On Fractal Properties for Pre-image Entropy

Abstract. During this paper, fractal dimension for pre-image entropy is introduced for continuous maps. Relationships among different types of pre-image entropy dimension are studied and an inequality relating them is given. Some basic examples are provided to compare those values of polynomial growth type with the pre-image entropy dimension. After that, this study constructs a symbolic subspace to attain any value between 0 and 1 for pre-image entropy dimension.

Key words and phrases fractal dimension, power rule, forward generator, polynomial growth type.

1 Introduction

Topological entropies represent the orbit complexity of a dynamical system (X, T), where T is a continuous map from the compact metric space X to itself, and have been extensively studied (e.g. see [9, 11, 16, 17, 18]). This value measures the exponential growth rate of the number of separating orbits for an arbitrary topological dynamical system. Moreover, topological entropy is one of the fundamental dynamical invariants associated with a continuous map. These entropy invariants have been discussed extensively and numerous early studies have been cited in [17, 24].

When a mapping T is invertible, it is clear to show that topological entropy $h_{top}(T^{-1})$ equals $h_{top}(T)$. However, if the mapping T is non-invertible, several possibilities lead to entropy-like invariants for non-invertible maps. During 1990s, several authors have studied entropy-like invariants for non-invertible maps, in particular Hurley's two point-wise entropies $h_m(T)$ and $h_p(T)$ in [15], which are vaguely analogous to topological entropy. Cheng and Newhouse [5] later defined two other such invariants, one topological and the other measure-theoretic in nature, and proved a "pre-image

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variational principle" relating these two invariants. A good survey of pre-image entropy for continuous map can be found in [5, 15, 18, 21, 25].

Although systems with positive entropy are much more complicated than those with zero entropy, zero entropy systems have various levels of complexity, and have been studied by [2, 4, 8, 10, 13, 20, 22]. Previous authors adopted various methods to classify zero entropy dynamical systems. For instance, Carvalho [2] introduced the notion of entropy dimension to distinguish the zero topological entropy systems and obtained some basic properties of entropy dimension. Ferenczi and Park [10] investigated a new entropy-like invariant for the action of $\mathbb{Z}$ or $\mathbb{Z}^d$ on a probability space.

Researchers have used several notions, including sequence entropy [12] and maximal pattern entropy[14] to analyze topological entropy zero systems. Pre-image entropy is another fundamental dynamical invariant associated with a continuous map. This type of entropy roughly measures the complexity of the structure of the map from the backward orbit. Zero pre-image entropy generally means that the continuous map exhibits a less complicated dynamical behavior everywhere at each backward orbit. Motivated by the technique of entropy dimension [4], the similar approximation can be discussed for zero pre-image entropy.

The purpose of this paper is to introduce the pre-image entropy dimension to quantify the complexity of zero pre-image entropy measurable dynamics. It should be a measure of the growth behavior of uncertainty from all backward orbits at each point. Section 2 first defines the pre-image entropy dimension and then gives some properties of it. Relationship among different forms of entropy dimension is discussed and section 3 obtains an inequality about them. This study considers some popular examples to compare those values between polynomial growth type of pre-image entropy and pre-image entropy dimension. Section 4 reveals if these values are the same or different. Section 5 presents an example to show that every value in (0,1) can be attained by the pre-image entropy dimension of dynamical systems.

2 Fractal dimension

This section introduces the upper and lower pre-image entropy dimension of a dynamical system, and then gives those basic propositions of pre-image entropy dimension and the formula for power inequality.

Note that for a nonnegative sequence $\{a_n\}_{n\geq 1}$, if

$$\limsup_{n\to\infty} \frac{1}{n^s} a_n \text{ is finite for } s = s_0,$$

with simple calculation, then the value of

$$\limsup_{n\to\infty} \frac{1}{n^{s_0}} a_n$$

is zero for all $s_0 > s$. If

$$\limsup_{n\to\infty} \frac{1}{n^s} a_n = \infty,$$

then

$$\limsup_{n\to\infty} \frac{1}{n^{s_0}} a_n = \infty$$

for all $s_0 \leq s$.

This indicates that the value of

$$\limsup_{n\to\infty} \frac{1}{n^s} a_n$$

jumps from ∞ to 0 at both sides of one critical point s, which is similar to a fractal measure and will be useful in the following. Reference materials can be found in [8, 23].

2.1 Some definitions

In this paper, a topological dynamical system (TDS, for short) is a pair (X, T), where (X, d) is a compact metric space and $T : X \to X$ is a continuous function. Before defining the notion of the fractal dimension of pre-image entropy for a TDS, recall some notations of topological entropy. Given a TDS (X, T), denote by $\mathcal{C}_X$ the set of finite covers of X and $\mathcal{C}_X^0$ the collection of finite open covers of X. Given two covers $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X$, note that $\mathcal{U}$ is finer than $\mathcal{V}$ if for every $U \in \mathcal{U}$, there is a set $V \in \mathcal{V}$ such that $U \subseteq V$; this is denoted by $\mathcal{U} \succeq \mathcal{V}$. It is obvious that $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{U}$ and $\mathcal{U} \vee \mathcal{V} \succeq \mathcal{V}$ if $\mathcal{U} \vee \mathcal{V} = \{U \bigcap V : U \in \mathcal{U}, V \in \mathcal{V}\}$. Given two integers $m \leq n$ and a cover $\mathcal{U} \in \mathcal{C}_X$, let $\mathcal{U}_m^n = \bigvee_{i=m}^n T^{-i}\mathcal{U}$. If $\mathcal{U} \in \mathcal{C}_X^0$, then $\aleph(\mathcal{U}, A)$ denotes the number of sets in a subcover of $\mathcal{U}$ with the smallest cardinality on the subset A of X. According to this definition, it is obvious that $\aleph(\mathcal{U} \vee \mathcal{V}, A) \leq \aleph(\mathcal{U}, A)\aleph(\mathcal{V}, A)$.

Topological entropy considers the complexity of a given system by forward orbits on the whole space. This paper investigates the fractal dimension of pre-image entropy, which considers the complexity of a given system by backward orbits at each point. The definition can be given by using open covers as follows.

Definition 2.1. *Let (X, T) be a TDS, $\mathcal{U}$ be a finite open cover of X and $s \geq 0$ be a real number. The* upper and lower s-pre-image entropy *of T with respect to $\mathcal{U}$ are defined as*

$$\overline{D}(s, T, \mathcal{U}) = \limsup_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x), \tag{2.1}$$

and

$$\underline{D}(s, T, \mathcal{U}) = \liminf_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x). \tag{2.2}$$

Here, $T^{-k}x$ means $T^{-k}(x)$. When $s = 1$, $\overline{D}(s, T, \mathcal{U})$ is just the *pre-image entropy* of T with respect to $\mathcal{U}$, (usually denoted by $h_{pre}(T, \mathcal{U})$), see [5]. It is clear that $\overline{D}(s, T, \mathcal{U}) \leq \overline{D}(s_0, T, \mathcal{U})$ when $s \geq s_0 \geq 0$ and the graph of $\overline{D}(s, T, \mathcal{U})$ against s shows that there is a critical value of s at which $\overline{D}(s, T, \mathcal{U})$ jumps from ∞ to 0. Therefore, define the *upper pre-image entropy dimension* of T with respect to $\mathcal{U}$ from this critical value by

$$\overline{D}(T, \mathcal{U}) = \inf\{s \geq 0 : \overline{D}(s, T, \mathcal{U}) = 0\} = \sup\{s \geq 0 : \overline{D}(s, T, \mathcal{U}) = \infty\}$$

Similarly, the *lower pre-image entropy dimension* of T w.r.t. $\mathcal{U}$ is

$$\underline{D}(T, \mathcal{U}) = \inf\{s \geq 0 : \underline{D}(s, T, \mathcal{U}) = 0\} = \sup\{s \geq 0 : \underline{D}(s, T, \mathcal{U}) = \infty\}$$

It is then clear that $0 \leq \overline{D}(T, \mathcal{U}) \leq \underline{D}(T, \mathcal{U})$. If $\overline{D}(T, \mathcal{U}) = \underline{D}(T, \mathcal{U}) = s$, then this cover $\mathcal{U}$ has pre-image entropy dimension s.

Definition 2.2. *If (X, T) is a TDS, then the* upper and lower pre-image entropy dimension *of T are defined as*

$$\overline{D}_{pre}(T) = \sup\{\overline{D}(T, \mathcal{U}) : \mathcal{U} \in \mathcal{C}_X^0\}, \tag{2.3}$$

and

$$\underline{D}_{pre}(T) = \sup\{\underline{D}(T, \mathcal{U}) : \mathcal{U} \in \mathcal{C}_X^0\}, \tag{2.4}$$

It is clear that $0 \leq \underline{D}_{pre}(T) \leq \overline{D}_{pre}(T)$. When $0 \leq \underline{D}_{pre}(T) = \overline{D}_{pre}(T)$, this value represents the pre-image entropy dimension of (X, T) and is denoted by $D_{pre}(T)$.

2.2 Dynamical properties

The following propositions are the basic properties of pre-image entropy dimension in the topological version. Statement 3 of the following lemma shows that the stability property of upper pre-image entropy dimension of open covers is true.

Lemma 2.1. *If (X, T) is a TDS and $\mathcal{U}, \mathcal{V} \in \mathcal{C}_X^0$, then the following statements are true.*

(1) If $\mathcal{U} \preceq \mathcal{V}$, then $\overline{D}(T, \mathcal{U}) \leq \overline{D}(T, \mathcal{V})$ and $\underline{D}(T, \mathcal{U}) \leq \underline{D}(T, \mathcal{V})$;

(2) For any $0 \leq m \leq n$, we have $\overline{D}(T, \mathcal{U}) = \overline{D}(T, \mathcal{U}_m^n)$ and $\underline{D}(T, \mathcal{U}) = \underline{D}(T, \mathcal{U}_m^n)$;

(3) $\overline{D}(T, \mathcal{U} \vee \mathcal{V}) = \max\{\overline{D}(T, \mathcal{U}), \overline{D}(T, \mathcal{V})\}$.

Proof. Since the proof is trivial for (1) and (2), we only prove (3) here. For any nonnegative real number s with $\max\{\overline{D}(T,\mathcal{U}), \overline{D}(T,\mathcal{V})\} < s$, note that

$$\limsup_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x) = 0,$$

and

$$\limsup_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{V}, T^{-k}x) = 0,$$

which implies

$$\begin{aligned} & \limsup_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}(\mathcal{U}\vee\mathcal{V}), T^{-k}x) \\ \leq & \limsup_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x) + \limsup_{n\to\infty} \frac{1}{n^s} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{V}, T^{-k}x) \\ = & \ 0 \end{aligned}$$

Therefore, $\overline{D}(T,\mathcal{U}\vee\mathcal{V}) < s$. Since s is arbitrary,

$$\overline{D}(T,\mathcal{U}\vee\mathcal{V}) \leq \max\{\overline{D}(T,\mathcal{U}), \overline{D}(T,\mathcal{V})\}.$$

From (1), we conclude that $\overline{D}(T,\mathcal{U}\vee\mathcal{V}) = \max\{\overline{D}(T,\mathcal{U}), \overline{D}(T,\mathcal{V})\}$. □

Notice that if $a_n = \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x)$, then $\{a_n\}_{n\geq 1}$ is a monotonically increasing sequence. Given a fixed positive integer $r \geq 1$, for any $n \in Z^+$, there exists $l \in N$ such that $lr \leq n \leq (l+1)r$. Thus,

$$\frac{(lr)^s}{((l+1)r)^s} \frac{a_{lr}}{(lr)^s} = \frac{a_{lr}}{((l+1)r)^s} \leq \frac{a_n}{n^s} \leq \frac{((l+1)r)^s}{(lr)^s} \frac{a_{(l+1)r}}{((l+1)r)^s} = \frac{a_{(l+1)r}}{(lr)^s}$$

which implies that

$$\limsup_{n\to\infty} \frac{a_n}{n^s} = \limsup_{n\to\infty} \frac{a_{nr}}{(nr)^s} \quad \text{and} \quad \liminf_{n\to\infty} \frac{a_n}{n^s} = \liminf_{n\to\infty} \frac{a_{nr}}{(nr)^s} \tag{2.5}$$

for any fixed positive integer r.

Obviously, pre-image entropy dimension as defined above is an invariant of topological conjugacy. This value reveals to consider those two pre-image entropy zero dynamical systems as being not the same or being not equivalent by different pre-image entropy dimension. The basic proposition of pre-image entropy dimension is the power rule. The inequality of the power rule can be shown as follows. The inverse part of inequality is still unknown.

Theorem 2.1. *For each positive integer r and $0 \leq s \leq 1$, we obtain*

$$\overline{D}(s,T^r,\mathcal{U}) \leq r^s \cdot \overline{D}(s,T,\mathcal{U})$$

and

$$\underline{D}(s,T^r,\mathcal{U}) \leq r^s \cdot \underline{D}(s,T,\mathcal{U}),$$

for any $\mathcal{U} \in \mathcal{C}_X^0$.

Proof. First show that

$$\overline{D}(s,T^r,\bigvee_{i=0}^{r-1} T^{-i}\mathcal{U}) = r^s \cdot \overline{D}(s,T,\mathcal{U})$$

for any open cover $\mathcal{U}$ of X. In fact, for each positive integer r and $0 \leq s \leq 1$,

$$\begin{aligned} \overline{D}(s,T^r,\bigvee_{i=0}^{r-1} T^{-i}\mathcal{U}) &= \limsup_{n\to\infty} \frac{r^s}{(nr)^s} \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{j=0}^{n-1} T^{-rj}(\bigvee_{i=0}^{r-1} T^{-i}\mathcal{U}), T^{-k}x) \\ &= \limsup_{n\to\infty} \frac{r^s}{(nr)^s} \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{nr-1} T^{-i}\mathcal{U}, T^{-k}x) \\ &= r^s \cdot \limsup_{n\to\infty} \frac{1}{(nr)^s} \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{nr-1} T^{-i}\mathcal{U}, T^{-k}x) \\ &= r^s \cdot \overline{D}(s,T,\mathcal{U}) \text{ (using (2.5))} \end{aligned}$$

which implies

$$\overline{D}(s,T^r,\mathcal{U}) \leq \overline{D}(s,T^r,\bigvee_{i=0}^{r-1} T^{-i}\mathcal{U}) = r^s \cdot \overline{D}(s,T,\mathcal{U})$$

for any open cover $\mathcal{U} \in \mathcal{C}_X^o$. The same arguments yield that

$$\underline{D}(s,T^r,\mathcal{U}) \leq r^s \cdot \underline{D}(s,T,\mathcal{U}).$$

□

In a metric space (X,d), define the diameter of a cover $\mathcal{U}$ as

$$diam(\mathcal{U}) = \sup_{A\in\mathcal{U}} diam(A),$$

where $diam(A)$ denotes the diameter of the set A. If $\mathcal{U},\mathcal{V}$ are open covers of X and $diam(\mathcal{U})$ is less than a Lebesgue number for $\mathcal{V}$, then we have $\mathcal{U} \succeq \mathcal{V}$. The following lemma is a direct application of this fact.

Lemma 2.2. *If (X,T) is a TDS and $\{\mathcal{U}_n\}$ is a sequence of finite open covers of X with $\lim_{n\to\infty} diam(\mathcal{U}_n) = 0$, then*

$$\lim_{n\to\infty} \overline{D}(T,\mathcal{U}_n) = \overline{D}_{pre}(T) \text{ and } \lim_{n\to\infty} \underline{D}(T,\mathcal{U}_n) = \underline{D}_{pre}(T).$$

Except in very special cases, the pre-image entropy dimension cannot be easily computed by using those definitions and results, see [6] for some results of computing topological entropy. Thus, simplifying the definition is also important to us. Again, assume $T : X \to X$ is a continuous map. A finite open cover $\mathcal{U}$ of X is a *forward generator* for T if for every sequence $\{A_n\}_{n\geq 0}$ of members of $\mathcal{U}$, the set $\bigcap_{n=0}^{\infty} T^{-n}(\overline{A}_n)$ contains at most one point of X. See [3] and [21] for details.

Lemma 2.3. *Assume $T : X \to X$ is a continuous map of a compact metric space (X, d). Let $\mathcal{U}$ be a forward generator for T. For any $\varepsilon > 0$, there exist $N > 0$ such that each set in $\bigvee_{n=0}^{N} T^{-n}\mathcal{U}$ has diameter less than ε.*

A continuous map T from a compact metric space (X, d) to itself is said to be forward expansive if there exist $\delta > 0$ such that, for any distinct $x \neq y \in X$, the forward images $T^n x$ and $T^n y$ are more than δ apart, for some n. The following lemma is well-known, e.g. see [24].

Lemma 2.4. *A continuous map T from a compact metric space (X, d) to itself is forward expansive if and only if it has a forward generator.*

Using the second item of lemma 2.2 and the above lemmas, the following conclusion is easily obtained.

Theorem 2.2. *Let $T : X \to X$ be a forward expansive continuous map of the compact metric space (X, d). Then there exists a forward generator $\mathcal{U}$ such that $\overline{D}(T, \mathcal{U}) = \overline{D}_{pre}(T)$ and $\underline{D}(T, \mathcal{U}) = \underline{D}_{pre}(T)$.*

3 Relationships among different definitions

This section discusses the relations of different definitions of pre-image entropy dimensions.

3.1 Basic definitions

For convenience, we first recall some concepts and notations adopted by Dou, Huang and Park [8]. Consider an increasing sequence of integers $S = \{s_i\}_{i=1}^{\infty} = \{s_1 < s_2 < s_3 ...\}$. For $\gamma \geq 0$, denote

$$\overline{D}(S, \gamma) = \limsup_{n\to\infty} \frac{n}{(s_n)^{\gamma}}$$

and

$$\underline{D}(S, \gamma) = \liminf_{n\to\infty} \frac{n}{(s_n)^{\gamma}}$$

It is clear that $\overline{D}(S,\gamma) \leq \overline{D}(S,\acute{\gamma})$ if $\gamma \geq \acute{\gamma} \geq 0$. Similarly, graphs of $\underline{D}(S,\gamma)$ and $\overline{D}(S,\gamma)$ against γ show that there is a critical value of γ at which $\overline{D}(S,\gamma)$ and $\underline{D}(S,\gamma)$ jump from ∞ to 0. Therefore, the upper dimension of this sequence S is given by

$$\overline{D}(S) = \inf\{\gamma \geq 0 : \overline{D}(S,\gamma) = 0\} = \sup\{\gamma \geq 0 : \overline{D}(S,\gamma) = \infty\}$$

and the lower dimension of S is defined by

$$\underline{D}(S) = \inf\{\gamma \geq 0 : \underline{D}(S,\gamma) = 0\} = \sup\{\gamma \geq 0 : \underline{D}(S,\gamma) = \infty\}$$

It is then clear that $0 \leq \underline{D}(S) \leq \overline{D}(S) \leq 1$. If $\overline{D}(S) = \underline{D}(S) = \gamma$, this sequence S has dimension γ.

The following discussion investigates the dimension of a special kind of sequence, which we call *the pre-image entropy generating sequence.*

Let (X,T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. An increasing sequence of integers $S = \{s_i\}_{i=1}^{\infty} = \{s_1 < s_2 < \cdots\}$ is a *pre-image entropy generating sequence* of $\mathcal{U}$ if

$$\liminf_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x) > 0.$$

Denote by $\mathcal{E}(T,\mathcal{U})$ the set of all pre-image entropy generating sequences of $\mathcal{U}$ and by $\mathcal{P}(T,\mathcal{U})$ the set of sequence $S = \{s_i\}_{i=1}^{\infty} = \{s_1 < s_2 < \cdots\}$ of $\mathbb{Z}^+$ with the property that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x) > 0.$$

In other words, $\mathcal{P}(T,\mathcal{U})$ is the set of increasing sequence of integers along which $\mathcal{U}$ has positive pre-image entropy.

Definition 3.1. *Let (X,T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. Define*

$$\overline{D}_e(T,\mathcal{U}) = \begin{cases} \sup_{S\in\mathcal{E}(T,\mathcal{U})} \overline{D}(S) & \text{if } \mathcal{E}(T,\mathcal{U}) \neq \emptyset \\ 0 & \text{if } \mathcal{E}(T,\mathcal{U}) = \emptyset \end{cases},$$

and

$$\overline{D}_p(T,\mathcal{U}) = \begin{cases} \sup_{S\in\mathcal{P}(T,\mathcal{U})} \overline{D}(S) & \text{if } \mathcal{P}(T,\mathcal{U}) \neq \emptyset \\ 0 & \text{if } \mathcal{P}(T,\mathcal{U}) = \emptyset \end{cases}.$$

Similarly, define $\underline{D}_e(T,\mathcal{U})$ and $\underline{D}_p(T,\mathcal{U})$ by changing the upper dimension to the lower dimension.

Definition 3.2. *Let (X,T) be a TDS, define*

$$\overline{D}_e(X,T) = \sup_{\mathcal{U}\in\mathcal{C}_X^0} \overline{D}_e(T,\mathcal{U}) \text{ and } \overline{D}_p(X,T) = \sup_{\mathcal{U}\in\mathcal{C}_X^0} \overline{D}_p(T,\mathcal{U}).$$

Similar definition for $\underline{D}_e(X,T)$ and $\underline{D}_p(X,T)$.

The following proposition shall explain why this study defines the entropy generating sequence as lim inf instead of lim sup.

Theorem 3.1. *Let (X,T) be a TDS. Then*

$$\overline{D}_p(T,\mathcal{U}) = \begin{cases} 1 & \text{if } \mathcal{P}(T,\mathcal{U}) \neq \emptyset \\ 0 & \text{if } \mathcal{P}(T,\mathcal{U}) = \emptyset \end{cases} \quad \text{for } \mathcal{U} \in \mathcal{C}_X^o.$$

Proof. Assume that $\mathcal{P}(T,\mathcal{U}) \neq \emptyset$. Thus, there exists $S = \{s_i\}_{i=1}^\infty = \{s_1 < s_2 < \cdots\} \subset \mathbb{Z}^+$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=1}^n T^{-s_i}\mathcal{U}, T^{-k}x) = a > 0.$$

Next, take a subsequence of positive integers $\{n_i\}_{i\geq 1}$ such that it attains the previous limit superior. Without loss of generality, for each $j \in \mathbb{N}$, let's assume that

(i) $n_{j+1} \geq 2s_{n_j}$;

(ii) $n_{j+1} \geq n_1 + \sum_{i=1}^j (n_{i+1} - s_{n_i})$.

Otherwise, we can choose a subsequence of $\{n_j\}_{j\geq 1}$ to satisfy the above two conditions. Put

$$F = S \cup \{1, 2, \cdots, n_1\} \cup \bigcup_{i=1}^\infty \{s_{n_i} + 1, s_{n_i} + 2, \cdots, n_{i+1}\}.$$

To simplify, write $F = \{t_1 < t_2 < \cdots\}$, which implies that

$$\begin{aligned} \limsup_{n\to\infty} \frac{1}{n} \log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=1}^n T^{-t_i}\mathcal{U}, T^{-k}x) &\geq \limsup_{j\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=1}^{n_j} T^{-s_i}\mathcal{U}, T^{-k}x)}{|F \cap [0, 1, \cdots, s_{n_j}]|} \\ &\geq \limsup_{j\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=1}^{n_j} T^{-s_i}\mathcal{U}, T^{-k}x)}{2n_j} \\ &= \frac{a}{2} > 0, \end{aligned}$$

where the second inequality follows from condition (ii). Therefore, $F \in \mathcal{P}(T,\mathcal{U})$. Since $n_{j+1} \geq 2s_{n_j}$ for each $j \in \mathbb{N}$, it is easy to see that the upper density of F is at least of $\frac{1}{2}$, hence $\overline{D}(F) = 1$. This implies $\overline{D}_p(T,\mathcal{U}) = 1$. □

3.2 Relationships

In the following, those relationships among these different kinds of dimensions are investigated.

Theorem 3.2. *Let (X,T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. Then,*

$$\underline{D}_e(T,\mathcal{U}) \leq \overline{D}_e(T,\mathcal{U}) \leq \underline{D}_p(T,\mathcal{U}) \leq \overline{D}(T,\mathcal{U}).$$

Proof. 1) $\underline{D}_e(T,\mathcal{U}) \le \overline{D}_e(T,\mathcal{U})$ is obvious by Definition 3.1.

2) To show that $\overline{D}_e(T,\mathcal{U}) \le \underline{D}_p(T,\mathcal{U})$, it is sufficient to assume that $\overline{D}_e(T,\mathcal{U}) > 0$.

Given $\tau \in (0, \overline{D}_e(T,\mathcal{U}))$. There exists $S = \{s_i\}_{i=1}^{\infty} = \{s_1 < s_2 < \cdots\} \in \mathcal{E}(T,\mathcal{U})$ with $\overline{D}(S) > \tau$, i.e. $\limsup_{n\to\infty} \frac{n}{(s_n)^\tau} = \infty$. Hence,

$$\limsup_{n\to+\infty} \frac{n}{n + s_n^\tau} = 1. \tag{3.6}$$

Let $F = S \cup \{\lfloor n^{\frac{1}{\tau}} \rfloor : n \in \mathbb{N}\}$. Clearly $\underline{D}(F) \ge \tau$. Let $F = \{t_1 < t_2 < \cdots\}$. Then for each $n \in \mathbb{N}$ there exists a unique $m(n) \in \mathbb{N}$ such that $s_n = t_{m(n)}$. Since

$$\{s_1, s_2, \cdots, s_n\} \subseteq \{t_1, t_2, \cdots, t_{m(n)}\} \subseteq \{s_1, s_2, \cdots, s_n\} \cup \{\lfloor k^{\frac{1}{\tau}} \rfloor : k \le s_n^\tau\},$$

we have $n \le m(n) \le n + s_n^\tau$. Combining this with (3.6) leads to

$$\limsup_{n\to\infty} \frac{n}{m(n)} = 1. \tag{3.7}$$

This implies

$$\begin{aligned}
&\limsup_{m\to+\infty} \frac{\log \sup_{x\in X, k\ge 1} \aleph(\bigvee_{i=1}^{m} T^{-t_i}\mathcal{U}, T^{-k}x)}{m} \\
&\ge \limsup_{n\to\infty} \frac{\log \sup_{x\in X, k\ge 1} \aleph(\bigvee_{i=1}^{m(n)} T^{-t_i}\mathcal{U}, T^{-k}x)}{m(n)} \\
&\ge \limsup_{n\to\infty} \frac{\log \sup_{x\in X, k\ge 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x)}{n} \frac{n}{m(n)} \\
&\ge (\liminf_{n\to\infty} \frac{\log \sup_{x\in X, k\ge 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x)}{n}) \cdot (\limsup_{n\to\infty} \frac{n}{m(n)}) \\
&= \liminf_{n\to\infty} \frac{\log \sup_{x\in X, k\ge 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x)}{n} \quad \text{(by (3.7))} \\
&> 0 \qquad \text{(since } S \in \mathcal{E}(T,\mathcal{U})\text{)}.
\end{aligned}$$

Thus, $F \in \mathcal{P}(T,\mathcal{U})$. Hence $\underline{D}_p(T,\mathcal{U}) \ge \underline{D}(F) \ge \tau$. Since τ is arbitrary, we obtain $\overline{D}_e(T,\mathcal{U}) \le \underline{D}_p(T,\mathcal{U})$.

3) Assume by contradiction that $\underline{D}_p(T,\mathcal{U}) > \overline{D}(T,\mathcal{U})$. By assumption there exists $\tau \in (0,1)$ such that $\underline{D}_p(T,\mathcal{U}) > \tau > \overline{D}(T,\mathcal{U})$. On the one hand,

$$\limsup_{m\to\infty} \frac{1}{m^\tau} \log \sup_{x\in X, k\ge 1} \aleph(\bigvee_{i=1}^{m} T^{-i}\mathcal{U}, T^{-k}x) = 0 \tag{3.8}$$

since $\tau > \overline{D}(T,\mathcal{U})$.

On the other hand, since $\underline{D}_p(T,\mathcal{U}) > \tau$ there exists $S = \{s_1 < s_2 < \cdots\} \in \mathcal{P}(T,\mathcal{U})$ with $\underline{D}(S) > \tau$, i.e. $\liminf_{n\to\infty} \frac{n}{s_n^\tau} = \infty$. Hence, there exists $c > 0$ such that $\frac{n}{s_n^\tau} \ge c$ for all sufficiently large n. It follows that

$$\begin{aligned}
&\limsup_{m\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=1}^{m} T^{-i}\mathcal{U}, T^{-k}x)}{m^\tau} \\
&\geq \limsup_{n\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=1}^{s_n} T^{-i}\mathcal{U}, T^{-k}x)}{s_n^\tau} \\
&\geq \limsup_{n\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x)}{n} \frac{n}{s_n^\tau} \\
&\geq \limsup_{n\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=1}^{n} T^{-s_i}\mathcal{U}, T^{-k}x)}{n} \cdot c \\
&> 0 \qquad \text{(since } S \in \mathcal{P}(T,\mathcal{U})\text{)},
\end{aligned}$$

which contradicts (3.8). □

Theorem 3.3. *Let (X,T) be a TDS and $\mathcal{U} \in \mathcal{C}_X^o$. Then*

$$\underline{D}_e(T,\mathcal{U}) \leq \underline{D}(T,\mathcal{U}) \leq \overline{D}(T,\mathcal{U}).$$

Proof. By the definitions, it is clear $\underline{D}(T,\mathcal{U}) \leq \overline{D}(T,\mathcal{U})$. It suffices to show that $\underline{D}_e(T,\mathcal{U}) \leq \underline{D}(T,\mathcal{U})$. Without loss of generality, assume that $\underline{D}_e(T,\mathcal{U})) > 0$. Otherwise, there is nothing to prove. For any $\tau \in (0, \underline{D}_e(T,\mathcal{U}))$, there exists $S = \{s_1 < s_2 < \cdots\} \in \mathcal{E}(T,\mathcal{U})$ such that $\underline{D}(S) > \tau$, that is, $\liminf_{n\to\infty} \frac{n}{s_n^\tau} = \infty$. Hence, there is $c > 0$ such that $\frac{n}{s_n^\tau} \geq c^{-\tau}$, i.e. $s_n \leq cn^{\frac{1}{\tau}}$ for sufficiently large n.

For $m \in \mathbb{N}$ with $m \geq s_1$, there exists a unique $n(m) \in \mathbb{N}$ such that $s_{n(m)} \leq m-1 < s_{n(m)+1} \leq c(n(m)+1)^{\frac{1}{\tau}}$. Since $S \in \mathcal{E}(T,\mathcal{U})$,

$$\begin{aligned}
&\liminf_{m\to\infty} \frac{1}{m^\tau} \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{m-1} T^{-i}\mathcal{U}, T^{-k}x) \\
&\geq \liminf_{m\to\infty} \frac{1}{c^\tau(n(m)+1)} \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{j=1}^{n(m)} T^{-s_j}\mathcal{U}, T^{-k}x) \\
&\geq \frac{1}{c^\tau} \liminf_{k\to\infty} \frac{1}{k} \log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{j=1}^{k} T^{-s_j}\mathcal{U}, T^{-k}x) > 0.
\end{aligned}$$

This implies that $\underline{D}(T,\mathcal{U}) \geq \tau$. Finally, as τ is arbitrary, we conclude that

$$\underline{D}_e(T,\mathcal{U}) \leq \underline{D}(T,\mathcal{U}).$$

□

Proving that $\overline{D}_e(X,T) = \underline{D}_p(X,T) = \overline{D}(X,T)$ requires the concept of "independent" and some related combinatorial lemma.

Let (X,T) be a TDS. Let $A_1, A_2, \cdots, A_k$ be k-subsets of X and $W \subseteq \mathbb{Z}_+$. Note that $\{A_1, A_2, \cdots, A_k\}$ is *independent* along W if for any $s \in \{1, 2, \cdots, k\}^W$ we have $\bigcap_{w \in W} T^{-w} A_{s(w)} \neq \emptyset$.

Next, recall the following lemma from [8].

Lemma 3.1. *[8] Let (X,T) be a TDS and $A_1, A_2, \cdots, A_k$ be k-pairwise disjoint non-empty closed subsets of X $(k \geq 2)$, $\mathcal{U} = \{A_1^c, A_2^c, \cdots, A_k^c\}$. For $\tau \in (0,1]$, $0 < \eta < \tau$ and $c > 0$ there exists $N \in \mathbb{N}$ (depending on k, τ, η, c) such that if a finite subset B of $\mathbb{Z}_+$ satisfies $|B| \geq N$ and $\aleph(\bigvee_{i \in B} T^{-i}\mathcal{U}, X) \geq e^{c|B|^\tau}$, then there exists $W \subseteq B$ with $|W| \geq |B|^\eta$ such that $\{A_1, A_2, \cdots, A_k\}$ is independent along W.*

Let (X,T) be a TDS. Let $A_1, A_2, \cdots, A_k$ be k subsets of X and $W \subseteq \mathbb{Z}_+$. We say $\{A_1, A_2, \cdots, A_k\}$ is *pre-image independent* along W, if for any $s \in \{1, 2, \cdots, k\}^W$, then $\bigcap_{w \in W} T^{-w} A_{s(w)} \bigcap T^{-k}x \neq \emptyset$ for some k and x.

The following lemma is obtained by a proof similar to that of lemma 3.1.

Lemma 3.2. *Let (X,T) be a TDS, and let A_1, A_2 be 2-pairwise disjoint non-empty closed subsets of X, $\mathcal{U} = \{A_1^c, A_2^c\}$. For $\tau \in (0,1]$, $0 < \eta < \tau$ and $c > 0$ there exists $N \in \mathbb{N}$ (depending on τ, η, c) such that if a finite subset B of $\mathbb{Z}_+$ satisfies $|B| \geq N$ and $\aleph(\bigvee_{i \in B} T^{-i}\mathcal{U}, T^{-k}x) \geq e^{c|B|^\tau}$ for some k and x, then there exists $W \subseteq B$ with $|W| \geq |B|^\eta$ such that $\{A_1, A_2\}$ is pre-image independent along W.*

Based on the lemma above, the following theorem holds.

Theorem 3.4. *Let A_1, A_2 be 2-pairwise disjoint non-empty closed subsets of a TDS (X,T) and $\mathcal{U} = \{A_1^c, A_2^c\}$. Then there exists a sequence $F \in \mathcal{E}(T,\mathcal{U})$ such that $\overline{D}(F) = \overline{D}(T,\mathcal{U})$ when $\mathcal{E}(T,\mathcal{U})$ is nonempty. Hence, by Theorem 3.2, $\overline{D}(T,\mathcal{U}) = \overline{D}_e(T,\mathcal{U})$.*

Proof. If $\overline{D}(T,\mathcal{U}) = 0$ and $\mathcal{E}(T,\mathcal{U})$ is nonempty, then any sequence in $\mathcal{E}(T,\mathcal{U})$ will have upper dimension zero.

Assume that $\overline{D}(T,\mathcal{U}) > 0$ and let $\{\tau_j\} \subset (0, \overline{D}(T,\mathcal{U}))$ be a sequence of strictly increasing real numbers such that $\lim_{j \to \infty} \tau_j = \overline{D}(T,\mathcal{U})$. Then, choose $a > 0$ so that

$$\limsup_{n \to +\infty} \frac{1}{n^{\tau_j}} \log \sup_{x \in X, k \geq 1} \aleph(\bigvee_{i=1}^{n} T^{-i}\mathcal{U}, T^{-k}x) > a \text{ for } j \in \mathbb{N}.$$

Let $\tau_{j-1} < \eta_j < \tau_j$ for $j \in \mathbb{N}$. By Lemma 3.2, there exists $N_j \in \mathbb{N}$ such that for every finite set B with $|B| \geq N_j$ and $\aleph(\bigvee_{i \in B} T^{-i}\mathcal{U}, T^{-k}x) \geq e^{\frac{a}{2}|B|^{\tau_j}}$ for some $x \in X$ and $k \geq 1$, we can find $W \subseteq B$ with $|W| \geq |B|^{\eta_j}$ and $\{A_1, A_2\}$ pre-image independent along W.

Take $1 = n_1 < n_2 < \cdots$ such that $(n_{j+1} - n_j)^{\eta_j} \geq j n_j + N_j$ and

$$\sup_{x \in X, k \geq 1} \aleph(\bigvee_{i=n_j+1}^{n_{j+1}} T^{-i}\mathcal{U}, T^{-k}x) \geq \sup_{x \in X, k \geq 1} \aleph(\bigvee_{i=1}^{n_{j+1}-n_j} T^{-i}\mathcal{U}, T^{-k}x) \geq e^{\frac{a}{2}(n_{j+1}-n_j)^{\tau_j}}$$

for each $j \in \mathbb{N}$. For each $j \in \mathbb{N}$ there exists $W_j \subseteq \{n_j+1, n_j+2, \cdots, n_{j+1}\}$ with $|W_j| \geq (n_{j+1}-n_j)^{\eta_j}$ and $\{A_1, A_2\}$ pre-image independent along W_j.

For each set W_j and $s = (s(z))_{z\in B} \in \{1,2\}^B$, we can find $x_s \in \bigcap_{z\in B} T^{-z}A_{s(z)} \cap T^{-k}x_j$ for some $x_j \in X$ and $k \geq 1$. Let $X_j = \{x_s : s \in \{1,2\}^{W_j}\}$. It is clear that for any $s \in \{1,2\}^{W_j}$ we have $|\bigcap_{z\in W_j} T^{-z}A^c_{s(z)} \cap X_j \cap T^{-k}x_j| = 1$. Combining this fact with $|X_j| = 2^{|W_j|}$ leads to

$$\aleph(\bigvee_{z\in W_j} T^{-z}\mathcal{U}, T^{-k}x_j) \geq 2^{|W_j|}. \tag{3.9}$$

Put $F = \bigcup_{i=1}^{\infty} W_j$ and write $F = \{t_1 < t_2 < \cdots\}$. For $n \in \mathbb{N}$ with $n \geq |W_1|$ there exists a unique $k(n) \in \mathbb{N}$ such that $\sum_{i=1}^{k(n)} |W_i| \leq n < \sum_{i=1}^{k(n)+1} |W_i|$. Thus,

$$\begin{aligned}\sup_{x\in X,k\geq 1} \aleph(\bigvee_{j=1}^{n} T^{-t_j}\mathcal{U}, T^{-k}x) &\geq \max\{\sup_{x\in X,k\geq 1} \aleph(\bigvee_{w\in W_{k(n)}} T^{-w}\mathcal{U}, T^{-k}x), \\ &\qquad \sup_{x\in X,k\geq 1} \aleph(\bigvee_{w\in W_{k(n)+1}\cap\{t_1,\cdots,t_n\}} T^{-w}\mathcal{U}, T^{-k}x)\} \\ &\geq \max\{2^{|W_{k(n)}|}, 2^{n-\sum_{i=1}^{k(n)}|W_i|}\} \quad \text{(by (3.9))} \\ &\geq 2^{\frac{n-\sum_{i=1}^{k(n)-1}|W_i|}{2}}.\end{aligned}$$

Hence,

$$\begin{aligned}\liminf_{n\to+\infty} \frac{1}{n}\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{j=1}^{n} T^{-t_j}\mathcal{U}) &\geq \liminf_{n\to+\infty} \frac{n-\sum_{i=1}^{k(n)-1}|W_i|}{2n}\log 2 \\ &\geq \liminf_{n\to+\infty}(\frac{1}{2} - \frac{\sum_{i=1}^{k(n)-1}|W_i|}{2|W_{k(n)}|})\cdot\log 2 \\ &\geq \liminf_{n\to+\infty}(\frac{1}{2} - \frac{\sum_{i=1}^{k(n)-1}(n_{i+1}-n_i)}{2|W_{k(n)}|})\cdot\log 2 \\ &\geq \liminf_{n\to+\infty}(\frac{1}{2} - \frac{n_{k(n)}}{2(n_{k(n)+1}-n_{k(n)})^{\eta_{k(n)}}})\cdot\log 2 \\ &\geq \liminf_{n\to+\infty}(\frac{1}{2} - \frac{1}{2k(n)})\cdot\log 2 = \frac{1}{2}\log 2 > 0.\end{aligned}$$

This shows $F \in \mathcal{E}(T,\mathcal{U})$.

Note that

$$\begin{aligned}\limsup_{m\to+\infty} \frac{m}{t_m^{\eta_j}} &\geq \limsup_{k\to+\infty} \frac{|W_1|+|W_2|+\cdots+|W_k|}{t^{\eta_j}_{|W_1|+|W_2|+\cdots+|W_k|}} \\ &\geq \limsup_{k\to+\infty} \frac{|W_1|+|W_2|+\cdots+|W_k|}{n_{k+1}^{\eta_j}} \geq \limsup_{k\to+\infty} \frac{|W_k|}{n_{k+1}^{\eta_j}} \\ &\geq \limsup_{k\to+\infty} \frac{(n_{k+1}-n_k)^{\eta_k}}{n_{k+1}^{\eta_j}} \geq 1,\end{aligned}$$

we have $\overline{D}(F) \geq \eta_j$. Hence, $\overline{D}(F) = \overline{D}(T,\mathcal{U})$. This completes the proof. □

Remark 1. *1. Lemma 3.2 and Theorem 3.4 are also true for the case* $\mathcal{U} = \{A_1^c, A_2^c, \cdots, A_k^c\}$, *where* $A_1, A_2, \cdots, A_k$ *are k-pairwise disjoint non-empty closed subsets of* X.

2. Following the proof of Theorem 3.4, it is easy to show that

$$\underline{D}(T,\mathcal{U}) = \underline{D}_e(T,\mathcal{U});$$

3. Notice that $\overline{D}_e(T,\mathcal{U}) \leq \overline{D}_p(T,\mathcal{U})$. *Hence,* $\overline{D}(T,\mathcal{U}) = \overline{D}_e(T,\mathcal{U}) \leq \overline{D}_p(T,\mathcal{U})$.

Theorem 3.5. *Let* (X,T) *be a TDS. Then,*

1. $\overline{D}_e(X,T) = \underline{D}_p(X,T) = \overline{D}_{pre}(T)$.
2. $\underline{D}_e(X,T) \leq \underline{D}(X,T) \leq \overline{D}_{pre}(T) \leq \overline{D}_p(X,T)$.

where those notations come from definition 2.2 and 3.2.

4 Polynomial growth type

Another conjugacy invariant that measures the orbit complexity of a dynamical system (X,T) is defined by the polynomial growth type in [1, 19]. With the same concept and notation, the upper and lower polynomial growth types of pre-image entropy for TDS (X,T) are defined as follows:

$$\overline{P}_{pre}(T,\mathcal{U}) = \limsup_{n\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x)}{\log n},$$

and

$$\underline{P}_{pre}(T,\mathcal{U}) = \liminf_{n\to\infty} \frac{\log \sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x)}{\log n},$$

Set

$$\overline{P}_{pre}(T) = \sup\{\overline{P}_{pre}(T,\mathcal{U}) : \mathcal{U} \in \mathcal{C}_X^0\} \text{ and } \underline{P}_{pre}(T) = \sup\{\underline{P}_{pre}(T,\mathcal{U}) : \mathcal{U} \in \mathcal{C}_X^0\},$$

Since

$$\lim_{n\to\infty} \frac{\log n}{n^r} = 0, \text{ for any } r > 0,$$

the velocity of n^r to approach ∞ is much faster than that of $\log n$. More precisely, it is obvious if the pre-image entropy dimension $\overline{D}_{pre}(T) = s > 0$ or $\underline{D}_{pre}(T) = s > 0$, then the value of $\overline{P}_{pre}(T)$ and $\underline{P}_{pre}(T)$ is always equal to infinity.

Therefore, if $\overline{D}_{pre}(T) = 0$, then the cardinality $\sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x)$ is bounded or goes to ∞ very slowly. Suppose the cardinality $\sup_{x\in X,k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x)$

approaches ∞ as $n \to \infty$, it is meaningful to compare the velocity between $\log n$ and $\log \sup_{x \in X} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-n}x)$ as $n \to \infty$.

The following calculations evaluate some basic examples.

(1) For the two-sided shift T on finite symbolic space $X = \prod_{-\infty}^{\infty} Y$ where $Y = \{0, 1, 2, ..., k\}$,

$$\overline{D}_{pre}(T) = \underline{D}_{pre}(T) = \overline{P}_{pre}(T) = \underline{P}_{pre}(T) = 0.$$

(2) For the one-sided shift T on finite symbolic space $X = \prod_{-\infty}^{\infty} Y$ where $Y = \{0, 1, 2, ..., k\}$,

$$\overline{D}_{pre}(T) = \underline{D}_{pre}(T) = 1, \text{ but } \overline{P}_{pre}(T) = \underline{P}_{pre}(T) = \infty.$$

(3) Let $T_\lambda(x) = \lambda \min\{x, 1-x\}$ be a family of functions defined on $[0, 1]$, where $0 < \lambda \leq 2$. These functions are called tent maps.

When $1 < \lambda \leq 2$,

$$\overline{D}_{pre}(T) = \underline{D}_{pre}(T) = 1, \text{ but } \overline{P}_{pre}(T) = \underline{P}_{pre}(T) = \infty.$$

When $0 < \lambda \leq 1$, we can easily verify that T_λ is contractive, thus,

$$\overline{D}_{pre}(T) = \underline{D}_{pre}(T) = \overline{P}_{pre}(T) = \underline{P}_{pre}(T) = 0.$$

5 Examples

Let $T : X \to X$ be a homeomorphism of the compact metric space X. Since the pre-image of each point is only one point, it is trivial that pre-image entropy dimension $D_{pre}(T) = 0$. The Morse system also has zero pre-image entropy dimension, since its complexity function has linear growth rate (see, for example, [12]). Given each value between 0 and 1, this section will construct a topological dynamical system which attains this value for pre-image entropy dimension.

Example 5.1. *For any $0 < \tau < 1$, there exists a TDS which has pre-image entropy dimension τ.*

The TDS comes from a uniformly recurrent infinite 0-1 word constructed by Cassaigne, see [7]. The construction of the infinite word is as follows and all notations follow from [8].

Let $\mathbb{N}^*$ and $\{0, 1\}^*$ denote the collection of finite or infinite words over $\mathbb{N}$ and $\{0, 1\}$, respectively. The dyadic valuation word $\mathbf{v}$ is defined by the limit of a sequence of finite words z_j, where $z_0 = 0, z_1 = 1, z_{j+1} = z_j j z_j, j = 1, 2, \cdots$. Define inductively the substitution $\psi : \mathbb{N}^* \to \{0,1\}^*$ and the family $(x_k)_{k\in\mathbb{N}}$ of prefixes of the dyadic valuation word $\mathbf{v}$ as follows:

- $\psi(0) = 0, \psi(1) = 1$;
- x_k is the longest prefix of $\mathbf{v}$ such that

$$|\psi(x_k)| \leq \max(\varphi^{-1}(k+1) - \varphi^{-1}(k) - 1, 0), \text{ where } \varphi(t) = t^\tau;$$

- for all $j \geq 1$, $\psi(2j) = \psi(x_{\lfloor \log j \rfloor})0\psi(j)$ and $\psi(2j+1) = \psi(x_{\lfloor \log j \rfloor})1\psi(j)$.

Let $\mathbf{u} = \psi(\mathbf{v})$ and (X, T) be the TDS generated by $\mathbf{u}$ under the left shift. Note that $\mathbf{u}$ is a uniformly recurrent word and its complexity $p_{\mathbf{u}}(n)$ satisfies:

$$\log p_{\mathbf{u}}(n) \sim n^\tau,$$

and for the forward generating open cover $\mathcal{U} = \{[0]_X, [1]_X\}$ of X,

$$\log \sup_{x\in X, k\geq 1} \aleph(\bigvee_{i=0}^{n-1} T^{-i}\mathcal{U}, T^{-k}x) = \log p_{\mathbf{u}}(n)$$

Thus, it is clear that $\underline{D}(T,\mathcal{U}) = \overline{D}(T,\mathcal{U}) = \tau$. Since $\mathcal{U}$ is a generator, we have that $\overline{D}_{pre}(T) = \overline{D}(T,\mathcal{U}) = \tau$ and $\underline{D}_{pre}(T) = \underline{D}(T,\mathcal{U}) = \tau$.

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