

A new accurate formula for the large-angle period of a simple pendulum

ABSTRACT

This paper presents a numerical solution of the nonlinear differential equation governing the non-sinusoidal oscillatory motion for the large angle period of a simple pendulum. The numerical method is based on the discretization of motion equation according to an explicit finite difference scheme. Also, an approximation formula giving the period on the large oscillations amplitude is developed and compared with the numerical model. The results showed a good agreement with a deviation less than 0.063 %. The simple pendulum consists of a point mass attached to a massless and inextensible wire that is fixed at the upper end. The oscillations period value is calculated with a precision order of the one-tenth of the millisecond. The approximation formula developed in this work is simple, flexible and more accurate than other formula available in literature.

Keywords: Simple pendulum; nonlinear equation; numerical resolution; approximation formula.

Nomenclature

t	time step (s)
g	gravitation (m/s^2)
L	wire length (m)
M	point mass (kg)
n	time index
t	time (s)
T	oscillations period (s)
T_0	period of very small oscillations (s)

Greek symbols

	relative error (%)
a	absolute error (s)
$_{max}$	maximum relative error (%)
	instantaneous angle (rad)
θ_0	oscillations amplitude (rad)
θ_{0max}	oscillations amplitude of corresponds to
$_{max}$	(rad)

1. Introduction

Because the non-sinusoidal oscillatory motion of the simple pendulum is the first example of a nonlinear problem that university teachers discuss in their classes, and because there is no analytical solution of the differential equation of motion, the simple pendulum remains an attractive problem for scientists and academics. The simple pendulum periodic motion is harmonic only for small angle oscillations, when the approximation $\sin(\theta) \approx \theta$ is valid. Beyond this limit, the equation of motion is nonlinear [1, 2]. Therefore, in most of the papers published in the subject, various approximation formulas have been developed to express the pendulum period in term of large angle oscillations; we will cite the three simplest, where the third of them is the most accurate:

The most famous approximation formula for the large angle period is found by Borda [3], by substituting $\sin(\theta)$ by $\theta - \theta^3/6$, the resulting expression is:

$$T = 2\pi \sqrt{\frac{L}{g}} \times \left(1 + \frac{\theta_0^2}{16} \right). \quad (1.1)$$

The error of this approximation increases and exceeds the typical experimental error (0.1%) for amplitudes above 40°. The second formula is given by Kidd-Fogg [4], it has attracted much interest due to its simplicity:

$$T = 2\pi \sqrt{\frac{L}{g}} \times \left(\cos\left(\frac{\theta_0}{2}\right) \right)^{-0.5}. \quad (1.2)$$

The error of this approximation increases and exceeds the typical experimental error (0.1%) for amplitudes above 55°. The third formula established by Molina [5] is obtained by a simple linearization and the resulting expression is:

$$T = 2\pi \sqrt{\frac{L}{g}} \times \left(\frac{\sin(\theta_0)}{\theta_0} \right)^{-3/8}. \quad (1.3)$$

The error of this approximation increases and exceeds the typical experimental error (0.1%) for amplitudes above 70°. Indeed, some other proposed formulas are available in literature [9-12],

where the most accurate of them is most complicated to use, and vice versa

The objective of this paper is to use the finite differences method to resolve numerically the motion equation of a simple pendulum oscillating with any angle greater than 0° and lower or equal 90° , and subsequently to develop a more simplest and accurate formula for the oscillations period. Where the deviations from the exact results are of order less than the typical experimental error.

2. Mathematical modeling and resolution

We consider a simple pendulum consists of a point mass M attached to a massless and inextensible wire of length L , which is fixed at the upper end as illustrated in Fig. 1. If we move the pendulum away from an angle θ_0 with respect to the vertical axis and we release it, this last starts to oscillate.

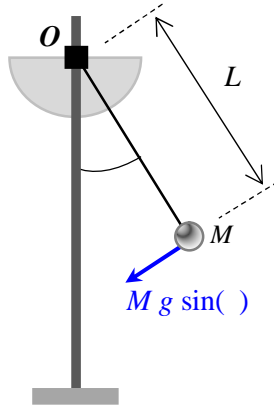


Fig. 1: Schematic of the considered problem.

By applying the theorem of angular momentum in which only the weight of the point mass has a non-zero moment, and by neglecting the opposite force due to the air resistance, the equation of motion verified by the angle $\theta(t)$ that the wire makes with the vertical position at the time t , is given as:

$$\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin\theta(t) = 0 \quad (2.1)$$

with two initial conditions

$$\theta(t=0) = \theta_0 \quad (2.2)$$

$$\left(\frac{d\theta(t)}{dt}\right)_{t=0} = 0 \quad (2.3)$$

Because of the existence of the nonlinear term ‘ $\sin(\theta)$ ’ in the motion equation, the simple pendulum behavior is not sinusoidal and it has nonlinear oscillations. Therefore, an analytical solution for the differential equation of motion is not available.

Moreover, the period of the oscillations depends, in addition to the length L and the gravity g , on the amplitude of oscillations θ_0 , as shown in the following equation:

$$T = T_0 \times f(\theta_0) \quad (3.1)$$

where

$$T_0 = 2\pi\sqrt{\frac{L}{g}}. \quad (3.2)$$

The adopted numerical solution is based on the discretization of equations system (2) according to an explicit finite difference scheme, as follow:

$$\frac{\theta^{n+1} - 2\theta^n + \theta^{n-1}}{\Delta t^2} + \frac{g}{L}\sin\theta^n = 0 \quad n = 1, 2, \dots \quad (4.1)$$

with two initial conditions

$$\theta^n(n=0) = \theta_0 \quad (4.2)$$

$$\left(\frac{\theta^{n+1} - \theta^n}{\Delta t}\right)_{n=0} = 0. \quad (4.3)$$

Then, the pendulum angle $\theta(t)$ is calculated at each time $t = n \times \Delta t$, where n is the time index and Δt is the time step, and we get:

$$\theta^{n+1} = -\frac{g}{L}\Delta t^2\sin\theta^n + 2\theta^n - \theta^{n-1} \quad n = 1, 2, \dots \quad (5.1)$$

with initial conditions

$$\theta^0 = \theta_0 \quad (5.2)$$

$$\theta^1 = \theta^0 \quad (5.3)$$

The instantaneous angle $\theta(t)$ obtained during the pendulum oscillation is shown in Fig. 2. The pendulum is initially released at different amplitudes that vary from 10° to 90° . The pendulum length, the gravity and time step values are $L = 1 \text{ m}$, $g = 9.81 \text{ m/s}^2$ and $\Delta t = 10^{-4} \text{ s}$, respectively. It can be seen that the pendulum oscillates periodically between $-\theta_0$ and θ_0 , and the oscillations period increases with the increasing of the oscillations amplitude. Indeed, this figure confirms that the oscillations period is strongly depending on the initial angle θ_0 . In other way, the choosing of $\Delta t = 10^{-4} \text{ s}$ was permissive, because of the numerical solution can be obtained with any accuracy. Then, for a desired precision, it is necessary to identify the right value of the time step size.

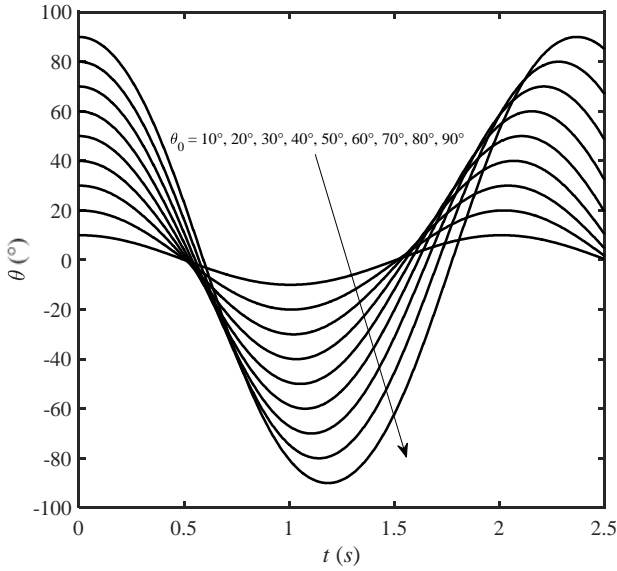


Fig. 2: Timewise evolution of the pendulum position for different initial amplitudes.

3. Looking for accurate results

In order to obtain an accurate numerical solution, we decided to only accept numbers with at least four certain digits after the decimal point. In the table 1,

are shown the period values for different amplitudes and different time step sizes; the pendulum length and the gravity values are $L = 1 \text{ m}$ and $g = 9.81 \text{ m/s}^2$, respectively. By analyzing the period value at each time step and for all amplitudes, it is noted that only the last digit is affected by the decreasing in time step size, and the other ones remain invariant. It means that in each period value, the only uncertain digit is the last one. As example, for $\theta_0 = 90^\circ$, the uncertain digit is 4, 7, 8, 5 and 5 for time step equal to $5 \times 10^{-2} \text{ s}$, $5 \times 10^{-3} \text{ s}$, $5 \times 10^{-4} \text{ s}$, $5 \times 10^{-5} \text{ s}$ and $5 \times 10^{-6} \text{ s}$, respectively. Therefore, the decreasing in the time step size improves the accuracy of the period value, which can be obtained with at least four certain digits after the decimal point when the time step size is less than or equal $5 \times 10^{-6} \text{ s}$ ($\Delta t \leq 5 \times 10^{-6} \text{ s}$). Indeed, when the four digits after the decimal point in the period value are certain, that signifies that period is accurate in order of the one-tenth of a millisecond. Moreover, in Fig. 3, the period and the dimensionless period evolutions as function of time step size are displayed for different initial angles θ_0 . One can conclude that the numerical solution convergence is obtained with the desired accuracy, and this can be done independently of the pendulum length and the gravity values.

Table 1: Period values for different amplitudes and different time steps, $L = 1 \text{ m}$, and $g = 9.81 \text{ m/s}^2$.

	T (s)								
	$\theta_0 = 10^\circ$	20°	30°	40°	50°	60°	70°	80°	90°
$t = 5 \times 10^{-2} \text{ s}$	2.05	2.05	2.05	2.1	2.15	2.2	2.25	2.3	2.4
$5 \times 10^{-3} \text{ s}$	2.01	2.025	2.045	2.07	2.11	2.155	2.215	2.285	2.37
$5 \times 10^{-4} \text{ s}$	2.01	2.0215	2.041	2.069	2.106	2.153	2.211	2.2820	2.368
$5 \times 10^{-5} \text{ s}$	2.0099	2.0215	2.041	2.06895	2.10595	2.1529	2.211	2.2819	2.36785
$5 \times 10^{-6} \text{ s}$	2.009895	2.021455	2.04099	2.06894	2.105935	2.152875	2.21098	2.28189	2.367845

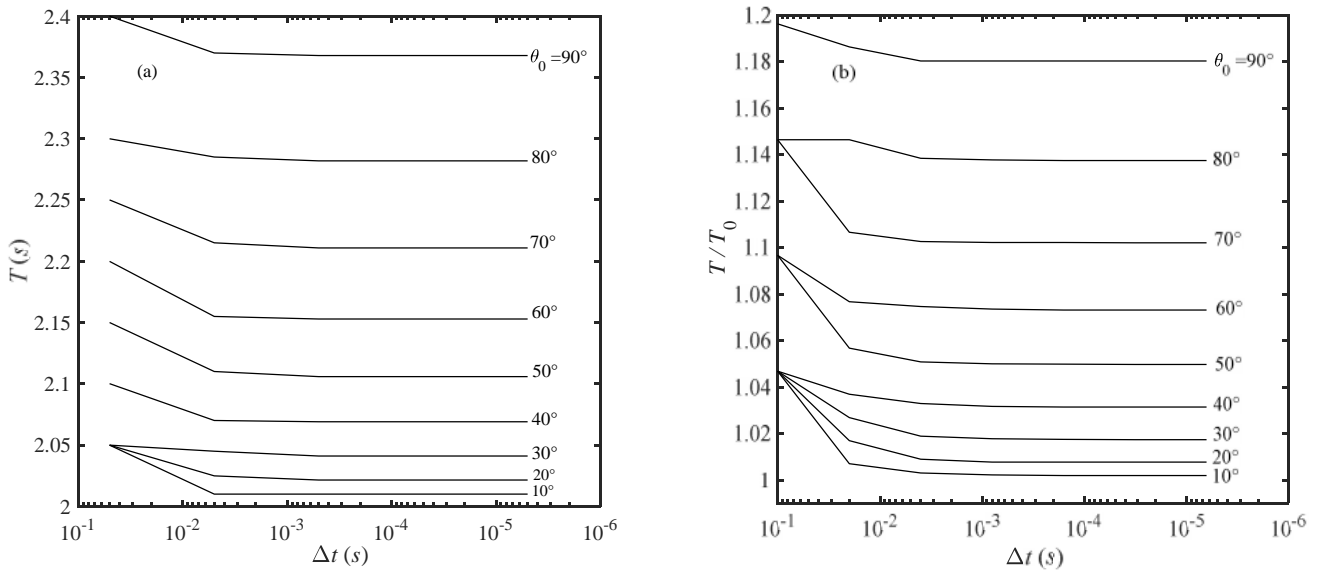


Fig. 3: Time step size effects on the numerical solution convergence.

4. Model validation

To validate the quality of the present numerical solution, a comparison with results available in literature is done. Thornton and Marion [13] expressed the period oscillation in term of elliptical integral of the first kind, which can be evaluated numerically or by expanding it in a power series. At first, the comparison is shown in Table 2, where the present period values for different amplitudes are confronted to those of Thornton and Marion [13]; and the two models coincide quite for at least four digits after the decimal points. Secondly, the comparison is

enlarged to cover all amplitudes between 0° and 90° , and the results is exposed in Fig. 4a, which shown that the agreement between the two methods is excellent. Indeed, the absolute and relative errors between the two solutions are also evaluated in Fig. 4b and Fig. 4c, respectively. The absolute error is always less than 5×10^{-6} s, which means that the accuracy is of the order of the one-tenth of the millisecond. The relative errors is at all less than 2.5×10^{-3} %, which is 400^{-1} times the typical experimental error (0.1%). Consequently, the present numerical solution is of high level of confidence.

Table 2: Comparison of period values for different amplitudes.

θ_0 ($^{\circ}$)	10°	20°	30°	40°	50°	60°	70°	80°	90°
T (s)									
Present study	2.009895	2.021455	2.04099	2.06894	2.105935	2.152875	2.21098	2.28189	2.367845
T (s)									
[13]	2.009893	2.021451	2.0409899	2.068938	2.1059346	2.1528747	2.210976	2.281886	2.367842

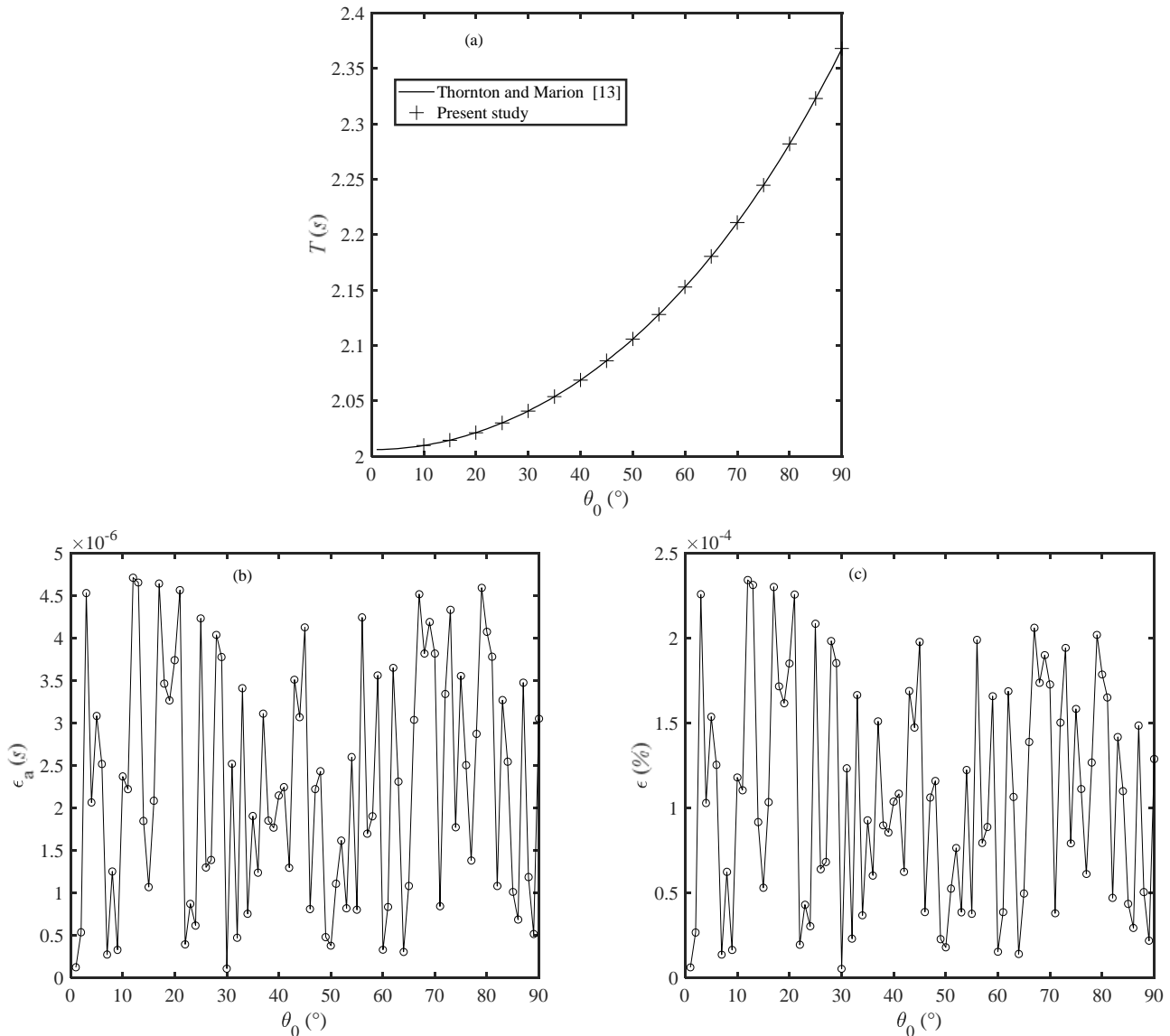


Fig. 4: Comparison of the present numerical solution with literature results [13].

5. Evaluation of the simplest existing formulas

The three simplest approximation formulas available in literature and detailed above, namely formula of Molina [5], Kidd and Fogg [4] and Borda [3] are confronted to the numerical solution in Fig. 5. The Fig. 5a shows the evolution of the dimensionless period as function of the oscillations amplitude; the dimensionless period is obtained using the present numerical solution and the three cited approximations. It can be noted that the three approximations overlap for small amplitudes, where the divergence between the numerical solution and the Borda [3] formula begins to appear at $\theta_0 = 30^\circ$, the divergence of the Kidd and Fogg [5] formula begins at $\theta_0 = 45^\circ$ and that of Molina [3] starts at $\theta_0 = 60^\circ$. Then, this last approximation formula is the

most accurate. Indeed, the placement of the curves close to each other is not sufficient to evaluate the error order; and to do so, it is necessary to calculate and plot the relative error ($\epsilon = |T_{numerical} - T_{approximation}| / T_{numerical}$) between the exact solution and each approximation formula, as shown in Fig. 5b. From this figure; it is clear that the three approximations reaches an error low than 0.1% (the typical experimental error) for $\theta_0 < 40^\circ$. Then, the increasing in the amplitude value increases the divergence, when the Molina formula reaches an error $\epsilon = 0.36\%$, the Kidd and Fogg formula reaches an error $\epsilon = 0.75\%$ and Borda formula reaches an error $\epsilon = 2.22\%$ for an amplitude $\theta_0 = 90^\circ$. Therefore, these approximations are of half-efficiency.

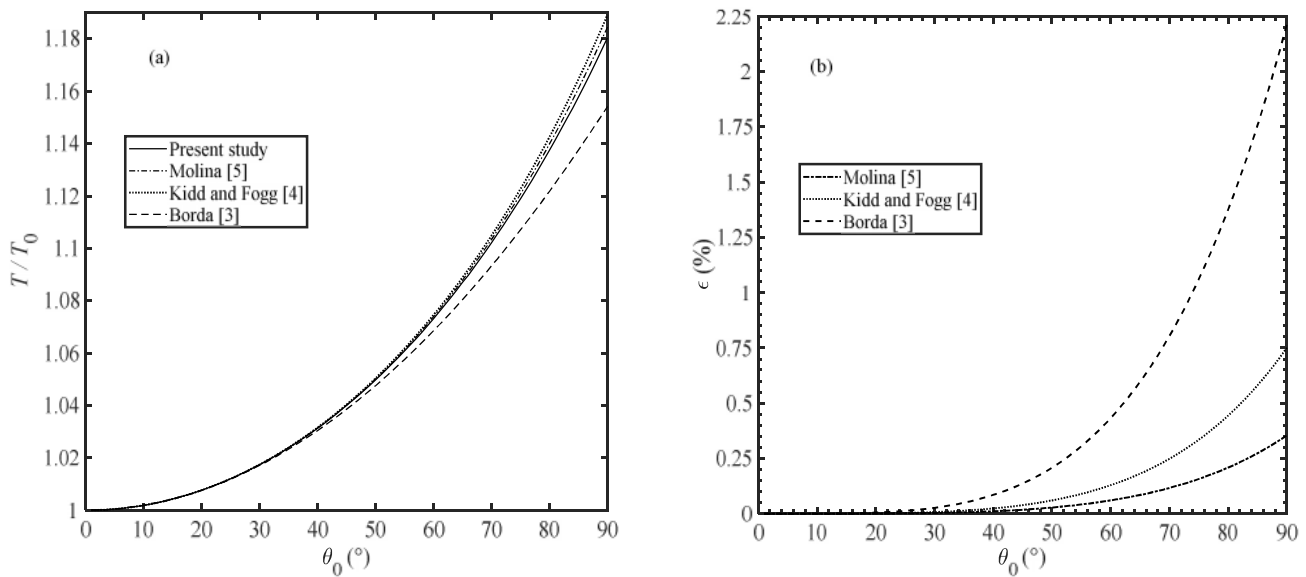


Fig. 5: Comparison of present results with those of approximation formulas available in literature.

6. Development of a new approximation formula

Based on Molina correlation, a new useful, flexible and more accurate correlation is established and expressed as:

$$T = T_0 \times \left(\frac{\theta_0}{\sin(\theta_0)} \right)^C, \quad (6.1)$$

where C is a constant to be determined. First, we define the relative error that depends on the amplitude θ_0 and C constant, as:

$$\epsilon(\theta_0, C) = \frac{|T_{numerical} - T_{proposed_correlation}|}{T_{numerical}}, \quad (6.2)$$

and the maximum relative error that is calculated basing on the relative error by varying the amplitude from 1° to 90° :

$$\epsilon_{\max}(C) = \max_{\theta_0 = 0^\circ \rightarrow 90^\circ}(\epsilon). \quad (6.3)$$

Indeed, we are looking for the C value that corresponds the low value of the maximum relative error. In Fig. 6a is plotted the relative error between the present numerical solution and the proposed correlation, as function of the C constant and for different amplitudes. We note that for each given amplitude value, the relative error, ϵ , is minimum for a C value between 0.36 and 0.38, approximatively. This minimum value is not constant but it varies in dependence with the oscillations amplitude. That is why the maximum relative error is plotted in Fig. 6b as function of C constant, and it is obtained that the low value of the maximum relative error is equal to 0.06222%, and it corresponds to $C = 0.36854$ so, this is the optimal value that which minimizes the relative error of the proposed correlation. Indeed, if we need

to consider the relative error between the present numerical solution and the one obtained by elliptic integral [13], which equal to 0.0025%, the relative error of the proposed formula become:

$$\varepsilon_{\max} = 0.06222 + 0.0025 \quad (6.4)$$

then,

$$\varepsilon_{\max} = 0.06472 \% , \quad (6.5)$$

which is less than 0.065 % and therefore less the typical experimental error (0.1%). In other hand, the amplitude at which the maximum relative error occurs, $\theta_{0\max}$, is plotted as function of C value in Fig. 6c. As can be seen, the amplitude $\theta_{0\max}$ is equal to 90° beyond the range of C value between 0.35919 and 0.36853, and it decreases from 90° to 59° monotonously inside this range; the returning of $\theta_{0\max}$ to the value of 90° is not monotonic but it is jumping from 59° to 90° .

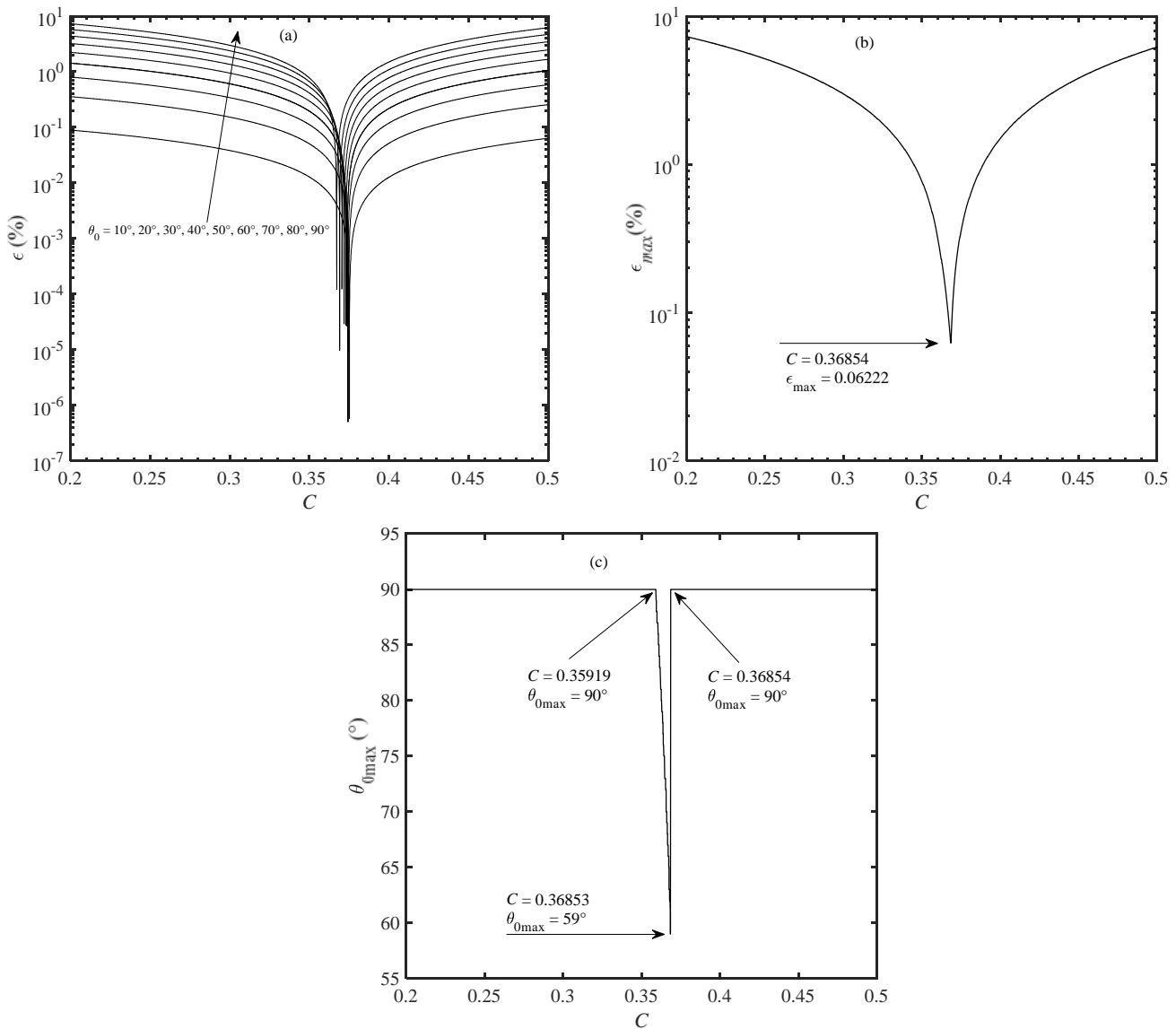


Fig. 6: Evolutions of relative error, maximum error and its corresponding amplitude as function of C value.

7. Evaluation of the new approximation formula

The relative errors of the proposed and Molina correlations with respect of the amplitude θ_0 are shown in Fig. 7a. It can be showed that at all amplitude the proposed correlation error is less than 0.063 % in contrary of the Molina correlation that exceeds the 0.1% for $\theta_0 > 68^\circ$. In fact, the Molina correlation become more accurate for $\theta_0 < 60^\circ$, but

this show has no significant effects because at all the error is less than 0.1% and the nonlinearity problem becomes more important for higher value of θ_0 . Finally, in Fig. 7b is compared the dimensionless period evolution of the numerical solution, proposed correlation and Molina approximation as function of θ_0 . It is clear that proposed correlation coincide more quite with the present numerical solution.

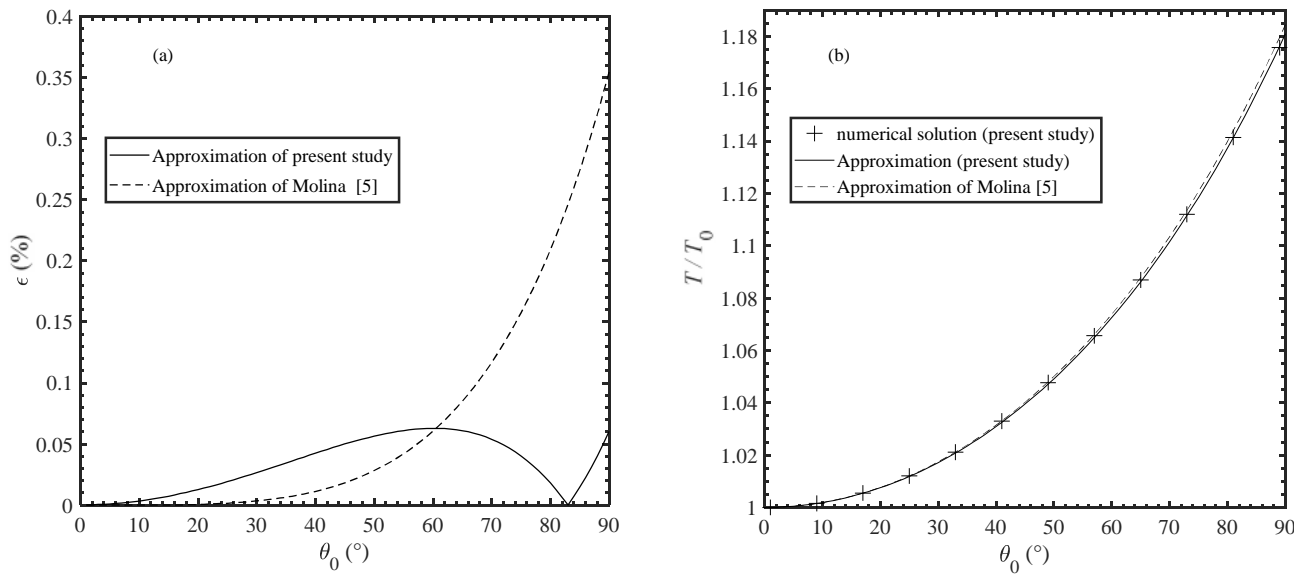


Fig. 7: Evolutions of the relative errors and dimensionless periods of the developed and Molina formulas as function of θ_0 .

8. Conclusion

In the present study, a numerical simulation of a simple pendulum's motion is developed for the large angle period and a new approximation formula giving the period pendulum oscillations as function of the amplitude is carried out with better accuracy by comparison with other approximation approaches available in literature. Then, teachers and students can easily use the developed formula to calculate the oscillations period with high confidence level. Indeed, it is the first correlation that gives an error less than the typical experimental error along the initial amplitude range between 0° and 90° .

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