

## SPECTRAL RADIUS OF A NORMAL OPERATOR

**Abstract**

The spectral radius  $r(T)$  of an operator  $T$  clearly satisfies the relation  $0 \leq r(T) \leq \|T\|$ . The spectral mapping theorem implies that  $r(T^n) = (r(T))^n$  for every positive integer  $n$ . It frequently turns out that it is easy to compute the spectral radius of an operator even if it is hard to find the spectrum. This is often made easy by the spectral radius formula. Let  $H$  be a Hilbert space and  $T$  be a bounded linear operator in  $H$ . In this paper we show that if  $T$  is normal, then  $T^n$  is normal for each  $n \in \mathbb{N}$  and  $\|T^n\| = \|T\|^n$ . Consequently, we use the spectral radius formula to show that  $r(T) = \|T\|$ . Moreover, we show that if  $X$  is a Complex Banach space and  $T$  is bounded in  $X$  then there is a  $\lambda$  belonging to the spectrum of  $T$  such that  $|\lambda| = r(T)$ . Let  $H$  be a Complex Hilbert space and  $T$  be a bounded operator in  $H$  which is normal; we show that  $\|T\| = \sup\{|Tx, x| : x \in H \text{ and } \|x\| = 1\}$  and the residual spectrum of  $T$  is void.

**Subject Classification:** xxxxxx

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# 1 Introduction

Important results in spectral theory can be found by studying the power series and Laurent developments ([?] and [?]) of the resolvent  $(\lambda I - T)^{-1}$ . The best theorems relate to the Neumann expansion ([?] and [?]) of the resolvent

$$(\lambda I - T)^{-1} = R(\lambda) = \sum_{n=1}^{\infty} \lambda^{-n} T^{n-1}$$

which is valid whenever the series is convergent in the uniform topology. By extension of a theorem of a classical function theory [?], about the radius of convergence of a power series, we have convergence of the Neumann series if  $|\lambda| > r(T)$  and divergence if  $|\lambda| < r(T)$ , where

$$r(T) = \lim_n \sup \|T^n\|^{\frac{1}{n}}.$$

The number  $r(T)$  is called the spectral radius of  $T$  and this equation was first proved by Gelfand[?] in relation to Banach Algebra. Krein and Rutmann [?] further obtained the result that if  $T$  is a compact and positive cone which is total, and if the spectral radius of  $T$  is positive, then  $r(T)$  is a pole of the resolvent.

In this paper we will use the spectral radius formulae to show that the spectral radius of a normal operator is equal to its operator norm. Most definitions in this paper can be found in [?], [?], [?] and [?]

## 1.1 Some results on spectra

**Proposition 1.** *Let  $X$  be a Complex Banach space and  $T \in B(X)$  if  $S \in B(X)$  is invertible then  $\sigma(T) = \sigma(S^{-1}TS)$ .*

*Proof.* We need to prove that  $\rho(T) = \rho(S^{-1}TS)$ . Let  $\lambda \in \rho(T)$ . Hence  $(\lambda I - T)^{-1}$  exists and is in  $B(X)$ . Now,

$$S^{-1}(\lambda I - T)S = S^{-1}(\lambda S - TS) = \lambda I - S^{-1}TS.$$

Note that  $S^{-1}, T, S \in B(X) \Rightarrow S^{-1}TS \in B(X)$ . Since  $S$  is invertible,  $S^{-1} \in B(X)$  and  $(S^{-1})^{-1} = S \in B(X)$ .

The product  $S^{-1}(\lambda I - T)S$  being a product of invertible operators is invertible. Hence

$$\begin{aligned} \lambda I - S^{-1}TS &\text{ is invertible} \\ \Rightarrow \lambda &\in \rho(S^{-1}TS). \end{aligned}$$

Thus

$$\rho(T) \subseteq \rho(S^{-1}TS) \tag{1.1}$$

It remains to establish the reverse inclusion which is a consequence of (??) itself.

For consider the operator  $S^{-1}TS \in B(X)$  instead of  $T$ . Then by (??)

$$\begin{aligned} \rho(S^{-1}TS) &\subseteq \rho((S^{-1})^{-1}(S^{-1}TS)(S^{-1})) \\ \text{i.e } \rho(S^{-1}TS) &\subseteq \rho(T) \end{aligned}$$

Thus  $\rho(S^{-1}TS) = \rho(T)$ . Taking complements we get  $\sigma(S^{-1}TS) = \sigma(T)$ . □

**Proposition 2.** *Let  $X$  be a Banach space and  $p$  be a polynomial with complex coefficients. Let  $T \in B(X)$ . Then*

$$\sigma(p(T)) = p(\sigma(T))$$

where  $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$

**Elucidation**

Suppose  $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_n\lambda^n$  where  $a_0, a_1, a_2, \dots, a_n \in \mathbb{C}$  and  $\lambda$  is an indeterminate if  $T \in B(X)$ , then  $p(T)$  stands for the bounded linear operator.

$$a_0I + a_1T + a_2T^2 + \dots + a_nT^n$$

where  $I$  is the identity operator on  $X$ .

So  $\sigma(p(T))$  is the spectrum of the bounded linear operator  $p(T)$ .

Whereas  $p(\sigma(T))$  is the set  $\{p(\lambda) : \lambda \in \sigma(T)\}$  of complex numbers.

This result is a restricted form of the famous result known as the **Spectral Mapping Theorem** [?].

**Application**

Suppose  $p(\lambda) = \lambda^n$  and  $T \in B(X)$ . Therefore  $p(T) = T^n$ .  $T \in B(X) \Rightarrow T^n \in B(X)$ . By the Spectral Mapping Theorem[?] we get.

$$\begin{aligned} \sigma(T^n) &= p(\sigma(T)) = \{\lambda^n : \lambda \in \sigma(T)\} \\ &= (\sigma(T))^n \end{aligned}$$

(where  $(\sigma(T))^n$  stands for the set  $\{\lambda^n : \lambda \in \sigma(T)\}$ )

In the general form of the spectral mapping theorem, the polynomial  $p$  may be replaced by a continuous or even measurable function.

*Proof of Proposition ??.* Let  $\lambda_o \in \mathbb{C}$ . Consider the equation  $p(\lambda) - p(\lambda_o) = 0$  we can write

$p(\lambda) - p(\lambda_o) = (\lambda - \lambda_o)q(\lambda)$  identically in  $\lambda$ .  $q(\lambda)$  being a polynomial  $(p(\lambda) - p(\lambda_o))$  vanishes identically when  $\lambda = \lambda_o$  therefore,  $\lambda_o$  is a root of the polynomial  $p(\lambda) - p(\lambda_o)$  therefore  $p(\lambda) - p(\lambda_o) = (\lambda - \lambda_o)q(\lambda)$ .

Hence

$$p(T) - Ip(\lambda_o) = (T - \lambda_oI)q(T)$$

where  $T$  is the given operator.

If  $\lambda_o \in \sigma(T)$ , then  $(T - \lambda_oI)$  is not invertible. Hence  $S = (T - \lambda_oI)q(T)$  is not invertible (for assume the converse, that is, if  $S$  is invertible then

$$S^{-1}(T - \lambda_oI)q(T) = I = (T - \lambda_oI)q(T)S$$

and this would require that  $T - \lambda_oI$  should be invertible that is,  $\lambda_o \notin \sigma(T)$ , a contradiction!).

Hence  $p(T) - p(\lambda_o)I$  is not invertible and consequently  $p(\lambda_o) \in \sigma(p(T))$ .

Since  $\lambda_o \in \sigma(T)$  so  $p(\lambda_o) \in p\sigma(T)$ . Thus

$$p(\sigma(T)) \subseteq \sigma(p(T)).$$

We need to establish the reverse inclusion

$$\sigma(p(T)) \subseteq p(\sigma(T)).$$

Let  $\lambda_o \in \sigma(p(T))$  so  $p(T) - \lambda_o I$  is not invertible.

Let  $\deg p(t) = n$  over  $\mathbb{K}$  and let  $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$  be the  $n$  zeros of the polynomial  $p(t) - \lambda_o$  (These zeros need not be distinct). We can write

$$p(t) - \lambda_o = \alpha(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

where  $\alpha$  is the non-zero element of  $\mathbb{K}$ .

Accordingly,

$$p(T) - \lambda_o I = \alpha(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)$$

For at least one  $j, (1 \leq j \leq n), (T - \lambda_j I)$  must be non-invertible (since if  $T - \lambda_k I$  was invertible  $\forall k = 1, 2, 3, \dots, n$  then the product  $\alpha(T - \lambda_1 I)(T - \lambda_2 I) \cdots (T - \lambda_n I)$  would be invertible, that is,  $p(T) - \lambda_o I$  would be invertible, a contradiction to the hypothesis:

$$(\lambda_o \in \sigma(p(T))).$$

Thus  $\lambda_j \in \sigma(T)$ . Therefore

$$p(\lambda_j) \in p(\sigma(T)) \text{ but } p(\lambda_j) = \lambda_o \text{ (for } \lambda_j \text{ is a root of } p(t) - \lambda_o \text{)}.$$

Thus

$$\lambda_o \in p(\sigma(T)).$$

Thus

$$\sigma(p(T)) \subseteq p(\sigma(T))$$

Hence

$$\sigma(p(T)) = p(\sigma(T)). \quad \square$$

**Proposition 3.** *Let  $X$  be a Complex Banach space and  $T \in B(X)$  which is invertible. Then  $\sigma(T^{-1}) = \{\sigma(T)\}^{-1}$  where*

$$\{\sigma(T)\}^{-1} = \{\lambda^{-1} : \lambda \in \sigma(T)\}$$

**Remark 1.** *{Since  $T$  is invertible,  $0 \notin \sigma(T)$  (for  $0 \in \sigma(T) \Leftrightarrow T - 0I$  is not invertible  $\Rightarrow T$  is not invertible. Therefore,  $T$  is invertible  $\Rightarrow 0 \notin \sigma(T)$ ). Hence if  $\lambda \in \sigma(T), \lambda \neq 0$  consequently  $\frac{1}{\lambda}$  exists  $\forall \lambda \in \sigma(T)$  and  $\{\sigma(T)\}^{-1}$  is meaningful}*

*Proof.* Suppose  $\lambda \in \sigma(T)$  (so  $\lambda \neq 0$ )

Now

$$\begin{aligned} \lambda I - T &= (\lambda T^{-1} T - T) \\ &= \lambda(T^{-1} - \lambda^{-1} I)T \end{aligned}$$

Since  $\lambda I - T$  is not invertible and  $T$  is invertible and  $\lambda \neq 0$ , so  $(T^{-1} - \lambda^{-1} I)$  must be non invertible, consequently,  $\lambda^{-1} \in \sigma(T^{-1})$  that is

$$\lambda \in \{\sigma(T^{-1})\}^{-1}.$$

Thus

$$\sigma(T) \subseteq \{\sigma(T^{-1})\}^{-1}$$

or

$$\{\sigma(T^{-1})\}^{-1} \subseteq \sigma(T) \tag{1.2}$$

Applying the same result to  $T^{-1}$  in place of  $T$  (Note  $T^{-1}$  is invertible and  $(T^{-1})^{-1} = T$ ). We get

$$\{\sigma(T^{-1})\}^{-1} \subseteq \{\sigma((T^{-1})^{-1})\} = \sigma(T) \tag{1.3}$$

(??) implies  $\sigma(T^{-1}) \supseteq \{\sigma(T)\}^{-1}$ .

(??) implies  $\sigma(T^{-1}) \subseteq \{\sigma(T)\}^{-1}$

that is

$$\sigma(T^{-1}) = \{\sigma(T)\}^{-1}. \tag{1.4}$$

□

**Proposition 4.** *Let  $H$  be a Complex Hilbert space and  $T \in B(H)$ .*

*Then*

$$\begin{aligned} \sigma(T^*) &= \overline{\sigma(T)} \\ \text{where } \overline{\sigma(T)} &= \{\bar{\lambda} : \lambda \in \sigma(T)\} \end{aligned}$$

*Proof.* Suppose  $\lambda \in \rho(T)$ . Then  $T - \lambda I$  is invertible. Now  $(T - \lambda I)^* = T^* - \bar{\lambda} I$  and from the result that  $A \in B(H)$  is invertible if and only if  $A^*$  is invertible. We note that  $T^* - \bar{\lambda} I$  is also invertible. Thus

$$\bar{\lambda} \in \rho(T^*).$$

So  $\lambda \in \rho(T) \Rightarrow \bar{\lambda} \in \rho(T^*).$

$$\Rightarrow \lambda \in \overline{\rho(T^*)}.$$

$$\rho(T) \subseteq \overline{\rho(T^*)} \tag{1.4}$$

Applying the result to  $T^*$  in place of  $T$  and noting that  $T^{**} = T$  we get that

$$\rho(T^*) \subseteq \overline{\rho(T)}. \tag{1.5}$$

But (??)  $\Rightarrow$

$$\overline{\rho(T)} \subseteq \rho(T^*). \tag{1.6}$$

(??) and (??)  $\Rightarrow \rho(T^*) = \overline{\rho(T)}$ .

Take complements with respect to  $\mathbb{C}$  to get the required result. □

## 1.2 Spectral radius

**Definition 1.** *Let  $X$  be a normed linear space and  $T \in B(X)$ . The number  $\sup\{|\lambda| : \lambda \in \sigma(T)\}$  (note  $\sigma(T) \neq \emptyset$ ) is called the **spectral radius** of  $T$  and represented by  $r(T)$*

It follows that  $r(T) \leq \|T\|$

Reason:

$$\lambda I - T = \lambda \left( I - \frac{T}{\lambda} \right) \text{ if } \lambda \neq 0 \tag{1.7}$$

Then  $\lambda I - T$  is invertible if and only if  $I - \frac{T}{\lambda}$  is invertible. {Since  $X$  is a Banach space,

we know that  $I - T$  is invertible if  $\|T\| \leq 1$  [?].

Looking at (??,) we conclude that  $I - \frac{T}{\lambda}$  is invertible if  $\|\frac{T}{\lambda}\| < 1$ , that is, if  $|\lambda| > \|T\|$ . Hence if  $\lambda \in \sigma(T)$ ; then  $|\lambda| \leq \|T\|$ . Consequently  $\sup\{|\lambda| : \lambda \in \sigma(T)\} \leq \|T\|$ .

That is,

$$r(T) \leq \|T\|.$$

The next result proposition ?? is proved for bounded linear operators in the Hilbert space context.

**Proposition 5.** *Let  $H$  be a complex Hilbert space and  $T \in B(H)$ .*

*Then*

$$\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \text{ exists and equals } r(T).$$

*Proof.* It follows from the Spectral Mapping Theorem that

$$\{r(T)\}^n = r(T^n) \forall n \in \mathbb{N}.$$

To see this, consider the polynomial  $p(t) = t^n$  and use the Spectral Mapping Theorem, VIZ.

$$\sigma(p(T)) = p(\sigma(T)) \forall T \in B(H)$$

that is

$$\sigma(T^n) = (\sigma(T))^n \text{ where } (\sigma(T))^n = \{\lambda^n : \lambda \in \sigma(T)\}$$

therefore

$$\begin{aligned} r(T^n) &= \sup\{|\mu| : \mu \in \sigma(T^n)\} \\ &= \sup\{|\lambda^n| : \lambda \in \sigma(T)\} \text{ for } \lambda(T^n) = (\sigma(T))^n \\ &= \sup\{|\lambda|^n : \lambda \in \sigma(T)\} \\ &= \sup\{|\lambda| : \lambda \in \sigma(T)\}^n \\ &= (r(T))^n. \end{aligned}$$

Thus (note  $r(T) \geq 0$ )

$$r(T) = \{r(T^n)\}^{\frac{1}{n}} \forall n \in \mathbb{N}.$$

We have already seen that for any  $A \in B(H)$ ,

$$r(A) \leq \|A\|.$$

Since  $T^n \in B(H)$  so  $r(T^n) \leq \|T^n\|$  and hence we get

$$r(T) \leq \|T^n\|^{\frac{1}{n}}$$

Letting  $n \rightarrow \infty$  the last line implies that

$$r(T) \leq \liminf \|T^n\|^{\frac{1}{n}} \tag{1.8}$$

(We are just considering the sequence of reals  $(\|T^n\|^{\frac{1}{n}})_{n=1}^{\infty}$  on the right hand side and hence it is relevant to invoke the limit infirmum of this sequence without consideration of the convergence of the latter).

Let  $\lambda \neq 0$  and  $\frac{1}{\lambda} \in \rho(T)$ . Now  $\frac{1}{\lambda}I - T = \frac{1}{\lambda}(I - \lambda T)$ . If  $\|\lambda T\| \leq 1$  then  $\lambda I - T$  is invertible and consequently  $\frac{1}{\lambda}$  would be in  $\rho(T)$  as asserted.

In such a situation the mapping, given any  $x, y \in H$ ,

$\lambda \mapsto \lambda \langle (I - \lambda T)^{-1}x, y \rangle$   
 would be analytic for all  $\lambda$  satisfying  $\frac{1}{|\lambda|} > r(T)$ . But

$$\lambda \langle (I - \lambda T)^{-1}x, y \rangle = \lambda \langle \sum_{n=0}^{\infty} (\lambda T)^n x, y \rangle = \lambda \sum_{n=0}^{\infty} \langle (\lambda T)^n x, y \rangle \tag{1.9}$$

for

$$(I - \lambda T)^{-1} = \sum_{n=0}^{\infty} (\lambda T)^n \text{ (Neumann series)}$$

therefore

$$(I - \lambda T)^{-1}x = \sum_{n=0}^{\infty} (\lambda T)^n x \text{ (strong convergence in } H \text{ on the right hand side).}$$

The convergence on the right hand side of (??) implies that the sequence of scalars

$$\langle (\lambda T)^n x, y \rangle_{n=1}^{\infty} \text{ (for any } x, y \in H \text{ fixed)}$$

must be bounded. Considering the functionals represented by the elements  $(\lambda T)^n x \in H$  we note that this sequence of functionals is bounded pointwise and hence as a consequence of the uniform boundedness principle ([?] and [?]) it follows that there exists a real  $M > 0$  such that

$$|\lambda|^n \|T^n\| \leq M \quad \forall n \in \mathbb{N}$$

for  $\lambda$  satisfying  $|\lambda| \leq \frac{1}{r(T)}$ .

**{Explanation**

$f_n y = \langle y, (\lambda T)^n x \rangle \Rightarrow \|f_n\| = (\|(\lambda T)^n x\|)_{n=1}^{\infty}$  is bounded at each  $x \in H \Rightarrow \|(\lambda T)^n\|$  is bounded  $\Rightarrow (\|\lambda^n T^n\|)$  is bounded  $\Rightarrow (|\lambda|^n \|T^n\|)$  is bounded}.

Hence  $|\lambda| \|T^n\|^{\frac{1}{n}} \leq M^{\frac{1}{n}} \forall \lambda$  and  $\forall \lambda$  satisfying  $|\lambda| \leq \frac{1}{r(T)}$ . Let  $n \rightarrow \infty$ , we know that

$$M^{\frac{1}{n}} \rightarrow 1 \text{ (from calculus)}$$

Hence

$$|\lambda| \limsup \|T^n\|^{\frac{1}{n}} \leq 1$$

since this is true  $\forall |\lambda| \leq \frac{1}{r(T)}$  we obtain

$$\frac{1}{r(T)} \limsup \|T^n\|^{\frac{1}{n}} \leq 1$$

that is

$$\limsup \|T^n\|^{\frac{1}{n}} \leq r(T) \tag{1.10}$$

Thus

$$r(T) \leq \liminf \|T^n\|^{\frac{1}{n}} \leq \limsup \|T^n\|^{\frac{1}{n}} \leq r(T)$$

hence

$$\liminf \|T^n\|^{\frac{1}{n}} = \limsup \|T^n\|^{\frac{1}{n}} = r(T),$$

that is

$$\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \text{ exists and equals } r(T). \quad \square$$

**Definition 2.** Let  $X$  be a normed linear space and  $T$  be a linear operator in  $X$ . A point  $\lambda \in \mathbb{K}$  (the underlying field) is called an **approximate eigenvalue** for  $T$  if for each real  $\epsilon > 0$  there is an  $x \in \mathfrak{D}_T$  such that  $x \neq 0$  and

$$\|(T - \lambda I)x\| \leq \epsilon \|x\|.$$

The collection of all approximate eigenvalues of  $T$  is called the **approximate point spectrum** of  $T$  and denoted by  $\Pi(T)$ .

Alternatively,  $\lambda \in \mathbb{K}$  is called the approximate eigenvalue of  $T$  if there exists a sequence  $(x_n)$  of elements of  $\mathfrak{D}_T$  such that  $\|x_n\| = 1 \forall n \in \mathbb{N}$  and

$$\|(T - \lambda I)x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

It is clear that, if  $\lambda$  is an eigenvalue, then  $\lambda \in \Pi(T)$ . For in this case there is an  $x \in \mathfrak{D}_T$  such that  $x \neq \bar{0}$  and  $(T - \lambda I)x = \bar{0}$ . Hence

$$\|(T - \lambda I)x\| = \|\bar{0}\| = 0.$$

Thus for any  $\epsilon > 0$  that we choose

$$\|(T - \lambda I)x\| < \epsilon \|x\|$$

this implies

$$\lambda \in \Pi(T).$$

Thus  $P\sigma(T) \subseteq \Pi(T)$ .

**Proposition 6.** *Let  $X$  be a Banach space and  $T \in B(X)$  then the following statements are equivalent*

(i)  $\lambda \in \Pi(T)$

(ii)  $\lambda \in P\sigma(T)$  or (if  $\lambda \notin P\sigma(T)$ ) then  $\lambda I - T$  has an unbounded inverse on the  $\mathfrak{R}_{\lambda I - T}$

*Proof.* i)  $\Rightarrow$  ii). Since  $P\sigma(T) \subseteq \Pi(T)$  a  $\lambda \in \Pi(T)$  could belong to  $P\sigma(T)$ . If  $\lambda \notin P\sigma(T)$  then  $(\lambda I - T)x \neq \bar{0}$  for all nonzero  $x \in \mathfrak{D}_T$ . Since  $\lambda \in \Pi(T)$  it follows that for each  $n \in \mathbb{N}$  there is  $x_n \in \mathfrak{D}_T$  such that  $\|x_n\| = 1$  and  $\|(\lambda I - T)x_n\| \leq \frac{1}{n}$  for all  $n \in \mathbb{N}$ . Hence we cannot find an  $\epsilon > 0$  such that

$\|(\lambda I - T)x\| < \epsilon \|x\|$  for all nonzero  $x \in \mathfrak{D}_T$ . Hence  $\lambda I - T$  is not bounded from below. Since  $\lambda \notin P\sigma(T)$  it follows that  $(\lambda I - T)^{-1}$  exists on  $\mathfrak{R}_{\lambda I - T}$  and the result is that  $\lambda I - T$  is not bounded from below implies that  $(\lambda I - T)^{-1}$  is unbounded.

ii)  $\Rightarrow$  (i): If  $\lambda \in P\sigma(T)$  then  $\lambda \in \Pi(T)$  for  $P\sigma(T) \subseteq \Pi(T)$ .

If on the other hand  $\lambda \notin P\sigma(T)$  then  $\lambda I - T$  has an inverse on the  $\mathfrak{R}_{\lambda I - T}$ . Since  $(\lambda I - T)^{-1}$  is unbounded, it follows that  $\lambda I - T$  is not bounded from below. Hence there is no  $\epsilon > 0$  such that

$$\|(\lambda I - T)x\| \geq \epsilon \|x\| \text{ for all nonzero } x \in \mathfrak{D}_T.$$

Thus we can find a sequence  $(x_n)$  of elements of the  $\mathfrak{D}_T$  of norm 1 such that

$$\|(\lambda I - T)x_n\| \rightarrow 0 \text{ that is } \lambda \in \Pi(T).$$

(If  $X$  is a Banach space and  $T \in B(X)$  then  $T$  has a bounded inverse on  $\mathfrak{R}_T$  iff  $T$  is bounded from below) □

It follows from proposition ?? that:

**Corollary 1.1.**  $\Pi(T) \subseteq \sigma(T)$

*Proof.* If  $\lambda \in \Pi(T)$  then  $\lambda \in P\sigma(T)$  or  $(\lambda I - T)$  has an unbounded inverse. If  $\lambda \in P\sigma(T)$  then  $\lambda \in \sigma(T)$ . If  $(\lambda I - T)$  has an unbounded inverse then  $\lambda \in R\sigma(T)$  or  $C\sigma(T)$ .

That is  $\lambda \in \sigma(T)$ .

Thus  $\Pi(T) \subseteq \sigma(T)$ . □

**Corollary 1.2.** *If  $T \in B(X)$ , then  $|\lambda| \leq \|T\| \forall \lambda \in \Pi(T)$ .*

*We have already seen that  $|\lambda| \leq \|T\| \forall \lambda \in \sigma(T)$ .*

*Since  $\Pi(T) \subseteq \sigma(T)$  we get*

$$|\lambda| \leq \|T\| \forall \lambda \in \Pi(T).$$

**Proposition 7.** *Let  $X$  be a Complex Banach space and  $T \in B(X)$ . Then  $\Pi(T)$  is a compact subset of  $\mathbb{C}$ .*

*Proof.* That  $\Pi(T)$  is a bounded subset of  $\mathbb{C}$  is immediate from the result that

$$|\lambda| \leq \|T\| \text{ for all } \lambda \in \Pi(T).$$

We shall show that  $(\Pi(T))^c$  is open in  $\mathbb{C}$ . Let  $\lambda_o \in (\Pi(T))^c$  that is,  $\lambda_o$  is not an approximate eigenvalue for  $T$ , that is, there is an  $\epsilon_o > 0$  such that for all non-zero  $x \in X (= \mathfrak{D}_T)$  we have

$$\begin{aligned} \|(\lambda_o I - T)x\| &\geq \epsilon_o \|x\| \\ \text{i.e } \|(\lambda_o I - T)x\| &\geq \epsilon_o \end{aligned}$$

for all unit vectors  $x \in X$ .

Take a  $\lambda$  such that  $|\lambda - \lambda_o| \leq \frac{\epsilon_o}{2}$ .

Then for all  $x \in X$  with  $\|x\| = 1$

$$\begin{aligned} \|(\lambda I - T)x\| &= \|(\lambda_o I - T)x - (\lambda_o - \lambda)x\| \\ &\geq \|(\lambda_o I - T)x\| - |\lambda_o - \lambda| \|x\| \\ &\geq \|(\lambda_o I - T)x\| - |\lambda_o - \lambda| \\ &\geq \epsilon_o - \frac{\epsilon_o}{2} = \frac{\epsilon_o}{2} \end{aligned}$$

This shows that  $\lambda \in (\Pi(T))^c$

Thus for  $\lambda_o \in (\Pi(T))^c$  there exists an open neighbourhood  $N(\lambda_o, \frac{\epsilon}{2})$  such that

$$N(\lambda_o, \frac{\epsilon}{2}) \subseteq (\Pi(T))^c.$$

Therefore  $(\Pi(T))^c$  is open in  $\mathbb{C}$  that is  $\Pi(T)$  is closed and bounded  $\Leftrightarrow \Pi(T)$  is a compact subset of  $\mathbb{C}$  (In Euclidean metric spaces  $\mathbb{R}^n, \mathbb{C}^n$  compactness  $\Leftrightarrow$  closedness and boundedness ([?], [?] and [?])).  $\square$

Note: For  $T \in B(X)$  ( $X$  is a Complex Banach space)  $\sigma(T)$  is a compact subset of  $\mathbb{C}$ . For  $\rho(T)$  is open, so  $\sigma(T)$  is closed and  $|\lambda| \leq \|T\| \forall \lambda \in \sigma(T)$  implies spectrum of  $T$  is bounded.  $\sigma(T)$  is bounded and closed  $\Leftrightarrow \sigma(T)$  is a compact subset of  $\mathbb{C}$ .

**Proposition 8.** *Let  $H$  be a Hilbert space and  $T \in B(H)$ . Then the following statements are equivalent.*

(i) *There is a  $\lambda \in \Pi(T)$  such that  $|\lambda| = \|T\|$*

(ii)  *$\|T\| = \sup\{ | \langle Tx, x \rangle | : x \in H \text{ and } \|x\| = 1 \}$*

*Proof.* i)  $\rightarrow$  ii) Since  $\lambda \in \Pi(T)$ , it follows that there is a sequence  $(x_n)$  of vectors of unit norm in  $H$  such that

$$\|(T - \lambda I)x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now,

$$\begin{aligned} |\lambda - \langle Tx, x \rangle| &= |\lambda \langle x_n, x_n \rangle - \langle Tx_n, x_n \rangle| \\ &= |\langle \lambda x_n - Tx_n, x_n \rangle| = |\langle (\lambda I - T)x_n, x_n \rangle| \\ &\leq \|(\lambda I - T)x_n\| \|x_n\| \text{ (by C.B.S)} \\ &= \|(\lambda I - T)x_n\| \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

Therefore  $\langle Tx_n, x_n \rangle \rightarrow \lambda$  as  $n \rightarrow \infty$ .

Thus the sequence  $(\langle Tx_n, x_n \rangle)$  is convergent in  $\mathbb{C}$  and hence is bounded.

Moreover by continuity of  $|\cdot| : \mathbb{C} \rightarrow \mathbb{R}$  it follows that

$|\langle Tx_n, x_n \rangle| \rightarrow |\lambda|$ . Take  $|\lambda| = \|T\|$  (since (i) is hypothesis). That is,  $(x_n)$ 's are such that  $\|x_n\| = 1$  for all  $n \in \mathbb{N}$  and  $|\langle Tx_n, x_n \rangle| \rightarrow \|T\|$ .

Since  $\forall x \in H$  with  $\|x\| = 1$ , we have

$$|\langle Tx, x \rangle| \leq \|Tx\| \|x\| \leq \|T\| \|x\|^2 = \|T\|.$$

It follows that

$$\{|\langle Tx, x \rangle| : x \in H \text{ and } \|x\| = 1\} \text{ is bounded above.}$$

Let

$$M = \sup\{|\langle Tx, x \rangle| : x \in H \text{ and } \|x\| = 1\}$$

clearly  $M \leq \|T\|$ . So

$$\|T\| \geq M = \sup\{|\langle Tx, x \rangle| : x \in H \text{ and } \|x\| = 1\} \geq \sup\{|\langle Tx_n, x_n \rangle|\}.$$

It follows from this that  $\|T\| = M$  (This is seen thus, since  $|\langle Tx_n, x_n \rangle| \rightarrow \|T\|$ , we can choose a subsequence  $(x_{n_k})$  of  $(x_n)$  such that

$$\sup\{|\langle Tx_{n_k}, x_{n_k} \rangle|\} = \|T\|$$

ii)  $\Rightarrow$  i).

Let  $\|T\| = \sup\{|\langle Tx, x \rangle| : x \in H \text{ and } \|x\| = 1\}$ . Hence there is a sequence  $(x_n)$  of unit vectors in  $H$  such that

$$|\langle Tx_n, x_n \rangle| \rightarrow \|T\| \text{ as } n \rightarrow \infty$$

Therefore the sequence  $(\langle Tx_n, x_n \rangle)$  is bounded in  $\mathbb{C}$ . Hence by the Bolzano Weirstrass Theorem ([?], [?] and [?]) there is a subsequence  $(x_{n_k})$  of  $(x_n)$  such that

$\langle Tx_{n_k}, x_{n_k} \rangle$  converges in  $\mathbb{C}$  to say  $\lambda_o$  that is

$$\langle Tx_{n_k}, x_{n_k} \rangle \rightarrow \lambda_o. \text{ By continuity of absolute value function } (|\cdot|) \text{ we get}$$

$$\begin{aligned} |\langle Tx_{n_k}, x_{n_k} \rangle| &\rightarrow |\lambda_o| \\ \text{Since } |\langle Tx_{n_k}, x_{n_k} \rangle| &\rightarrow \|T\| \text{ as } n \rightarrow \infty \text{ so} \\ |\langle Tx_{n_k}, x_{n_k} \rangle| &\rightarrow \|T\| \end{aligned}$$

But  $|\langle Tx_{n_k}, x_{n_k} \rangle| \rightarrow |\lambda_o|$ . Hence by uniqueness of the limit we get

$$|\lambda_o| = \|T\|.$$

Now we shall compute

$$\begin{aligned}
 0 \leq \|(T - \lambda_o)x_{n_k}\|^2 &= \langle Tx_{n_k} - \lambda_o x_{n_k}, Tx_{n_k} - \lambda_o x_{n_k} \rangle \\
 &= \langle Tx_{n_k}, Tx_{n_k} \rangle - \lambda_o \langle x_{n_k}, Tx_{n_k} \rangle - \overline{\lambda_o} \langle Tx_{n_k}, x_{n_k} \rangle + |\lambda_o|^2 \\
 &= \|Tx_{n_k}\|^2 - \lambda_o \langle x_{n_k}, Tx_{n_k} \rangle - \overline{\lambda_o} \langle Tx_{n_k}, x_{n_k} \rangle + |\lambda_o|^2 \\
 &\leq |\lambda_o|^2 - |\lambda_o|^2 - |\lambda_o|^2 + |\lambda_o|^2 \text{ (for } \|Tx_{n_k}\|^2 \leq \|T\|^2 \|x_{n_k}\|^2 \leq \|T\|^2 = |\lambda_o|^2) = 0
 \end{aligned}$$

Thus,

$$\|(T - \lambda_o)x_{n_k}\| \rightarrow 0 \text{ as } k \rightarrow \infty$$

Therefore

$$\lambda_o \in \Pi(T).$$

Thus there is a  $\lambda_o \in \Pi(T)$  such that  $|\lambda_o| = \|T\|$  and hence i) is proved.  $\square$

**Proposition 9.** *Let  $H$  be a Hilbert space and  $T \in B(H)$  be normal then  $\Pi(T) = \sigma(T)$*

*Proof.* We have seen that  $\Pi(T) \subseteq \sigma(T)$  hence we need to establish the reverse inequality  $\sigma(T) \subseteq \Pi(T)$ , which is equivalent to showing that

$$(\Pi(T))^c \subseteq (\sigma(T))^c.$$

Let  $\lambda \in (\Pi(T))^c$  that is  $\lambda \notin \Pi(T)$ . Hence there is a real number  $\epsilon_o > 0$  such that

$$\|(\lambda I - T)x\| \geq \epsilon_o \|x\| \forall x \in H (x \neq \bar{0}). \tag{1.11}$$

Thus  $(\lambda I - T)$  is bounded from below, hence  $(\lambda I - T)^{-1}$  exists on  $\mathfrak{R}_{\lambda I - T}$  and is bounded. We need to show that  $\overline{\mathfrak{R}_{\lambda I - T}} = H$  which is equivalent to showing that if  $\mathfrak{R}_{\lambda I - T}^\perp = \{\bar{0}\}$  that is  $y \perp \mathfrak{R}_{\lambda I - T}$  then  $y = \bar{0}$ .

Now  $T \in B(H)$  is normal  $\Rightarrow T - \lambda I$  is normal.

$\{T \in B(H) \text{ is normal } T \leftrightarrow T^* \text{ or } \|Tx\| = \|T^*x\|\}$

$(T - \lambda I)^* = T^* - \bar{\lambda}I$  since  $T \leftrightarrow T^*$  so

$$(T - \lambda I) \leftrightarrow (T^* - \bar{\lambda}I) \text{ for}$$

$$\begin{aligned}
 (T - \lambda I)(T^* - \bar{\lambda}I) &= TT^* - \bar{\lambda}T - \lambda T^* + |\lambda|^2 I \\
 &= T^*T = \bar{\lambda}T - \lambda T^* + |\lambda|^2 I \\
 &= (T^* - \bar{\lambda}I) = (T + \lambda I) \\
 \text{i.e } (T - \lambda I) &\leftrightarrow (T^* - \bar{\lambda}I)
 \end{aligned}$$

Hence

$$\|(T - \lambda I)x\| = \|(T^* - \bar{\lambda}I)y\| \forall x \in H \tag{1.12}$$

If  $y \in \mathfrak{R}_{\lambda I - T}^\perp = \mathfrak{R}_{\bar{\lambda}I - T^*}^\perp$  (standard result).

Hence  $(\bar{\lambda}I - T^*)y = \bar{0}$  that is  $\|(\bar{\lambda}I - T^*)y\| = 0$ . By (??)

$$\|(T - \lambda I)y\| = \|(T^* - \bar{\lambda}I)y\| = 0.$$

Therefore  $\|(T - \lambda I)y\| = 0$

By (??) we have

$$\|(T - \lambda I)y\| \geq \epsilon_o \|y\| \text{ (put } x = y \text{ in (??))}$$

Thus

$$0 = \|(T - \lambda I)y\| \geq \epsilon_o \|y\| \text{ and } \epsilon_o > 0$$

Hence  $\|y\| = 0$  that is  $y = \bar{0}$ . Consequently,  $\overline{\mathfrak{R}}_{\lambda I - T} = H$ . Therefore,

$$\lambda \in \rho(T) \text{ that is } \lambda \notin \sigma(T)$$

$$\Rightarrow \lambda \in (\sigma(T))^c.$$

Therefore  $(\Pi(T))^c \subseteq (\sigma(T))^c$  □

**Proposition 10.** *Let  $H$  be a Hilbert space.  $T \in B(H)$  is normal implies  $T^n$  is normal for each  $n \in \mathbb{N}$ .*

*Proof. Case I*

$T$  is normal  $\Rightarrow T^2$  is normal.

$T$  is normal  $\Rightarrow TT^* = T^*T$ .

$$(T^2)^* = (TT)^* = T^*T^* = (T^*)^2$$

so

$$\begin{aligned} (T^2)(T^2)^* &= (TT)(T^*T^*) = T(TT^*)T^* = (TT^*)(TT^*) \\ &= (T^*T)(T^*T) \\ &= T^*(TT^*)T \\ &= T^*(T^*T)T \\ &= (T^*T^*)(TT) \\ &= (T^2)^*(T^2) \\ &= (T^*)^2(T^2) \end{aligned}$$

that is,  $T^2 \leftrightarrow (T^2)^*$ , that is,  $T^2$  is normal.

By induction, it follows that  $T \in B(H)$  is normal implies that  $T^n$  is normal for each  $n \in \mathbb{N}$

Case II

Alternatively, it is easier to proceed thus if  $T$  is normal,  $T^n \leftrightarrow T^* \forall n \in \mathbb{N}$ . This is true when  $n = 1$  for  $T \leftrightarrow T^*$  since  $T$  is normal. Suppose for an  $r \in \mathbb{N}$   $T^r \leftrightarrow T^*$ . Then

$$\begin{aligned} T^{r+1}T^* &= (TT^r)T^* = T(T^rT^*) \\ &= T(T^*T^r) \text{ since by induction hypothesis } T^r \leftrightarrow T^* \\ &= (TT^*)T^r \\ &= (TT^*)T^r = (T^*T)T^r \\ &= T^*(TT^r) \\ &= T^*T^{r+1}. \end{aligned}$$

Thus  $T^r \leftrightarrow T^* \Rightarrow T^{r+1} \leftrightarrow T^*$ . Hence  $T^n \leftrightarrow T^*$  for all  $n \in \mathbb{N}$ . Next we show that

$T^n \leftrightarrow (T^*)^n$  for all  $n \in \mathbb{N}$ . Let  $T^n \leftrightarrow (T^*)^r$  for all  $r \in \mathbb{N}$ . Then

$$\begin{aligned} T^n(T^*)^{r+1} &= T^n((T^*)^r T^*) = (T^n(T^*)^r)T^* \\ &= ((T^*)^r T^n)T^* \text{ since } T^n \leftrightarrow (T^*) \\ &= (T^*)^r (T^n T^*) \\ &= (T^*)^r (T^* T^n) \text{ since we have already seen that } T^n \leftrightarrow T^* \text{ for all } n \in \mathbb{N} \\ &= ((T^*)^r T^*)T^n \\ &= (T^*)^{r+1} T^n \end{aligned}$$

Thus  $T^n \leftrightarrow (T^*)^{r+1}$  if  $T^n \leftrightarrow (T^*)^r$ . The result is true when  $r = 1$  therefore  $T^n \leftrightarrow (T^*)^r$  for all  $n \in \mathbb{N}$ . In particular  $T^n \leftrightarrow (T^*)^n$  for all  $n \in \mathbb{N}$ . But  $(T^n)^* = (TT\dots T)^* = T^* \dots T^* = (T^*)^n$ .

Thus we get

$$T^n \leftrightarrow (T^n)^*.$$

This shows that  $T^n$  is normal whenever  $T \in B(H)$  is normal. □

We are now in a position to prove;

**Proposition 11.** *Let  $H$  be a Hilbert space. If  $T \in B(H)$  is normal then  $r(T) = \|T\|$*

*Proof.*  $T \in B(H)$  is normal implies  $T^n$  is normal for  $n \in \mathbb{N}$ . We show that for a normal operator  $T$

$$\|T^n\| = \|T\|^n \text{ for all } n \in \mathbb{N}.$$

Obviously  $\|T^n\| = \|T\dots T\| \leq \|T\|\dots\|T\| = \|T\|^n$

That is

$$\|T^n\| \leq \|T\|^n \quad \forall n \in \mathbb{N}.$$

Hence we must prove the reverse inequality. Since  $T$  is normal, we have

$$\|Tx\| = \|T^*x\| \quad \forall x \in H$$

putting  $Tx$  in place of  $x$  we get

$$\|T(Tx)\| = \|T^*(Tx)\| \quad \forall x \in H$$

that is

$$\|T^2x\| = \|T^*Tx\| \quad \forall x \in H.$$

Taking the supremum of both sides over all  $x \in H$  satisfying  $\|x\| \leq 1$  we obtain

$$\|T^2\| = \|T^*T\|$$

But  $\|T^*T\| = \|T\|^2$

$$\therefore \|T^2\| = \|T\|^2 \tag{1.13}$$

if  $T$  is normal. Likewise since  $T^2$  is normal, replacing  $T$  by  $T^2$  in (??) we get

$$\|T^4\| = \|T^2\|^2 = (\|T\|^2)^2 = \|T\|^4.$$

In general it follows by induction that

$$\|T^m\| = \|T\|^m \text{ for all } m = 2^k \text{ where } k \in \mathbb{N}.$$

Considering  $n \in \mathbb{N}$  we can always write

$$n = 2^m - r \text{ for all } r \in \mathbb{N}$$

therefore

$$n + r = 2^m.$$

Then

$$\|T^{n+r}\| = \|T\|^{n+r}$$

that is,

$$\|T\|^{n+r} = \|T^{n+r}\| = \|T^n T^r\| \leq \|T^n\| \|T^r\| \leq \|T^n\| \|T\|^r$$

that is

$$\|T\|^n \|T\|^r \leq \|T^n\| \|T\|^r.$$

Cancelling  $\|T\|^r$  from both sides (of course  $\|T\| \neq 0$ )

we get

$$\|T\|^n \leq \|T^n\|$$

which is the required reverse inequality. Thus, if  $T \in B(H)$  is normal, then

$$\|T^n\| = \|T\|^n \quad \forall n \in \mathbb{N}.$$

Now

$$r(T) = \lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \lim_{n \rightarrow \infty} (\|T\|^n)^{\frac{1}{n}} = \lim_{n \rightarrow \infty} \|T\| = \|T\|. \quad \square$$

**Remark 2.** *There is an alternative method for arriving at the same result using proposition ???: If  $T \in B(X)$  ( $X =$  complex Banach space), then,*

$$\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} \text{ exists.}$$

*Consider a subsequence  $(\|T^{n_k}\|^{\frac{1}{n_k}})_{k=1}^{\infty}$  where  $n_k = 2^k$  of the given sequence  $(\|T^n\|^{\frac{1}{n}})_{n=1}^{\infty}$ . This subsequence is convergent to the same limit as that of  $\|T^n\|^{\frac{1}{n}}$ . Now we have seen that*

$$\|T^{n_k}\| = \|T\|^{n_k} \text{ (by proposition ??)}$$

therefore

$$\|T^{n_k}\|^{\frac{1}{n_k}} = \|T\|.$$

So  $\lim_{k \rightarrow \infty} \|T^{n_k}\|^{\frac{1}{n_k}} = \|T\|$ .

Therefore  $\lim_{n \rightarrow \infty} \|T^n\|^{\frac{1}{n}} = \|T\|$

**Proposition 12.** *Let  $X$  be a Complex Banach space and  $T \in B(X)$ . Then there is a  $\lambda \in \sigma(T)$  such that  $|\lambda| = r(T)$  (Note: When  $X$  is a Complex Banach space then  $\sigma(T) \neq \phi$ )*

*Proof.* By definition of  $r(T) = \sup\{|\lambda| : \lambda \in \sigma(T)\}$ . Then there exists a sequence  $(\lambda_n)$  of elements of  $\sigma(T)$  such that  $|\lambda_n| \rightarrow r(T)$  as  $n \rightarrow \infty$ . Hence the sequence  $(\lambda_n)$  is a bounded subset of  $\mathbb{C}$ . Hence by the Bolzano-Weierstrass Theorem[?], [?] and [?] there exists a subsequence  $(\lambda_{n_k})$  of  $(\lambda_n)$  such that  $(\lambda_{n_k})$  converges to say  $\lambda_o \in \mathbb{C}$ . Thus

$$\lambda_{n_k} \rightarrow \lambda_o$$

consequently by continuity on the modulus function  $|| : \lambda \mapsto |\lambda|$  we get

$$|(\lambda_{n_k})| \rightarrow |\lambda_o|.$$

Since  $(\lambda_{n_k}) \in \sigma(T)$  and  $\lambda_{n_k} \rightarrow \lambda_o$ , it follows that  $\lambda_o \in \overline{\sigma(T)}$ . But  $\sigma(T)$  is closed in  $\mathbb{C}$  (in usual topology).

So  $\overline{\sigma(T)} = \sigma(T)$

Hence  $\lambda_o \in \sigma(T)$ .

Since  $|\lambda_n| \rightarrow r(T)$  as  $n \rightarrow \infty$  so  $|\lambda_{n_k}| \rightarrow r(T)$  as  $k \rightarrow \infty$ .

Since  $(|\lambda_{n_k}|)$  is a subsequence of the convergent sequence  $(|\lambda_n|)$ . By uniqueness of the limit of a convergent sequence, we obtain

$$r(T) = |\lambda_o|.$$

Since  $\lambda_o \in \sigma(T)$  the assertion of the theorem follows (take  $\lambda$  in the statement of the theorem to be the  $\lambda_o$  obtained in the proof).  $\square$

**Proposition 13.** *Let  $H$  be a Complex Hilbert space and  $T \in B(H)$  be normal. Then*

$$\|T\| = \sup\{ | \langle Tx, x \rangle | : x \in H \text{ and } \|x\| = 1 \}$$

*Proof.* We have seen that if  $T \in B(H)$  is normal then  $r(T) = \|T\|$ . We also saw earlier that for a  $T \in B(H)$  which is normal,

$$\sigma(T) = \Pi(T).$$

In proposition ??, we saw that if  $T \in B(X)$  ( $X$  is a Banach space), then there is a  $\lambda \in \sigma(T)$  such that  $|\lambda| = r(T)$ .

Hence if  $H$  is a Hilbert space and  $T \in B(H)$  is normal, then from the result  $\sigma(T) = \Pi(T)$  we obtain that there is a  $\lambda \in \pi(T)$  such that  $|\lambda| = r(T)$ .

For such a  $T$  we know that  $r(T) = \|T\|$ . Hence we arrive at the assertion; If  $T \in B(H)$  is normal, there is a  $\lambda \in \Pi(T)$  such that

$$|\lambda| = \|T\|.$$

From an earlier result, for  $T \in B(H)$  the following statements are equivalent

- (i) There exists  $\lambda \in \Pi(T)$  such that  $|\lambda| = \|T\|$
- (ii)  $\|T\| = \sup\{ | \langle Tx, x \rangle | : x \in H \text{ and } \|x\| = 1 \}$

It then follows that if  $T \in B(H)$  is normal then

$$\|T\| = \{ | \langle Tx, x \rangle | : x \in H \text{ and } \|x\| = 1 \} \quad \square$$

**Remark 3.** *Since bounded self-adjoint operators and unitary operators are also normal, the result above for  $\|T\|$  holds for such operators.*

**Proposition 14.** *If  $T \in B(H)$  is normal, then  $R\sigma(T) = \phi$*

*Proof.* We have already seen that if  $T \in B(H)$  is normal, then

$$\sigma(T) = \Pi(T),$$

also  $P\sigma(T) \subseteq \Pi(T)$ . Suppose

$$\lambda \in \Pi(T). \text{ Then } \lambda \in P\sigma(T) \text{ or } \lambda \notin P\sigma(T).$$

If  $\lambda \notin P\sigma(T)$  then it must belong to  $C\sigma(T)$  or  $R\sigma(T)$ . In both of the latter cases, we know that  $(\lambda I - T)^{-1}$  would exist as a linear map on  $\mathfrak{R}_{\lambda I - T}$ , but if  $\lambda$  belonged to  $R\sigma(T)$  we would have

$$\overline{\mathfrak{R}_{\lambda I - T}} \neq H$$

Suppose  $\overline{\mathfrak{R}_{\lambda I - T}} \neq H$ . In such a case we would have  $\lambda \in R\sigma(T)$  or  $P\sigma(T)$ . Then there would exist a non-zero  $y \in H$  such that

$$y \perp \mathfrak{R}_{\lambda I - T}.$$

(A linear subspace ( $M$  of  $H$ ) is dense if and only if  $x \perp M \Rightarrow x = \bar{0}$ ) that is

$y \in \mathfrak{R}_{\lambda I - T}^\perp = \mathfrak{R}_{\bar{\lambda}I - T^*}$   
 hence  $(\bar{\lambda}I - T^*)y = \bar{0}$   
 Since  $T$  is normal, so is  $\lambda I - T$ , that is,  
 $\|(\lambda I - T)x\| = \|(\lambda I - T)^*x\|$  for all  $x \in H$ .  
 That is  $\|(\lambda I - T)x\| = \|(\bar{\lambda}I - T^*)x\|$  for all  $x \in H$ . In particular, put  $x = y$  we obtain  
 $\|(\lambda I - T)y\| = \|(\bar{\lambda}I - T^*)y\|$   
 But  $\|(\bar{\lambda}I - T^*)y\| = \|0\| = 0$ . Hence  $\|(\lambda I - T)y\| = 0$ , that is,  
 $(\lambda I - T)y = \bar{0}$ .  
 Since  $y \neq \bar{0}$  it follows that  $\lambda$  is an eigenvalue of  $T$ , that is  
 $\lambda \in P\sigma(T)$ .  
 So if  $\mathfrak{R}_{\lambda I - T} \neq H$ , then  $\lambda \in P\sigma(T)$  necessarily. Therefore  
 $R\sigma(T) = \phi$ . □

### 1.3 Conclusion

For an operator  $T$  in a complex Banach space  $X$ , we have proved the restricted form of the Spectral Mapping Theorem, which is,

$$\sigma(p(T)) = p(\sigma(T))$$

where  $p(\sigma(T)) = \{p(\lambda) : \lambda \in \sigma(T)\}$ . We have also shown that if  $H$  is a Hilbert space and  $T \in B(H)$  is normal then  $T^n$  is normal,  $\|T^n\| = \|T\|^n$ . Consequently, we have used the spectral radius formula to show that the spectral radius of  $T$  equals the norm of  $T$

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