

Perfect **i.i.d** processes



Abstract

This note proves a theorem about i.i.d. processes, when the index space is a measure space. A different version of this theorem also appears in Green [2019] in which the concept of a sample distribution limit corresponds to the concept of a perfect i.i.d process in this paper.

Introduction

Keywords:

1 Model

i.i.d processes; measure preserving transformation

Suppose (R, \mathcal{R}, ρ) is a probability space that we will call the *state space*; and (P, \mathcal{P}, π) be a probability space called the *index space*. The following definitions convey the prime theme of the paper.

Definition 1.1. A setting is defined as a pair $\langle (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) \rangle$ consisting of a state space and an index space.

Definition 1.2. A measure-preserving transformation is any measurable map $\psi : P \rightarrow R$ such that

$$(\forall B \in \mathcal{R})(\pi(\{p : \psi(p) \in B\}) = \rho(B)). \quad (1)$$

Definition 1.3. A setting $\langle (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) \rangle$ is said to admit a perfect i.i.d process if there exists a probability space $\langle \Omega, \mathcal{F}, \mathbb{P} \rangle$ and measurable functions $\{X_p\}_{p \in P}$ where $X_p : \Omega \rightarrow R$ such that

1. For any finite $P' \subset P$ and collection $\{B_p\}_{p \in P'} \subseteq \mathcal{R}$ we have that

$$\mathbb{P}(\cap_{p \in P'} \{X_p \in B_p\}) = \prod_{p \in P'} \rho(B_p).$$

2. There exists $A \in \mathcal{F}$ such that $\mathbb{P}(A) = 1$ and

$$A \subseteq \{\omega \in \Omega : X_p(\omega) \text{ is measure-preserving in } p\}.$$

Definition 1.4. An index space (P, \mathcal{P}, π) will be called fine if

1. For every $p \in P$, $\{p\} \in \mathcal{P}$.
2. For every $p \in P$, $\pi(\{p\}) = 0$.

It follows immediately that any fine index space (P, \mathcal{P}, π) is uncountably infinite. The following is the main theorem of the paper.

Theorem 1.1. Let $\langle (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) \rangle$ be a setting. Suppose that the index space (P, \mathcal{P}, π) is fine. Further, suppose that there exists a measure-preserving transformation $\psi : P \rightarrow R$. Then, the setting $\langle (R, \mathcal{R}, \rho), (P, \mathcal{P}, \pi) \rangle$ admits a perfect i.i.d process.

Proof. The proof proceeds in a few steps.

Step 1 : We argue that given a measure-preserving transformation $\psi : P \rightarrow R$; a countable subset $\hat{P} \subseteq P$; and any function $\hat{\psi} : \hat{P} \rightarrow R$, the map $\psi' : P \rightarrow R$ defined as

$$\psi'(p) = \begin{cases} \hat{\psi}(p) & \text{if } p \in \hat{P} \\ \psi(p) & \text{otherwise} \end{cases} \quad (2)$$

is also a measure-preserving transformation. This is true since the probability space (P, \mathcal{P}, π) is assumed to be fine. As \mathcal{P} includes all singleton sets, it follows that $\hat{P} \in \mathcal{P}$. Hence, ψ' is measurable. Further, since singletons have zero probability according to the probability measure π , implying that $\pi(\hat{P}) = 0$ (due to countable additivity), it follows that ψ' is also a measure-preserving transformation.

Step 2 : We now define the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Define Ω as

$$\Omega := \{\psi : P \rightarrow R : \psi \text{ is measure-preserving}\}.$$

By assumption, we have that $\Omega \neq \emptyset$. For a finite subset $\hat{P} \subseteq P$ and collection of sets $\{B_p\}_{p \in \hat{P}} \subseteq \mathcal{R}$, define the set

$$\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle := \{\psi \in \Omega : (\forall p \in \hat{P})(\psi(p) \in B_p)\}.$$

The collection of all such sets is defined as

$$\mathcal{S} := \left\{ \langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle : \text{finite } \hat{P} \subseteq P \text{ and collection } \{B_p\}_{p \in \hat{P}} \subseteq \mathcal{R} \right\}.$$

We show that \mathcal{S} is a semi-ring (see Aliprantis and Border [2006]). Further, we show that the following set function \mathbb{P}' defines a measure on \mathcal{S}

$$\mathbb{P}'(\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle) = \prod_{p \in \hat{P}} \rho(B_p).$$

We first prove that \mathcal{S} is a semi-ring. This follows simply from the following facts.

1. Suppose that we have a set of the form $\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle$ such that $B_p = \emptyset$ for some $p \in \hat{P}$. Hence,

$$\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle = \emptyset \in \mathcal{S}.$$

2. Suppose that $\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle, \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle \in \mathcal{S}$. Then, it is simple to prove that

$$\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle \cap \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle = \langle \hat{P} \cup \hat{P}', \{B_p \cap B'_p\}_{p \in \hat{P} \cap \hat{P}'} \cup \{B_p\}_{p \in \hat{P} \setminus \hat{P}'} \cup \{B'_p\}_{p \in \hat{P}' \setminus \hat{P}} \rangle.$$

Hence, we have shown that $\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle \cap \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle \in \mathcal{S}$.

3. Suppose that $\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle, \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle \in \mathcal{S}$. We wish to show that $\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle \setminus \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle$ may be represented as a finite union of pairwise disjoint sets in \mathcal{S} . Note that

$$\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle \setminus \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle = \langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle \cap (\Omega \setminus \langle \hat{P}', \{B'_p\}_{p \in \hat{P}'} \rangle).$$

Then, it follows that

$$\Omega \setminus \langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle = \bigcup_{Q \subseteq \hat{P}'; Q \neq \emptyset} \langle \hat{P}', \{\Omega \setminus B_p\}_{p' \in Q} \cup \{B_p\}_{p' \in \hat{P}' \setminus Q} \rangle,$$

which is a finite union of disjoint sets in \mathcal{S} . Hence, from 2., we have shown that $\langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle \setminus \langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle$ is a finite union of disjoint sets in \mathcal{S} as it may be represented as

$$\begin{aligned} & \langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle \setminus \langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle \\ &= \bigcup_{Q \subseteq \hat{P}'; Q \neq \emptyset} \langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle \cap \langle \hat{P}', \{\Omega \setminus B_p\}_{p' \in Q} \cup \{B_p\}_{p' \in \hat{P}' \setminus Q} \rangle \end{aligned}$$

and we have proved that \mathcal{S} is a semiring.

We show that \mathbb{P}' defines a measure on \mathcal{S} . Suppose, we have a countable collection of pairwise disjoint sets $\langle \hat{P}^i, \{B_p\}_{p \in \hat{P}^i} \rangle_{i=1}^\infty \subseteq \mathcal{S}$ and a set $\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle \in \mathcal{S}$ such that the following holds

$$\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle = \bigcup_{i=1}^\infty \langle \hat{P}^i, \{B_p\}_{p \in \hat{P}^i} \rangle. \tag{3}$$

We prove it also holds that

$$\mathbb{P}'(\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle) = \bigcup_{i=1}^\infty \mathbb{P}'(\langle \hat{P}^i, \{B_p\}_{p \in \hat{P}^i} \rangle).$$

We prove this as follows. Define the set $\hat{P}^* = \hat{P} \cup (\bigcup_{i=1}^\infty \hat{P}^i)$. Since \hat{P}^* is a countable union of finite sets, it is at most countable. We denote the probability space $(\otimes_{p \in \hat{P}^*} R, \otimes_{p \in \hat{P}^*} \mathcal{R}, \otimes_{p \in \hat{P}^*} \rho)$ as the product measure space where $\otimes_{p \in \hat{P}^*} R = \{\hat{\psi} : \hat{P}^* \rightarrow R\}$ is the product space corresponding to R with index set \hat{P}^* ; $\otimes_{p \in \hat{P}^*} \mathcal{R}$ is the product σ -field; $\otimes_{p \in \hat{P}^*} \rho$ is denoted as the associated product measure (see Halmos [2013]).

Define the map $T : \mathcal{S} \rightarrow \otimes_{p \in \hat{P}^*} \mathcal{R}$ as

$$T(\langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle) = \{\hat{\psi} : \hat{P}^* \rightarrow R : (\forall p \in \hat{P}' \cap \hat{P}^*)(\hat{\psi}(p) \in B_p)\}.$$

Hence, it follows that $T(\langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle) \in \otimes_{p \in \hat{P}^*} \mathcal{R}$. If we have $\hat{P}' \subseteq \hat{P}^*$, then by the definition of the product measure space $(\otimes_{p \in \hat{P}^*} R, \otimes_{p \in \hat{P}^*} \mathcal{R}, \otimes_{p \in \hat{P}^*} \rho)$, we have that :

$$\otimes_{p \in \hat{P}^*} \rho(T(\langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle)) = \prod_{p \in \hat{P}'} \rho(B_p) = \mathbb{P}'(\langle \hat{P}', \{B_p\}_{p \in \hat{P}'} \rangle).$$

Hence, from countable additivity of the product measure $\otimes_{p \in \hat{P}^*}$ and by applying equality 3, to prove that \mathbb{P}' defines a measure on \mathcal{S} , it suffices to show that for the given pairwise disjoint collection $\langle \hat{P}^i, \{B_p\}_{p \in \hat{P}^i} \rangle_{i=1}^\infty \subseteq \mathcal{S}$, the following holds :

$$T\left(\bigcup_{i=1}^\infty \langle \hat{P}^i, \{B_p\}_{p \in \hat{P}^i} \rangle\right) = \bigcup_{i=1}^\infty T(\langle \hat{P}^i, \{B_p\}_{p \in \hat{P}^i} \rangle).$$

We prove that $T(\cup_{i=1}^{\infty} \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle) \subseteq \cup_{i=1}^{\infty} T(\langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$. Suppose $\hat{\psi} \in T(\cup_{i=1}^{\infty} \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$. By equality 3, we have that $\hat{\psi} \in T(\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle)$. Then, from Step 1, it follows that there exists a measure-preserving transformation ψ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^*$. Hence, by equality 3 we get that $\psi \in \langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle = \cup_{i=1}^{\infty} \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle$. This implies there exists a \hat{P}^i such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^i$. Hence, this shows that $\hat{\psi} \in \cup_{i=1}^{\infty} T(\langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$. We have, hence established that $T(\cup_{i=1}^{\infty} \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle) \subseteq \cup_{i=1}^{\infty} T(\langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$.

This way, appropriately we may also that prove that, $\cup_{i=1}^{\infty} T(\langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle) \subseteq T(\cup_{i=1}^{\infty} \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$. Suppose $\hat{\psi} \in \cup_{i=1}^{\infty} T(\langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$. Then, there exists a \hat{P}^i such that $\hat{\psi} \in T(\langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$. From Step 1 again, it follows that there exists a measure-preserving transformation ψ such that $\psi(p) = \hat{\psi}(p)$ for all $p \in \hat{P}^*$. This means that $\psi \in \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle$. By equality 3, this implies $\psi \in \langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle$. Hence, $\hat{\psi} \in T(\langle \hat{P}, \{B_p\}_{p \in \hat{P}} \rangle)$, which means $\hat{\psi} \in T(\cup_{i=1}^{\infty} \langle \hat{P}^i, \{B_p^i\}_{p \in \hat{P}^i} \rangle)$.

Hence, \mathbb{P}' defines a measure on \mathcal{S} .

The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is then completely defined as $\mathcal{F} := \sigma(\mathcal{S})$ and \mathbb{P} is defined to be the extension of the measure \mathbb{P}' on the defined σ -field \mathcal{F} by the Caratheodory Extension Theorem.

Step 3 : We finish the proof of the theorem. We have defined the appropriate probability space $(\Omega, \mathcal{F}, \mathbb{P})$. For $p \in P$, define

$$X_p(\psi) := \psi(p) \text{ and } A = \Omega.$$

This completes the proof of the theorem. □

References

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