ON MOMENT CONVERGENCE RESULT FOR PROPERLY NORMALISED DELAYED SUMS

ABSTRACT

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common distribution function F. Let (S_n) be the partial sum sequence. Set $T_n = S_{n+a_n} - S_n = \sum_{k=n+1}^{n+a_n} X_k$. The sum T_n is called a (forward) delayed sum. We obtain a moment convergence result for the delayed sums, when the random variables are in the domain of normal attraction of a stable law with index α, $1 < \alpha < 2$. This result plays a vital role in studying a local limit theorem.

Key Words: Domain of normal attraction, stable law, delayed moment convergence, local limit theorem.

1. INTRODUCTION

Let $\{X_n, n \geq 1\}$ be a sequence of independent and identically distributed(i.i.d) random variables (r.v.s) with a common distribution function(d.f) F. Let (S_n) be the partial sum sequence. Let F belongs to the domain of normal attraction (DNA) of a stable law with index α, $1 < \alpha < 2$. Let $\{A_n, n \geq 1\}$ and $\{B_n, n \geq 1\}$ be the sequences of constants. Set $Y_n = \frac{S_n}{B_n} - A_n$ and $T_n = S_{n+a_n} - S_n = \sum_{k=n+1}^{n+a_n} X_k$.

The sum T_n is called a (forward) delayed sum and (a_n) be a sequence of real constants with $\limsup_{n \to \infty} \frac{a_n}{n} < \infty$.

When F belongs to the DNA of a stable law with index α, $1 < \alpha < 2$, Owen [8] obtained a moment convergence result for (Y_n). Considering the works of Owen [8], we envisage that the moment convergence result holds for properly normalized delayed sums. In the next section a needed lemma and the main theorems were presented. Throughout the paper c, with or without a suffix, stand for positive constant.

2. MOMENT CONVERGENCE RESULT

Lemma

There exists positive constants k and l_0, both independent of n, such that, for $t > l_0$,

$$P\left(\left|\frac{T_n}{n^{\frac{1}{\alpha}}}\right| > t\right) \le \frac{k}{t^{\alpha}} \tag{1}$$

Proof

For fixed n and t>1, define $Y_i = \begin{cases} X_i, & \text{if } |X_i| < t\,n^r \\ 0, & \text{otherwise} \end{cases}$,where $r = n^{\frac{1}{\alpha}}$. Then we have,

$$Y_i = \begin{cases} X_i, & \text{if } -t\,n^r < X_i < t\,n^r \\ 0, & \text{if } |X_i| \ge t\,n^r \text{ or } X_i \in \left(-\infty\,,\, tn^r\right] \cup \left[tn^r,\, \infty\right) \end{cases}$$

which implies

$$\sum_{i=n+1}^{n+a_n} Y_i = \begin{cases} \sum\limits_{i=n+1}^{n+a_n} X_i, & \text{if } -(a_n\text{-}1)t\,n^r < \sum\limits_{i=n+1}^{n+a_n} X_i < (a_n\text{-}1)t\,n^r \\ 0, & \text{if } \sum\limits_{i=n+1}^{n+a_n} X_i \le -(a_n\text{-}1)t\,n^r \text{ or } \sum\limits_{i=n+1}^{n+a_n} X_i \ge (a_n\text{-}1)t\,n^r \end{cases}$$

Let $V_i = X_i - Y_i$, $T_{1,n} = Y_{n+1} + Y_{n+2} + \ldots + Y_{n+a_n}$ and $T_{2,n} = V_{n+1} + V_{n+2} + \ldots + V_{n+a_n}$.

Observe that

$$\left\{\left|\frac{T_n}{n^r}\right| > t\right\} \Leftrightarrow \left\{\frac{T_n}{n^r} \in \left(\left(-\infty, t\right] \cup \left[t, \infty\right)\right)\right\} \Leftrightarrow \sum_{i=n+1}^{n+a_n} X_i < -t\,n^r \text{ (or) } \sum_{i=n+1}^{n+a_n} X_i > t\,n^r \tag{2}$$

$$\left\{\left|\frac{T_{1,n}}{n^r}\right| > t\right\} = \left\{\frac{T_{1,n}}{n^r} < -t \text{ (or) } \frac{T_{1,n}}{n^r} > t\right\} = \left\{\sum_{i=n+1}^{n+a_n} Y_i < -t\,n^r \text{ (or) } \sum_{i=n+1}^{n+a_n} Y_i > t\,n^r\right\} \tag{3}$$

and $\left\{T_{2,n} \ne 0\right\} = \left\{\sum\limits_{i=n+1}^{n+a_n} V_i \ne 0\right\} = \left\{\sum\limits_{i=n+1}^{n+a_n} \left(X_i - Y_i\right) \ne 0\right\} = \left\{\sum\limits_{i=n+1}^{n+a_n} X_i \ne \sum\limits_{i=n+1}^{n+a_n} Y_i\right\} = \left\{\sum\limits_{i=n+1}^{n+a_n} Y_i = 0\right\}$

From the definition of Y_i, we have

$$\left\{\sum_{i=n+1}^{n+a_n} Y_i = 0\right\} \Leftrightarrow \left\{\sum_{i=n+1}^{n+a_n} X_i \le -(a_n\text{-}1)tn^r \text{ (or) } \sum_{i=n+1}^{n+a_n} X_i > (a_n\text{-}1)tn^r\right\} \tag{4}$$

By the arguments (2), (3) and (4),we get that

$\left\{\left|\frac{T_n}{n^r}\right|>t\right\} \subset \left\{\left(\left|\frac{T_{1,n}}{n^r}\right|>t\right) \cup \left(T_{2,n} \neq 0\right)\right\}$ and hence $P\left\{\left|\frac{T_n}{n^r}\right|>t\right\} \leq P\left(\left|\frac{T_{1,n}}{n^r}\right|>t\right) + P\left(T_{2,n} \neq 0\right)$ (5)

Consider $P\left(T_{2,n} \neq 0\right) = P\left(\sum_{i=n+1}^{n+a_n} V_i \neq 0\right) = a_n P\left(V_1 \neq 0\right)$, where V_i's are i.i.d r.v.s

$$= a_n P\left(X_1 - Y_1 \neq 0\right) = a_n P\left(Y_1 \neq X_1\right) = a_n P\left(Y_1 = 0\right) = a_n P\left(\left|X_1\right| > t n^r\right)$$, by definition of Y_i.

Since X_n's are in domain of normal attraction of a stable law with index α ,1<α<2,we have, for some constant c>0 such that

$$F(x) \leq c\left|x\right|^{-\alpha} \text{ , if } x < 0 \text{ and } 1 - F(x) \leq c x^{-\alpha} \text{ , if } x > 0 \tag{6}$$

Since $\limsup_{n \to \infty} \frac{a_n}{n} < \infty$ and using (6), one can find some constant $c_1 > 0$, such that

$$P\left(T_{2,n} \neq 0\right) = a_n P\left(\left|X_1\right| > t n^r\right) \square \frac{c}{t^\alpha} \frac{a_n}{n} < \frac{c_1}{t^\alpha}. \tag{7}$$

Notice that $E(X_1) = 0$ implies $0 = E(X_1) = \int_{-\infty}^{\infty} x_1 dF(x) = \int_{|x|>tn^r} x_1 dF(x) + \int_{|x|<tn^r} x_1 dF(x)$

$$\Leftrightarrow \int_{|x|<tn^r} x_1 dF(x) = -\int_{|x|>tn^r} x_1 dF(x) \Leftrightarrow \left|E(Y_1)\right| = \left|-\int_{|x|>tn^r} x dF(x)\right| = \int_{|x|>tn^r} x dF(x).$$

Consider EY_1 and integrating by parts, we have,

$$EY_1 = \int_{|x|>tn^r} x dF(x) = x F\Big|_{|x|>tn^r} - \int_{|x|>tn^r} F dx = xF\Big|_{-\infty}^{-tn^r} + xF\Big|_{tn^r}^{\infty} - \int_{-\infty}^{-tn^r} F dx - \int_{tn^r}^{\infty} F dx$$

Again using (6), there exists some positive constant $c_2(>c_1)$,such that,

$$EY_1 \leq c_2\ x|x|^{-\alpha}\Big|_{-\infty}^{-tn^r} + x\left(1 - c_2 x^{-\alpha}\right)\Big|_{tn^r}^{\infty} - \int_{-\infty}^{-tn^r} c_2 |x|^{-\alpha} dx - \int_{tn^r}^{\infty} (1 - c_2 x^{-\alpha}) dx$$

$$\leq c_2(-tn^r)(tn^r)^{-\alpha}\Big| + \infty + x\Big|_{tn^r}^{\infty} - c_2 x^{-\alpha+1}\Big|_{tn^r}^{\infty} - c_2\left(\frac{(-x)^{-\alpha+1}}{-\alpha+1}\right)\Bigg|_{-\infty}^{tn^r} - \int_{tn^r}^{\infty} dx + c_2 \int_{tn^r}^{\infty} x^{-\alpha} dx$$

$$\leq c_2\left(tn^r\right)^{-\alpha+1} + \infty + \infty - tn^r - \infty + c_2\left(tn^r\right)^{-\alpha+1} - c_2 \frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} - \infty - x\Big|_{tn^r}^{\infty} + c_2 \frac{x^{-\alpha+1}}{-\alpha+1}\Bigg|_{tn^r}^{\infty}$$

$$\leq c_2\left(tn^r\right)^{-\alpha+1} - tn^r + c_2\left(tn^r\right)^{-\alpha+1} - c_2 \frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} - \infty + tn^r + \infty - c_2 \frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1}$$

$\le c_2\left(tn^r\right)^{-\alpha+1} + c_2\left(tn^r\right)^{-\alpha+1} - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} - c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1}$, which implies,

$\left|EY_1\right| \le 2\,c_2\left(tn^r\right)^{-\alpha+1} + 2\,c_2\frac{\left(tn^r\right)^{-\alpha+1}}{-\alpha+1} \le \frac{2c_2}{t^{\alpha-1}n^{1-r}}\left(1+\frac{1}{\alpha-1}\right) \le \frac{c_3\alpha}{(\alpha-1)t^{\alpha-1}n^{1-r}}$, where $c_3(>c_2)$ is some positive constant. Observe that

$$\left|E\left(\frac{T_{1,n}}{n^r}\right)\right| = \left|\frac{1}{n^r}E\left[Y_{n+1} + Y_{n+2} + ... + Y_{n+a_n}\right]\right| \le \frac{1}{n^r}\left[\left|EY_{n+1}\right| + \left|EY_{n+2}\right| + ... + \left|EY_{n+a_n}\right|\right] \le \frac{1}{n^r}\left[\frac{c_3\ n\ \alpha}{(\alpha-1)t^{\alpha-1}n^{1-r}}\right]$$

$$= \frac{c_3\alpha}{(\alpha-1)t^{\alpha-1}}.$$

Since α>1 and t>1, we must have $\left|E\left(\frac{T_{1,n}}{n^r}\right)\right| \le \frac{c_3\alpha}{\alpha-1}$. Let $B = \frac{c_3\alpha}{\alpha-1}$ and s=1+B. Henceforth, we take t>s.

Then, $P\left(\left|\frac{T_{1,n}}{n^r}\right| > t\right) \le P\left(\left|\frac{T_{1,n}}{n^r} - \frac{EY_1}{n^{1-r}}\right| > t - B\right)$ (8)

From Chebyshev's inequality, we get, $P\left(\left|\frac{T_{1,n}}{n^r} - \frac{EY_1}{n^{1-r}}\right| > t - B\right) \le \frac{V\left(\frac{T_{1,n}}{n^r}\right)}{\left(t - B\right)^2}$. (9)

Observe that $V\left(\frac{T_{1,n}}{n^r}\right) = \frac{1}{n^{2r}}V(T_{1,n}) = \frac{n}{n^{2r}}V(Y_1) = n^{1-2r}V(Y_1)$.

Since $V(Y_1) \le E(Y_1^2) = \int_{|y|\le tn^r} y^2 dF(y) = \int_{-tn^r}^{tn^r} y^2 dF(y)$. Using integration by parts, we get,

$$V(Y_1) \le y^2F(y)\Big|_{-tn^r}^{tn^r} - \int_{-tn^r}^{tn^r} F(y)dy^2 \le y^2F(y)\Big|_{-tn^r}^{0} + y^2F(y)\Big|_{0}^{tn^r} - \int_{-tn^r}^{0} 2yF(y)dy - \int_{0}^{tn^r} 2yF(y)dy.$$

From the fact (6), we have for some constant $c_4>0$,

$$V(Y_1) \le y^2c_4y^{-\alpha}\Big|_{-tn^r}^{0} + y^2(1 - c_4\ y^{-\alpha})\Big|_{0}^{tn^r} - 2\int_{-tn^r}^{0} y\ c_4\ y^{-\alpha}dy - 2\int_{0}^{tn^r} y(1 - c_4y^{-\alpha})dy$$

$$\le 0 - c_4(tn^r)^2(tn^r)^{-\alpha} + (tn^r)^2 - c_4(tn^r)^2(tn^r)^{-\alpha} - 2c_4\frac{y^{-\alpha+2}}{-\alpha+2}\Bigg|_{-tn^r}^{0} - 2\left(\frac{y^2}{2}\right)\Bigg|_{0}^{tn^r} + 2c_4\left(\frac{y^{-\alpha+2}}{-\alpha+2}\right)\Bigg|_{0}^{tn^r}$$

$$\le -2c_4\ (tn^r)^{2-\alpha} + (tn^r)^2 - \frac{2c_4}{-\alpha+2}\left[0 - (tn^r)^{2-\alpha}\right] - (tn^r)^2 + \frac{2c_4}{-\alpha+2}\left[(tn^r)^{2-\alpha}\right]$$

$$\le -2c_4(tn^r)^{2-\alpha} + \frac{2c_4}{-\alpha+2}(tn^r)^{2-\alpha} + \frac{2c_4}{-\alpha+2}(tn^r)^{2-\alpha} = -2c_4(tn^r)^{2-\alpha} + \frac{4c_4}{-\alpha+2}(tn^r)^{2-\alpha}$$

$$\le -2c_4(tn^r)^{2-\alpha}\left[\frac{2}{2-\alpha} - 1\right] = \frac{2\ \alpha\ c_4}{2-\alpha}(tn^r)^{2-\alpha}.$$

There exists a constant $c_5 > 0$, independent of n, such that $V(Y_1) \le c_5(tn^r)^{2-\alpha}$.

$V\left(\frac{T_{1,n}}{n^r}\right) = n^{1-2r}V(Y_1) \le n^{1-2r}c_5(tn^r)^{2-\alpha} \le c_5 t^{2-\alpha}$. From (8) and (9), we get,

$$P\left(\left|\frac{T_{1,n}}{n^r}\right| > t\right) \le c_5(1-B)^{-2}t^{2-\alpha} \le t^{-2}c_5\left(1-\frac{B}{t}\right)^{-2}t^{2-\alpha} \le c_5\left(1-\frac{B}{t}\right)^{-2} t^{-\alpha} \le c_5\,(1-B)^{-2}\,t^{-\alpha} \qquad (10)$$

If we let $k=c_5(1-B)^{-2}+c_1$, then (7) and (10) give the result.

Theorem1

For each real number q with $0 \le q < \alpha$, there exists a finite positive real number Q, depending on q but independent of n, such that

$$E\left(\left|\frac{T_n}{n^r}\right|^q\right) \le Q \qquad (11)$$

In particular, there exists a constant M, independent of n, such that $E(|T_n|) \le Mn^r$ (12)

Proof. We can observe that the result is true for q=0. Choose q such that $0<q<\alpha$. Let N_0 and l_0 as in Lemma1.Then $E\left(\left|\frac{T_n}{n^r}\right|^q\right) = \int_0^{l_0} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) + \int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right)$.

Now $\int_0^{l_0} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) \le l_0^{\,q}$ and the next integral gets,

$$\int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) = \int_{l_0}^{\infty} x^q d\left(1-P\left(\left|\frac{T_n}{n^r}\right| > x\right)\right) = -\int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| > x\right)$$

$= -x^q\, P\left(\left|\frac{T_n}{n^r}\right| > x\right)\Bigg|_{l_0}^{\infty} + q\int_{l_0}^{\infty} x^{q-1}P\left(\left|\frac{T_n}{n^r}\right| > x\right)dx = -\infty + l_0^{\,q}\, P\left(\left|\frac{T_n}{n^r}\right| > l_0\right) + q\int_{l_0}^{\infty} x^{q-1}\frac{c_2}{x^{\alpha}}dx$, by Lemma1,

$$= -\infty + l_0^{\,q}P\left(\left|\frac{T_n}{n^r}\right| > l_0\right) + qc_2\int_{l_0}^{\infty} x^{q-1-\alpha}dx = -\infty + l_0^{\,q}\, P\left(\left|\frac{T_n}{n^r}\right| > l_0\right)\Bigg| + qc_2\,\frac{x^{q-\alpha}}{q-\alpha}\Bigg|_{l_0}^{\infty}$$

$= -\infty + l_0^{\,q}\frac{c_2}{l_0^{\,q}} + \infty + qc_2\,\frac{l_0^{\,q-\alpha}}{\alpha-q}$, again by Lemma1, we have

$$= l_0^{q-\alpha}\left(c_2 + \frac{qc_2}{\alpha-q}\right).$$

Now $\int_{l_0}^{\infty} x^q dP\left(\left|\frac{T_n}{n^r}\right| \le x\right) \le l_0^{\,q} + l_0^{q-\alpha}\left(c_2+\frac{qc_2}{\alpha-q}\right)$. Letting $Q = l_0^{\,q} + l_0^{q-\alpha}\left(c_2+\frac{qc_2}{\alpha-q}\right)$ gives (11) and with q=1, equation (12) follows from (11).

Theorem 2

$$\underset{n \to \infty}{\mathrm{Lim}}\, E\left(\frac{T_n}{n^r}\right) = E(Y) \qquad (13)$$

Moreover, for all q with 0≤q<α, we have, $\underset{n \to \infty}{\mathrm{Lim}}\, E\left(\left|\frac{T_n}{n^r}\right|^q\right) = E\left(|Y|^q\right)$ (14)

Proof

Choose p such that 0<q<p<α. Let u= $u = \frac{p}{q}$ and let G(t)=|t|$^{\alpha}$. Then $\frac{G(t)}{t} \to \infty$, as $t \to \infty$. By the convergence theorem, Loeve[5], page 183, we know that for (3) to hold $\left|\frac{T_n}{n^r}\right|$ should be uniformly integrable. Also Theorem 22, Meyer, Paul[6], shows that $\left|\frac{T_n}{n^r}\right|^q$ is uniformly integrable, whenever $\sup_n E\left(G\left(\left|\frac{T_n}{n^r}\right|^q\right)\right) < \infty$.

But observe that $E\left(G\left(\left|\frac{T_n}{n^r}\right|^q\right)\right) = E\left(\left|\frac{T_n}{n^r}\right|^q\right)$. From Theorem 1, we have $\sup_n E\left(\left|\frac{T_n}{n^r}\right|^q\right) \le Q$. Hence $\left|\frac{T_n}{n^r}\right|^q$ is uniformly integrable and in turn (13) is established.

In particular, when q=1, by observing that the uniform integrability of $|T_n|$ Implies that of T_n and again appealing to the convergence theorem, page183, Loeve [6], (14) gets established.

Note: Our results can be used to obtain a density version of a local limit theorem. See SujitK. Basu [7] and Gooty Divanji and Tadewas Koroto[2].

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