0.00129 \times 3.40325^{-u}$$

$$y_1(u) = 0.42071 \times (-0.88633)^{-u} + 0.55572 \times 0.96901^{-u} - 0.04023 \times 1.56583^{-u} + 0.00056 \times 3.40325^{-u}$$

Then solve a system of linear equations with $\alpha_{1,j}$, Table 3 lists the values of $\alpha_{1,j}$'s.

Table 3: Numerical results of $\alpha_{1,j}$ for $b = 10$.

θ	0.1	0.3	0.5	0.7	0.9
$\alpha_{1,0}$	0.001602	0.00103	0.00059	0.00027	5.109949×10^{-5}
$\alpha_{1,1}$	0.001703	0.00109	0.00063	0.00029	6.318151×10^{-5}

Explicit expressions for $\phi_2(u)$ is determined by (38), so we get the values of $\varphi_2(u)$ for $\theta = 0.5, c_1 = 2, c_2 = 1, p = 0.2, q = 0.8, \rho_1 = 0.3, \rho_2 = 0.6, v = 0.95, b = 10$, see Table 4.

Table 4: Numerical results of $\varphi_2(u)$ for $b = 10, \theta = 0.5$.

u	10	11	12	13	14
$\varphi_2(u)$	7.22942×10^{-4}	5.02029×10^{-4}	3.49790×10^{-4}	2.44063×10^{-4}	1.70395×10^{-4}
u	15	16	17	18	19
$\varphi_2(u)$	1.8993×10^{-4}	8.31054×10^{-5}	5.80441×10^{-5}	4.05411×10^{-5}	2.83162×10^{-5}

Especially, when $\omega(N_1,N_2)=1, b=10$, Figure 1 and Figure 2 depict the generating function of the time to ruin $m_b(u)$ as functions of u. Observing Figure 1 and Figure 2, for each fixed θ it is easy that a larger u corresponds to a smaller expected ruin time and $m_b(u)$ is a increasing function of θ when u is fixed.

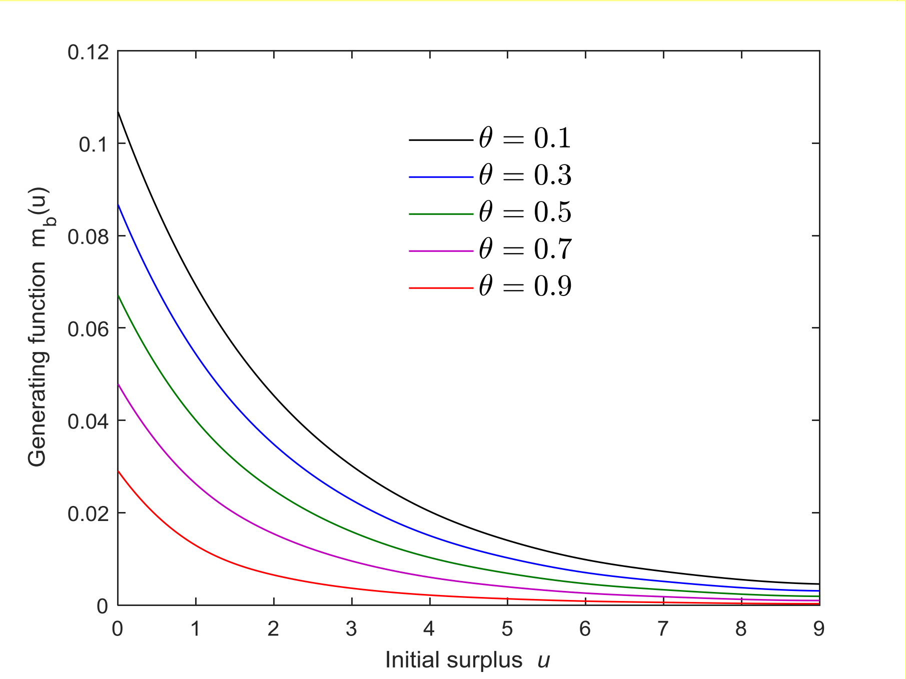

Figure 1: Numerical results of $m_b(u)$ for $b=10, u<b$.

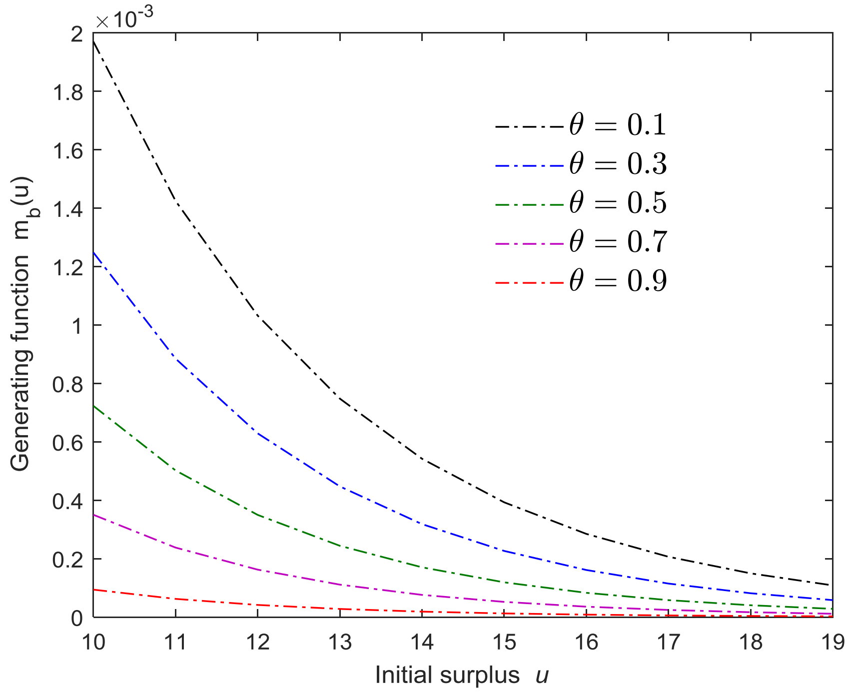

Figure 2: Numerical results of $m_b(u)$ for $b = 10, u \geq b$.

5. CONCLUSION

In this paper, we consider the compound binomial model with general premium rate and a constant dividend barrier.Using the roots of a generalization of Lundberg's fundamental equation and the general theory on difffference equations, we derive an explicit expression for the Gerber-Shiu discounted penalty function up to the time of ruin. In particular, a numerical example is provided to show that the formulae are readily programmable in practice. From the numerical example given above, we can see that the barrier level has a negative effffect on the total Gerber-Shiu discounted penalty function.

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