Approximation on differences of linear positive operators

Abstract. In the present paper, we study the approximation of differences of linear positive operators. Quantitative estimates for the differences of Baskakov with Baskakov-Szasz and Baskakov-Durrmeyer operators are discussed. We have also present difference properties of Baskakov-Szasz and genuine Baskakov-Durrmeyer operators. Finally, we obtain the quantitative estimate in terms of the weighted modulus of smoothness for these operators

Key words: Difference of operators, Linear positive operators, Approximation theory, Baskakov-operators, Modulus of continuity.

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1. Introduction

In the year 1953, Korovkin found the most powerful and easiest criterion in order to decide approximation process with linear positive operators on continuous functions. After that a considerable amount of research on linear positive operators has been done by various mathematicians. e.g. [2], [9], [11], [12], [13], [14], [15] etc.

The study on the difference of linear positive operators is an active area of research in recent years. Difference of linear positive operators was studied in the last few years. Such problem was initiated by A. Lupas [10]. The operators involved are usually on continuous functions defined on real intervals. Aral-Inoan-Rasa [3], Acu et. al. [1], V. Gupta [6], Gupta-Tachev [7] discovered some interesting results on difference of operators.

The aim of this paper is to study the approximation properties of difference on new generalization of the Baskakov operator. First, we recall classical Baskakov operators [4], which for $f \in C[0,\infty)$ are defined as

$$B_n(f,x) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k (1+x)^{-n-k} f\left(\frac{k}{n}\right). \quad \textbf{(1.1)}$$

First, we consider $F_{n,u}, G_{n,u}: D \to R,$ where D is a subspace of $C[0,\infty)$ having polynomials of degree up to four. Following operators have been defined for the difference of operators

$$M_n(f,x) = \sum_{u=0}^{\infty} p_{n,u} F_{n,u}(f), \;\; L_n(f,x) = \sum_{u=0}^{\infty} p_{n,u} G_{n,u}(f),$$

where $F_{n,u}(e_0) = G_{n,u}(e_0) = 1$.

We can define operators (1.1) as $B_n(f,x) = \sum_{u=0}^{\infty} p_{n,u}(x) F_{n,u}(f) = \sum_{u=0}^{\infty} p_{n,u}(x) f\left(\frac{k}{n}\right), \quad \textbf{(1.2)}$

where $p_{n,u}(x)$ is the Baskakov basis function and defined as $p_{n,u}(x) = \binom{n+k-1}{k} x^k (1+x)^{-n-k}$.

Notations: Throughout the present papers following notations are used $d^F = F(e_1)$, $\mu_k^F = F(e_1 - d^F e_0)^k, k \in N$

Applied operators:

Baskakov- Szasz operators: In [8] The Baskakov Szasz operators are defined as

$$S_n(f,x) = n\sum_{u=0}^{\infty} p_{n,u}(x) \int_0^{\infty} q_{n,u}(t) f(t)dt, \quad \textbf{(1.3)}$$

where $p_{n,u}(x)$ is defined in (1.2) and $q_{n,u}(t) = \frac{e^{-nt}(nt)^u}{u!}$.

Operators (1.3) can also be written as $S_n(f,x) = \sum_{u=0}^{\infty} p_{n,u}(x) G_{n,u}(f)$.

Baskakov Durrmeyer operators: These operators are defined as, see [5]

$$D_n(f,x) = \sum_{u=0}^{\infty} p_{n,u}(x) H_{n,u}(f),$$

where,
$$H_{n,u}(f) = \frac{1}{B(u,n+1)} \int_0^{\infty} \frac{t^{u-1}}{(1+t)^{n+u+1}} f(t)dt$$

2. Basic Results

Here we establish some lemmas and propositions, which are useful for the proof of main theorems

Proposition1: Denoting $F_{n,u}(f) = f(\frac{u}{n})$ such that $F_{n,u}(e_0) = 1$, $d^{F_{n,u}} := F_{n,u}(e_1)$ and considering $\mu_k^{F_{n,u}} = F_{n,u}(e_1 - d^{F_{n,u}} e_0)^k$, $k \in N$, then we have

$$\mu_2^{F_{n,u}} = F_{n,u}(e_1 - d^{F_{n,u}} e_0)^2 = 0$$

$$\mu_2^{F_{n,u}} = F_{n,u}(e_1 - d^{F_{n,u}} e_0)^4 = 0$$

Proposition 2 [6]: Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2$. Let $x \in [0,\infty)$, then for $n \in N$, we get

$$|(M_n - L_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'',\delta_1)}{2}\left(1+\gamma(x)\right) + 2\omega(f,\delta_2),$$

where, $\gamma(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_2^{F_{n,u}} + \mu_2^{G_{n,u}}\right)$

and $\delta_1^2 = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_4^{F_{n,u}} + \mu_4^{G_{n,u}}\right)$, $\delta_2^2 = \sum_{u=0}^{\infty} p_{n,u}(x)(d^{F_{n,u}} - d^{G_{n,u}})^2$.

$C_B[0,\infty)$is the class of continuous and bounded functions defined for $x \ge 0$, $\|\cdot\| = \sup_{x\in[0,\infty)} |f(x)| < \infty$.

Lemma 1: Some moments of the operators discussed in (1.2) are as follows

(i) $B_n(e_0,x) = 1,$
(ii) $B_n(e_1,x) = x,$
(iii) $B_n(e_2,x) = \frac{x}{n} + \frac{(n+1)x^2}{n},$
(iv) $B_n(e_3,x) = \frac{x}{n^2} + \frac{3(n+1)x^2}{n^2} + \frac{(n+1)(n+2)x^3}{n^2},$
(v) $B_n(e_4,x) = \frac{x}{n^3} + \frac{7(n+1)x^2}{n^3} + \frac{6(n+1)(n+2)x^3}{n^3} + \frac{(n+1)(n+2)(n+3)x^4}{n^3}.$

Following recurrence relation holds for these moments

$B_n(e_{m+1},x) = \frac{x(1+x)}{n} B_n'(e_m,x) + xB_n(e_m,x)$.

3. Difference between the operators

This section deals with the quantitative estimates for difference of Baskakov operators with Baskakov-Szasz type operators, Baskakov operators with Baskakov-Durrmeyer operators and with Baskakov-Szasz with Baskakov-Durrmeyer operators.

Proposition 3: By simple calculations as $e_k(t) = t^k$, where $k \in N^\circ$, we obtain

$$G_{n,u}(e_k) = \int_0^\infty q_{n,u}(t) t^k dt = \frac{(u+k)!}{u!\, n^k}$$

Hence $d^{G_{n,u}} = G_{n,u}(e_1) = \frac{u+1}{n}$

and $\mu_2^{G_{n,u}} = G_{n,u}(e_1 - d^{G_{n,u}} e_0)^2 = G_{n,u}(e_2,x) + \left(\frac{u+1}{n}\right)^2 - 2G_{n,u}(e_1,x)\left(\frac{u+1}{n}\right)$

$$= \frac{(u+2)(u+1)}{n^2} - \left(\frac{u+1}{n}\right)^2 = \frac{(u+1)}{n^2}$$

also $\mu_4^{G_{n,u}} = G_{n,u}(e_1 - d^{G_{n,u}} e_0)^4 = G_{n,u}(e_4,x) - 4G_{n,u}(e_3,x)\left(\frac{u+1}{n}\right) + 6G_{n,u}(e_2,x)\left(\frac{u+1}{n}\right)^2$

$$-4G_{n,u}(e_1,x)\left(\frac{u+1}{n}\right)^3 + G_{n,u}(e_0,x)\left(\frac{u+1}{n}\right)^4$$

$$= \frac{(u+4)(u+3)(u+2)(u+1)}{n^4} - 4\frac{(u+3)(u+2)(u+1)}{n^3}\left(\frac{u+1}{n}\right) + 6\frac{(u+2)(u+1)}{n^2}\left(\frac{u+1}{n}\right)^2$$

$$-4\left(\frac{u+1}{n}\right)\left(\frac{u+1}{n}\right)^3 + \left(\frac{u+1}{n}\right)^4 = \frac{3(u^2+4u+3)}{n^4}.$$

Theorem 1: (Difference between Baskakov-Szasz and Baskakov operators)

Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2$. Let $x \in [0,\infty)$, then for $n \in N$, we get

$$|(S_n - B_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'',\delta_1)}{2}\big(1+\gamma(x)\big) + 2\omega(f,\delta_2(x)),$$

where, $\gamma(x) = \frac{nx+1}{n^2}$, $\delta_1^2(x) = \frac{3x^2 n(n+1)+15nx+9}{n^4}$, $\delta_2^2(x) = \frac{1}{n^2}$.

Proof: Following propositions 1, 2 and 3 and also using Lemma 1, we have

$$\gamma(x) = \sum_{u=0}^\infty p_{n,u}(x)\left(\mu_2^{F_{n,u}} + \mu_2^{G_{n,u}}\right) = \sum_{u=0}^\infty p_{n,u}(x)\frac{(u+1)}{n^2}$$

$$= \frac{1}{n}B_n(e_1,x) + \frac{1}{n^2} = \frac{nx+1}{n^2}.$$

$$\delta_1^2(x) = \sum_{u=0}^\infty p_{n,u}(x)\left(\mu_4^{F_{n,u}} + \mu_4^{G_{n,u}}\right)$$

$$= \sum_{u=0}^\infty p_{n,u}(x)\mu_4^{G_{n,u}} = \frac{3x^2 n(n+1) + 15nx + 9}{n^4}$$

$$\delta_2^2(x) = \sum_{u=0}^\infty p_{n,u}(x)(d^{F_{n,u}} - d^{G_{n,u}})^2$$

$$= \sum_{u=0}^\infty p_{n,u}(x)\left[\frac{u}{n} - \frac{u+1}{n}\right]^2 = \frac{1}{n^2}.$$

Collecting above estimates, we get required result.

Proposition 4: By simple calculations with $e_k(t) = t^k, k \in N^\circ$, we obtain

$$H_{n,u}(e_k) = \frac{(u+r-1)!\,(n-k)!}{(u-1)!\,n!}$$

Hence $d^{H_{n,u}} = H_{n,u}(e_1) = \frac{u}{n}$.

$$\mu_2^{H_{n,u}} = H_{n,u}(e_1 - d^{H_{n,u}}e_0)^2 = H_{n,u}(e_2) - 2H_{n,u}(e_1)\left(\frac{u}{n}\right) + H_{n,u}(e_0)\left(\frac{u}{n}\right)^2$$
$$= \frac{u(u+1)}{n^2-n} - \left(\frac{u}{n}\right)^2 = \frac{u(u+n)}{n^2(n-1)},$$

and $\mu_4^{H_{n,u}} = H_{n,u}\left(e_1 - d^{H_{n,u}}e_0\right)^4 = H_{n,u}(e_4,x) - 4H_{n,u}(e_3,x)\left(\frac{u}{n}\right) + 6H_{n,u}(e_2,x)\left(\frac{u}{n}\right)^2$

$$-4H_{n,u}(e_1,x)\left(\frac{u}{n}\right)^3 + H_{n,u}(e_0,x)\left(\frac{u}{n}\right)^4$$
$$= \frac{(u+3)(u+2)(u+1)u}{n(n-1)(n-2)(n-3)} - 4\frac{(u+2)(u+1)u}{n(n-1)(n-2)}\left(\frac{u}{n}\right) + 6\frac{(u+1)u}{n(n-1)}\left(\frac{u}{n}\right)^2$$
$$-4\left(\frac{u}{n}\right)\left(\frac{u}{n}\right)^3 + \left(\frac{u}{n}\right)^4 = \frac{3\left(u^4(n+6)+2nu^3(n+6)+n^2u^2(n+8)+2n^3u\right)}{n^4(n-1)(n-2)(n-3)}.$$

Theorem 2: (Difference between Baskakov-Durrmeyer and Baskakov operators)

Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2$. Let $x \in [0,\infty)$, then for $n \in N$, we get

$$|(D_n - B_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'',\delta_1)}{2}\left(1+\gamma(x)\right),$$

where, $\gamma(x) = \frac{(n+1)x(1+x)}{n(n-1)}$ and

$$\delta_1^2(x) = \frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}[n^3x(x+1) + n^2(11x^2+11x+3) + n(36x^2+36x+7) + 6(6x^2 + 6x+1)].$$

Proof: Using Propositions 1, 3 and Lemma 1, we have the results

$$\gamma(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_2^{F_{n,u}} + \mu_2^{H_{n,u}}\right) = \frac{(n+1)(n+2)x}{n(n-1)}.$$

$$\delta_1^2(x) = \sum_{u=0}^{\infty} p_{n,u}(x)(\mu_4^{F_{n,u}} + \mu_4^{H_{n,u}}) = \sum_{u=0}^{\infty} p_{n,u}(x)\mu_4^{H_{n,u}} = \frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}$$

$[n^3x(x+1) + n^2(11x^2+11x+3) + n(36x^2+36x+7) + 6(6x^2+6x+1)]$.

Following proposition 2, we get the result.

Theorem 3: (Difference between Baskakov-Szasz and Baskakov Durrmeyer operators)

Let $f^{(r)} \in C_B[0,\infty), r = 0,1,2$. Let $x \in [0,\infty)$, then for $n \in N$, we get

$$|(S_n - D_n)f(x)| \le \frac{\|f''\|}{2}\gamma(x) + \frac{\omega(f'',\delta_1)}{2}\left(1+\gamma(x)\right) + 2\omega(f,\delta_2(x)),$$

where, $\gamma(x) = \frac{nx+1}{n^2} + \frac{(n+1)x(1+x)}{n(n-1)}$,

$\delta_1^2(x) = \frac{3x^2n(n+1)+15nx+9n}{n^4} + \frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}[n^3x(x+1) + n^2(11x^2+11x+3) + n(36x^2+36x+7) + 6(6x^2+6x+1)]$,

$\delta_2^2(x) = \frac{1}{n^2}$.

Proof: According to Lemma 1, Propositions 2, 3 and 4, we have

$$\gamma(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_2^{G_{n,u}} + \mu_2^{H_{n,u}}\right) = \frac{nx+1}{n^2} + \frac{(n+1)x(1+x)}{n(n-1)}$$

$$\delta_1^2(x) = \sum_{u=0}^{\infty} p_{n,u}(x)\left(\mu_4^{G_{n,u}} + \mu_4^{H_{n,u}}\right) = \frac{3x^2n(n+1)+15nx+9n}{n^4}$$

$$+\frac{3(n+1)x(x+1)}{n^3(n-1)(n-2)(n-3)}[n^3x(x+1) + n^2(11x^2+11x+3) + n(36x^2+36x+7) + 6(6x^2+6x+1)]$$

$$\delta_2^2(x) = \sum_{u=0}^{\infty} p_{n,u}(x)(d^{G_{n,u}} - d^{H_{n,u}})^2 = \frac{1}{n^2}.$$

Combining the estimates, according to proposition 2, we get the result.

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