Mixed-mode Oscillations in Filippov system

Abstract

The mechanism of the mixed mode oscillations of a class of non-smooth Filippov systems under multistable coexistence was studied in this paper. Based on a Lorenz-type chaotic model with multi-attractor coexistence, the Filippov system was established by introducing non-smooth terms as well as an external excitations. With multiple stable attractors in the discontinuous vector field, the parameter changes has lead to complex transition patterns between the attractors and the non-smooth interface, or between the attractors. When the order gap in the frequency domain exists in the vector field, the mixed-mode oscillations were aroused. Here we have taken several excitation amplitudes to cover different coexistence regions, a set of mixed mode oscillation patterns were obtained. Besides, the bifurcation set of two generalized autonomous subsystems and the coexistence region of attractors were discussed. Combined with the transformed phase diagram method, the bifurcation mechanism of bursting oscillation and the sliding dynamical behaviors of the systems at the discontinuous interface has revealed with slow varying parameters access in different regions of multistable attractors coexistence. The alternations between quiescent and spiking states become more frequent and complex, leading to the change of the structure of the bursting oscillation modes. Moreover, the non-smooth partition interface of the system yields multiple non-smooth bifurcations, which will also affected the oscillation modes of the generalized autonomous system.

Keywords: bursting oscillation, multiscale coexistence, multiscale coupling, nonsmooth system

1. Introduction

Multistable coexistence is a unique property of dynamical systems. It has been widely found in various fields of nature and engineering, such as circuit models [1], biological population model [2] and the food chain model [3] *etc*. When the initial state changes, the multistable coexistence system may be influenced by different attractors, which may appear completely different dynamical behaviors, leading to the dynamical behaviors of the system becoming unpredictable [4]. Therefore, studying the dynamical behaviors and attractive domains of systems with multistable coexistence is meaningful.Up to now, the research about multistable systems has focused on the solution of attractors or characterizing the attractive domain [5]. However, in practical engineering, systems often exist multiscale coupling or non-smooth factors [6] etc., leading to the more complex behaviors. Multi-timescale coupling phenomena are common in the field of living organisms [7], physics [8], chemistry [9] *etc*. In a period of dynamic behavior, there will be bursting oscillation behaviors that alternate between large oscillation (spiking state) and small oscillation (quiescent state) [10]. Due to a lack of theoretical foundation, early scholars mostly used the approximate solution and numerical simulation [11] methods to study the bursting oscillation mechanism. It is until 2000, Rinzel proposed the fast and slow analysis method [12], which provided theoretical support for the bursting oscillation mechanism and given the classification of the bursting oscillation form, after that more scholars began to study it. When there exists an order gap between the exciting frequency and the natural frequency, implying the coupling of two scales in the frequency domain exists, slow-fast behaviors may also appear [13]. Such as Han Xiujing [14] et al found that bursting oscillations of Duffing oscillator can be controlled by forcing frequency and forcing amplitude.

It is worth noting that when there is two-scale coupling, the limit cycles coexist with the equilibrium points, two-dimensional torus in the vector field, implying the system exhibits both mixed mode oscillations and quasi-periodic oscillations[15]. It is shown that multistable coexistence in the multiscale coupled system may directly affect the fast and slow behavior.b Moreover, when the non-smooth term exists, the discontinuity of the vector field makes the system controlled by different subsystems. Some dynamic behaviors such as sliding bifurcation on the non-smooth interface may appear, making the switching mechanism between the stable attractors more complicated [16]. However, most of the current works are mainly aimed at systems containing single stable attractor, which mainly show the transition modes of quiescent states and spiking states with single type in Filippov system under two-scale coupling [17]. However, for common dynamical systems, such as chaotic attractors of Lorentz type, stable equilibrium points often coexist with limit cycles and other forms of stable attractors, which undoubtedly leads to more complex mixed oscillation modes.

Here, we introduce a Lorentz-type chaotic system containing multistable coexistence attractors [18]. With the variation of the parameters, it is not difficult to find that the number of equilibrium points may change. When the system takes specific parameters, we are able to find its corresponding multistable coexistence region and further analyze its dynamical behaviors. A novel non-smooth dynamical model is developed by introducing the non-smooth factor and the periodic exciting term:

$$\begin{aligned} \dot{x} &= ax - yz, \\ \dot{y} &= -by + xz, \\ \dot{z} &= -cz + xyz + d + g(x) + A\sin(\Omega t), \end{aligned} \tag{1.1}$$

Describe the model

Where, $w = A\sin(\Omega t)$ is the periodic exciting term, A represents the excitation amplitude, and the non-smooth term $g(x) = -3\,\mathrm{sgn}\,x$. Because of the existence of the slowly varying excitation term w, when the exciting frequency Ω is far less than the natural frequency of the system, denoted by W_N, the coupling of two scales in frequency domain exists, The occurrence of different scale coupling will lead to bursting oscillations phenomenon, which usually be represented as a transition process between large and small oscillations. Furthermore, the non-smooth boundary, defined as $\Sigma \equiv \{(x, y, z), x = 0\}$, divides the phase space into two regions, described by $D_1 \equiv \{(x, y, z), x > 0\}$ and $D_2 \equiv \{(x, y, z), x < 0\}$, in which the trajectory is governed by different subsystem, respectively. Because of the trajectories of non-smooth systems are represented by two different subsystems, more complicated dynamical behaviors can be obtained.

The whole exciting term $w = A\sin(\Omega t)$ can be regarded as a generalized state variable, which forms the slow subsystem, leading to the so-called fast subsystem in generalized autonomous form. Based on the above, we select specific parameters with the excitation frequency at $\Omega = 0.01$. With the overlap of the transformed phase diagram and the bifurcation diagram, we deeply study the dynamic mechanism of the Filippov system in this paper.

2. Mixed-mode oscillations with different excitation amplitudes

Two typical situations are examined below.

Case 1: A=3.2

The system presents a quasi-periodic movement on the right side of the interface (x> 0), and the radius of movement increases with the amplitude of the external excitation. As shown in Fig.1, the D_2 region (x< 0) occurs two spiking states, represent by SP_2 and SP_3 respectively. Further observation shows that SP_3 belongs to the quasi-periodic type, its frequency different with SP_2. The inconsistent frequency indicates that the system trajectory from the spiking state(SP_2) to a short silence and then tends to the limit cycle quickly. Furthermore, more smooth bifurcations appear with the increased of external excitation amplitude, leading the SP_3 turn to the QS_1。

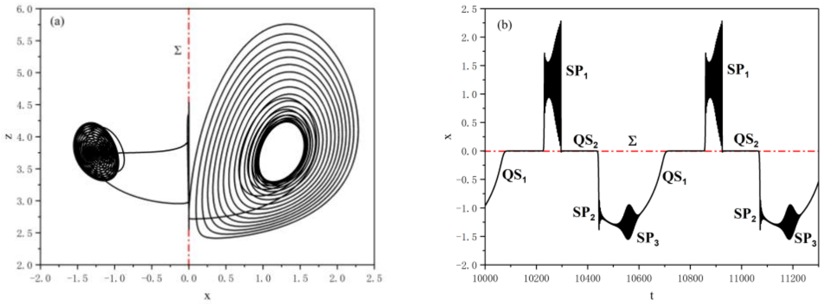

Fig.1 A=3.2 (a)Phase portrait on the (x, z) plane ;(b) Time histories of x.

Case 2: A=3.9

When the external excitation amplitude increases to A=3.9, the periodic oscillation in the D_1 region (x> 0) disappears and turns to two spiking states, as shown in Fig.2.Among that the SP_2 consistent with case A=3.2, the trajectory tends to a smooth equilibrium point form the SP_2, and the two spiking states are connected in a quiescent state QS_2, indicating that more local bifurcations occur in the system.

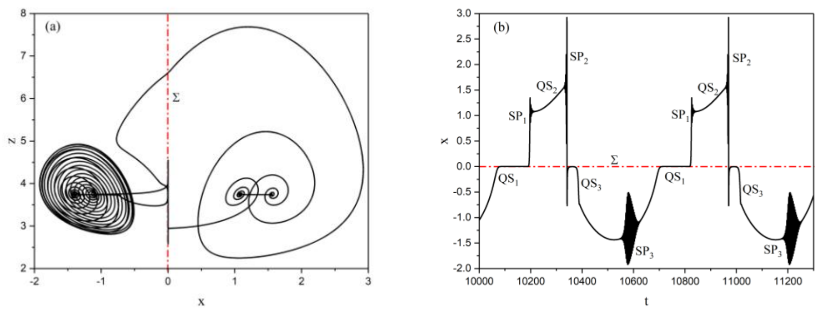

Fig.2 A=3.9 (a) Phase portrait on the (x, z) plane ;(b) Time histories of x.

The trajectory in the D_2 region (x< 0) has also changed. Compared with the case 1 for A=3.2, the number of spiking states decreases. The original oscillations toward the stable equilibrium point disappear, indicating that the trajectory may move to the limit cycle via an additional stable attractor. In order to reveal the mechanism of the dynamics of the full system, now we turn to the attractors and their bifurcations of the fast subsystem with the variation of the slow-varying parameter w.

3. Equilibrium branches and bifurcations (Mention type of bifurcation)

To simplify the analysis, here we fix the parameters at a=2, b=7, c=3, and d=8, and we give the equilibrium cycles and bifurcations of the subsystems, as shown in Fig 3. Where, all black solid lines indicate stable equilibrium branches, and black dashed lines indicate unstable equilibrium branches. Furthermore, all pseudo-equilibrium points are marked with red solid and dashed lines. Due to the non-smooth interface, trajectory actually move within a subsystem of x> 0 in D_1 region, likewise, trajectory actually move within a subsystem of x <0 in D_1 region.

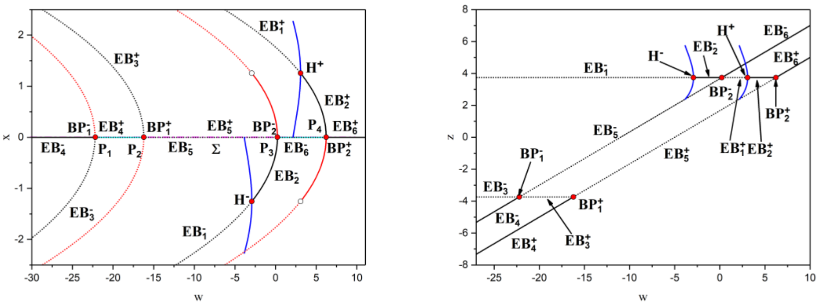

Fig.3 Equilibrium branches and bifurcations.

When x> 0, the trajectory is governed by the subsystem D_1, which the corresponding equilibrium points are represented as the EB^-. Similarly, the equilibrium points of subsystem D_2 can be represented as EB^+. Sub-critical pitchfork bifurcation occurs at the point P_1 and P_2 and super-critical pitchfork bifurcation occurs at the point P_3 and P_4.Hopf bifurcations occur at the points H^+ on the equilibrium branch EB_2^+ and H^- on the equilibrium branch EB_2^-, resulting in stable limit cycles. In order to show the system in the phase space clearly, we omit all the pseudo-equilibrium points in the (w, z) diagram.

As can be seen from the figure, if we take a certain range, the system will have multiple stable coexisted attractors, such as when w>2.5, the EB_2^+ coexists with the EB_6^-. When w is between P_3 and P_4, the stable limit cycle exists with the equilibrium points. Therefore, we combine the bifurcation diagram with the transformed phase portrait to conduct a mechanistic analysis of all of the above mixed-mode oscillation patterns.rewrite it is not clear

4. Analysis of the mixed-mode oscillation mechanism

We overlap the transformed phase portrait with the bifurcation diagram, from which we can obtain the transition mechanism of the trajectory. Moreover, due to the overlap of the non-smooth interface with the equilibrium branches in the x-plane, we introduce the overlap of the transformed phase portrait and equilibrium branches on the (w, x) plane and the (w, z) plane. Combine the two diagrams, we can distinguish the sliding phenomenon at the non-smooth interface, and the movement along the equilibrium point.

Case 1: A=3.2

When the amplitude A increases to 3.2, as shown in Fig 4. Assuming when the trajectory starts from the minimum value of w, because of the Hopf bifurcation at the point H^-, it turns to oscillate according to the limit cycle LC_2, yielding spiking oscillations SP_3. The trajectory may finally settle down to the stable equilibrium branch $EB_2^-$and turns to move almost strictly along the branch until the pitchfork bifurcation occurs. Then, the trajectory move along the stable equilibrium branch EB_6^- until the point M_1, as shown in Fig5(b). When the trajectory moves to the point M_2, it enter the subsystem in D_2 region. The trajectory slides along the unstable equilibrium branch EB_5^+ in the (w, z) plane, corresponding to the (w, z) plane it slides along the boundary. The trajectory starts moving to the right as it moves to the w maximum. At the point M_3, affected by the non-smooth sliding bifurcation, it jumps to the equilibrium point. Before the trajectory fully converging to the stable equilibrium branch EB_2^+, the Hopf bifurcation occurs at the point H^+, then the trajectory oscillates according to the stable limit cycle LC_1, resulting in repetitive spiking oscillations SP_1.

As the w continues to decrease, the homoclinic bifurcation of the limit cycle appears, leading to the trajectory ends the oscillation along the limit cycle. It turns to move along EB_6^- until the point P_3,where the pitchfork bifurcation occurs, leading to the trajectory moves along the EB_5^-,appearing in quiescent state QS_2.Then the non-smooth sliding bifurcation occurs, the trajectory jumps to the focal point in the stable equilibrium EB_2^-,yielding spiking oscillations SP_2.With the w continues to decrease, the trajectory arrives at the starting point, which finishes one period of the bursting oscillations.

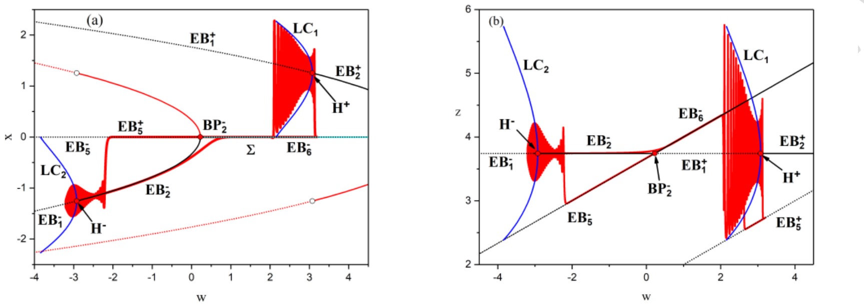

Fig.4 A=3.2 Overlap of the transformed phase portrait and equilibrium branches on (w, x) plane (a) and on (w, z) plane (b).

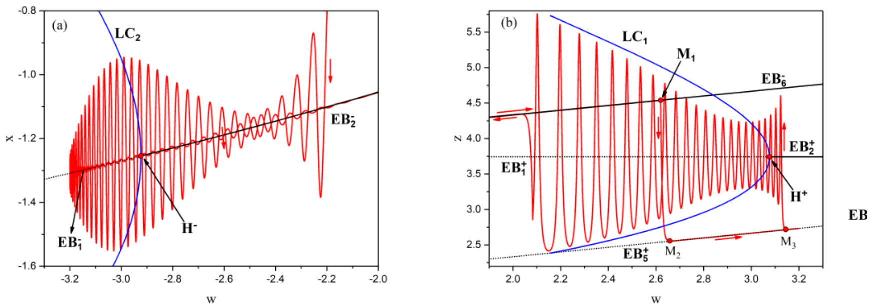

Fig.5 A=3.2 Locally enlarged parts of (w, x) (a) and (w, z) (b).

Note that the slow-passage effect affects the movement of the system trajectory when the excitation amplitude increases further, and below we analyze the case of A =3.9.

Case 2: A=3.9

As shown in Figure 6, the system shows a completely different fast and slow behavior in the coexistence region of EB_2^+ and EB_6^-,directly leading to the alternation between the quiescent states and the spiking states of multiple modes. When the trajectory starts from the minimum value of w, as the case 1 for A=3.2, it turns to oscillate according to the limit cycle LC_2 via the Hopf bifurcation at the point H^-, yielding spiking oscillations SP_3. Then the trajectory moves along the stable equilibrium branch EB_2^- until the pitchfork bifurcation occurs at the point BP_2^-,it turns to move along the stable equilibrium branch EB_2^-. As the parameter w increases to 2.8, the trajectory turns to be governed by the subsystem in D_1 region. Then the trajectory slides along the non-smooth interface in the (w, x) plane, leading to it moves along the unstable equilibrium branch EB_5^+ in the (w, z) plane. With the parameter w reaches the maximum for w=3.9, the trajectory turns to left. With the influence of sliding bifurcation, it exits the non-smooth interface, instead converges to the stable equilibrium point EB_2^+, representing in the spiking state SP_1. The trajectory moves along the stable equilibrium branch EB_2^+ until the Hopf bifurcation occurs at the point H^+. Due to the slow passage effect, the trajectory continues to move along the unstable equilibrium branch EB_2^-, then it oscillates according to the stable limit cycle LC_1 with the same frequency of LC_1, resulting in repetitive spiking oscillations SP_2.Meanwhile, with the oscillation amplitude increases, the trajectory will contact the non-smooth partition interface, leading to the generation of the crossing sliding bifurcation, as shown in Fig 11. Then the trajectory tends to the boundary, manifesting as a large sharp peak and rapidly crossing, eventually converging to the stable equilibrium EB_6^-.Other movement processes are repeated with the above situation, so we will not repeat.

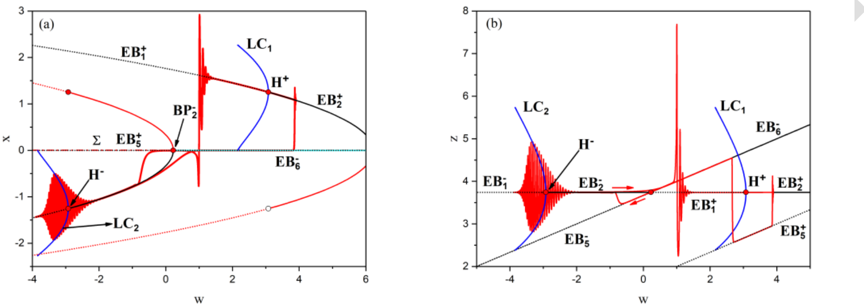

Fig.6A=3.9 Overlap of the transformed phase portrait and equilibrium branches on (w, x) plane (a) and on (w, z) plane (b).

From the above analysis, we can find that when the slow variable parameter w visits different multistable coexistence regions, leading to more spiking states and quiescent states appear, which can also affect the structure of mixed mode oscillations. Moreover, due to the existence of the non-smooth partition interface, the trajectory will also be affected by multiple non-smooth bifurcations during the alternations between the spiking states and quiescent states, which makes the mechanism of the mixed-mode oscillation more complicated.

5. Conclusion

For a class of multistable coexisting Lorenz chaotic systems, a periodic external excitation and non-smooth term are introduced to establish a novel non-smooth Filippov system. We studied the bursting oscillation mechanism of non-smooth Filippov systems with two-scale coupling. The system trajectory shows more alternations between quite states and spiking states in multistable coexistence regions, due to the system trajectory switches between the subsystems in the boundary. In order to analyze the multistable coexistence region and subsystems, here we made the overlap of the transformed phase portrait and equilibrium branches as well as the bifurcations in the (w, x) and (w, z) plane. The analysis shows that sliding bifurcation can take place in trajectory movements. In addition, the slow passage effect also plays an important role in the bursting oscillations, leading to a new quiescent state trajectory that connecting two spiking states being appear, implying the system trajectory will present more abundant dynamical behaviors.Rewrite this section based on your objective

.

References

1 Ju Y, Li Y, Tan J, et al. Transition mechanism and dynamic behaviors of a multi-stable piezoelectric energy harvester with magnetic interaction[J]. Journal of Sound and Vibration, 2021, 501: 116074.

2 Laurent M, Kellershohn N. Multistability: a major means of differentiation and evolution in biological systems[J]. Trends in biochemical sciences, 1999, 24(11): 418-422.

3 Adhikary P D, Mukherjee S, Ghosh B. Bifurcations and hydra effects in Bazykin s predator–prey model[J]. Theoretical population biology, 2021, 140: 44-53.

4 Chudzik A, Perlikowski P, Stefanski A, et al. Multistability and rare attractors in van der Pol–Duffing oscillator[J]. International Journal of Bifurcation and Chaos, 2011, 21(07): 1907-1912.

5 Han X, Bi Q. Generation of hysteresis cycles with two and four jumps in a shape memory oscillator[J]. Nonlinear Dynamics, 2013, 72(1): 407-415.

6 Tong X, Li F, Liu M, et al. Thermal fatigue resistance of non-smooth cast iron treated by laser cladding with different self-fluxing alloys[J]. Optics & Laser Technology, 2010, 42(7): 1154-1161.

7 Wang Y, Rubin J E. Multiple timescale mixed bursting dynamics in a respiratory neuron model[J]. Journal of Computational Neuroscience, 2016, 41(3): 245-26

8 Ahmed W, Maqsood A, Riaz R. Multiple time scale analysis of pressure oscillations in solid rocket motors[J]. Results in Physics, 2018, 8: 1068-1075

9 Li X H, Bi Q S. Single-Hopf bursting in periodic perturbed Belousov-Zhabotinsky reaction with two time scales[J]. Chinese Physics Letters, 2013, 30(1): 010

10 Bertram R, Rubin J E. Multi-timescale systems and fast-slow analysis[J]. Mathematical biosciences, 2017, 287: 105-121.

11 Verhulst, F. Singular perturbation methods for slow–fast dynamics.Nonlinear Dynamics, 2007;50(4), 747-753.

12 Rinzel J. Bursting Oscillations in An Excitable Membrane Model. In: Sleeman B D, Jarvis R J, eds. Ordinary and Partial Differential Equations. Berlin: Springer, 1985. 304–316

13 Bi Q, Li S, Kurths J, et al. The mechanism of bursting oscillations with different codimensional bifurcations and nonlinear structures[J]. Nonlinear Dynamics, 2016, 85(2): 993-1005.

14 Han, Xiujing, & Qinsheng Bi. Bursting oscillations in Duffing’s equation with slowly changing external forcing. *Communications in Nonlinear Science and Numerical Simulation* 16.10 2011: 4146-4152.

15 Ceni A, Olmi S, Torcini A, et al. Cross frequency coupling in next generation inhibitory neural mass models[J]. Chaos: An Interdisciplinary Journal of Nonlinear Science, 2020, 30(5): 053121.

16 di Bernardo M, Kowalczyk P, Nordmark A. Sliding bifurcations: a novel mechanism for the sudden onset of chaos in dry friction oscillators[J]. International journal of Bifurcation and chaos, 2003, 13(10): 2935-2948.

17 Qu R, Wang Y, Wu G, et al. Bursting oscillations and the mechanism with sliding bifurcations in a Filippov dynamical system[J]. International Journal of Bifurcation and Chaos, 2018, 28(12): 1850146.

18 Lai, Q., Akgul, A., Li, C., Xu, G., & Çavuşoğlu, Ü. A new chaotic system with multiple

attractors: Dynamic analysis, circuit realization and S-Box design. Entropy, 2017;20(1), 12.