Logistics Financial Function of the Fractal Dispersion of the Hausdorff Measure prior to Crash market Signal

Abstract

The world of finance works better through logistics. We consider the logistics function in large market crashes corresponding to values of packing dimension of R^n or $\alpha(max)$ by analyzing the fractal dispersion of Hausdorff prior to market signal with constraint of a set zero heat capacity. Also advocated is a stochastic Ito's iterated procedure for locating optimal market crises signal relative to the Heat equation to give an early warning of them. The resultant differential equation was solved to obtain the recurrence formula.

Keywords: Fractal Dispersion, Hausdorff Measure, Crash Market Signal, Logistic Financial Function, Confluent Hyper-geometric Equation.

1.0 Introduction

There are many ways of trying to predict and manage market crises. The oldest and perhaps the simplest approach in financial markets is termed fundamental analysis [1]. This approach involves an investigative analysis of a company, industry or market and or the economy, around a proposed investment or trade, which can reveal the cause of stock price changes. The result can then be used to try and predict a stock's next move. The assumption in this approach is that if one can understand the underlying cause early enough, then one can forecast the event and take the appropriate action to manage the crises or investment. Major trading houses still employ large numbers of fundamental analysts to find discernible patterns [2] which are at times quite accurate. However, on an open market, this type of analysis can be severely limited. The precise mechanism that links news to price is often inconsistent and is open to individual interpretation. The second oldest form of analysis is called technical analysis [3]. This deals with the recognition of patterns and the study of price, volume and indicator charts, in search of clues as to whether to buy or sell. After falling out of favour in the 1980s, the financial world expanded again in the 1990s as the public took to the internet to trade stocks and shares online. From this evolution, modern financial theory was born, involving analysis methods that emerged from the mathematics of chance, stochastic field and modelling. In this context, market crises may be considered to be a measurable quantity, and is therefore manageable. Published articles outlining technical trading approaches can be traced to the 1960s, where a trading strategy was proposed for copper futures, for example [4]. In 1988, a study was undertaken to apply technical trading rules to twelve futures markets which obtained returns of up to 5.6% over a period of six years [5]. In 2007, an extensive review of technical trading strategies applied to options and futures markets was carried out [6]. It was found that the average annual profit for almost 100 different approaches was between 4% and 6%. The term fractal geometry was first coined by Benoit Mandelbrot [7]. The term fractal is derived from the Latin adjective fractus which means to create irregular fragments. In the 1960s, Benoit Mandelbrot considered the basis for a selfsimilar geometry in papers such as ‘How long is the Coast of Britain? Statistical self-Similarity and Fractional Dimension [8]. In 1975, he consolidated the results by coining the word ‘fractal’. By 1982, he expanded these ideas in his book ‘The Fractal Geometry of Nature [7], a work which introduced fractal geometry into the main stream of professional mathematics.

The confluent hyper-geometric equation is an important differential equation that is used in many areas of classical and quantum physics, chemistry and engineering [9]. This equation also arises in optics [10,11], classical electrodynamics [9,11], classical waves [12,13], diffusion [14], fluid flow [15], heat transfer [16], general relativity [17,18], semi-classical quantum mechanics [19], quantum chemistry [20,21], graphic design [22], finance and many other areas. The solutions of the confluent hyper-geometric equation depend in an essential way on whether or not $\beta, \beta + 2$ and $\beta - (\beta + 2)$ are integers. A comprehensive discussion of the history of the hyper-geometric function, from which confluent hyper-geometric functions are descended, has been written by Dutka [23]. Unlike most other special functions, which were defined as the solutions of their corresponding differential equation, the hyper-geometric functions were first defined in terms of their power series and the differential equation that they satisfy was discovered later. Since the hyper-geometric functions are not defined or are not different for some integer values of their parameters, this introduces challenges with describing all of the solutions of the corresponding differential equation. Othman [24] considered the use of confluent hyper-geometric functions in determining the bound states of the alternative coulomb potential. This work deals with the logistic function in large market crashes corresponding to values of packing dimension by analyzing the fractal dispersion of Hausdorff prior to market signal with constant of a set zero heat capacity.

2.0 Basic Tools and Preliminaries

One of the several different techniques in investigating the size of a zero Hausdorff measure in R^n is the notion of packing dimension [25] defined through the packing measure as follows. Start with a class φ of monotone function $h: (0, \delta) \to (0,1)$ which is non-decreasing, right continuous and satisfies $h(0^+) = 0$ and for which there is a constant.

$$c_1 > 0: h(2s) \leq c_1 h(s) \; for \; 0 < s < \frac{\delta}{2} \qquad (1)$$

Packing dimension is now obtained by two definitions. We first define the actual pre-measure

$$h - \bar{p}(E) = \lim_{\sigma \to 0} \sup\left\{\sum_{i=1}^{\infty} h(2r_i): B_{r_i}(x_i)\right\}$$

(2)

disjoint $x_i \in E, r_i < \delta$, where $B_{r_i}(x_i)$ denotes the open ball centered on x_i with radius r_i. Next, we define the expected outer-measure.

$$h - p(E) = \inf\{\sum_{i=1}^{\infty} h - \bar{p}(E_i): E \subset \bigcup_{i=1}^{\infty} E_i\} \qquad (3)$$

Equation (3) can be generalized to Hausdorff measure using maximal packing of the outer-measure of E so that if $h(s) = s^n$, then $h - p(\,)$ on R^n is n-dimensional Hausdorff measure.

Definition 2.1 : In Hausdorff, let ϕ be a Borel measure on R^n and $h \in \phi$, then for any Borel set $E \subseteq R^n$,

$$h - p(E) \geq c_1 \phi(E)\ inf_{x \in E}(D_n(x))^{-1} \tag{4}$$

where c_1 is a constant and

$$D_n(x) = \lim_{r \to 0} \inf \frac{\phi(B_r(x))}{h(2r)} = \frac{h(s)}{s^n} = \frac{\mu(B(x,r))}{r^{\alpha}(|\log r|^{\lambda})}$$
(5)

For the analysis of subset B of R^n of zero Hausdorff measure, we have $\frac{h(s)}{s^n} \to \infty$ as $s \to 0$, so that if $h(s) = s^{\alpha}, \alpha > 0$, $h - m(B) = h - p(E)$ turns out to be either zero or infinity. In other words, to measure the Borel subset of $E \subseteq R^n$, we need $\lim_{s \to 0} \frac{s^n}{h(s)} = 0$ so that if $h(s) = s^{\alpha}, \alpha > 0$, there is a unique value α for which the packing measure $h - p(E)$ drops from infinity to zero. Define $Dim\ E$, the packing measure of E as

$$\inf\{\alpha > 0 : s^{\alpha} - p(E) = 0\} = p(E) \Rightarrow \lim\inf \frac{\phi(B_r(x))}{h(2r)} \tag{6}$$

$$\sup\{\alpha > 0 : s^{\alpha} - p(E) = \infty\} = h \Rightarrow \sup \frac{\phi(B_r(x))}{h(2r)} \tag{7}$$

3.0 Applications

Consider a market comprising h unit of asset in long position and one unit of the option in short position. At time T, the market value is assumed to be

$$h - p(E) = h - m(B)\ or\ h(s) - \bar{v}. \tag{8}$$

After an elapse Δt, the value of the market will change by an amount $h(\Delta s + D\Delta t) - \Delta \bar{v}$ in view of the dividend received on h unit held. By Ito's lemma and relative to the heat equation

$$h(\mu s \Delta t + \sigma s \Delta z + D\Delta t) - \left\{\left(\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial s}\mu s + \frac{1}{2}\frac{\partial^2 \bar{v}}{\partial s^2}\sigma^2 s^2\right)\Delta t + \frac{\partial \bar{v}}{\partial s}\sigma s \Delta z\right\} \tag{9}$$

or

$$\left\{h\mu s + hD - \left(\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial s}\mu s + \frac{1}{2}\frac{\partial^2 \bar{v}}{\partial s^2}\sigma^2 s^2\right)\right\}\Delta t + \left(h\sigma s - \frac{\partial \bar{v}}{\partial s}\sigma s\right)\Delta z \tag{10}$$

Let $h = \frac{\partial \bar{v}}{\partial s}$, the uncertainty term disappears and the market in this case is temporarily riskless. It should therefore grow in value by the riskless rate in force. That is

$$\left\{h\mu s + hD - \left(\frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial s}\mu s + \frac{1}{2}\frac{\partial^2 \bar{v}}{\partial s^2}\sigma^2 s^2\right)\right\}\Delta t = (h(s) - \bar{v})r\Delta t \tag{11}$$

Hence,

$$D\frac{\partial \bar{v}}{\partial s} - \left(\frac{\partial \bar{v}}{\partial t} + \frac{1}{2}\frac{\partial^2 \bar{v}}{\partial s^2}\sigma^2 s^2\right) = \left(\frac{\partial \bar{v}}{\partial s}s - \bar{v}\right)r \tag{12}$$

So

$$\frac{\partial \bar{v}}{\partial t} + (rs - D)\frac{\partial \bar{v}}{\partial s} + \frac{1}{2}\frac{\partial^2 \bar{v}}{\partial s^2}\sigma^2 s^2 = r\bar{v} \tag{13}$$

Take $z = \frac{\alpha}{s}$; $\bar{v}(s) = z^{\beta} w(z)$. Thus

$$\frac{dz}{ds} = -\frac{\alpha}{s^2} = -\frac{1}{\alpha}z^2 \tag{14}$$

$$\frac{d\bar{v}}{ds} = -\frac{1}{\alpha}z^2\left(\beta z^{\beta-1}w + z^{\beta}\frac{dw}{dz}\right) = -\frac{1}{\alpha}\left(\beta z^{\beta+1}w + z^{\beta+2}\frac{dw}{dz}\right). \tag{15}$$

Hence,

$$\frac{d^2\bar{v}}{ds^2} = -\frac{1}{\alpha}z^2\left(\beta(\beta+1)z^{\beta}w + \beta z^{\beta-1}\frac{dw}{dz} + (\beta+2)z^{\beta+1}\frac{dw}{dz} + z^{\beta+2}\frac{d^2w}{dz^2}\right) \tag{16}$$

Since $\bar{v}$ is not dependent on r, substituting into the given differential equation gives

$$rz^{\beta}w = \frac{\sigma^2}{2}\left(\beta(\beta+1)z^{\beta}w + \beta z^{\beta+1}\frac{dw}{dz} + 2(\beta+1)z^{\beta+1}\frac{dw}{dz} + z^{\beta+2}\frac{d^2w}{dz^2}\right)$$
$$+\left(\frac{r\alpha}{z} - D\right)\left(-\frac{1}{\alpha}\right)\left(\beta z^{\beta+1}w + z^{\beta+2}\frac{dw}{dz}\right) \tag{17}$$

Cancelling by z^{β} and collecting like terms give

$$0 = \frac{\sigma^2}{2}z^2\frac{d^2w}{dz^2} + \frac{dw}{dz}\left(\sigma^2(\beta+1)z - rz + \frac{D}{\alpha}z^2\right) + w\left(\left(\frac{\sigma^2}{2}\right)\beta(\beta+1) - r\beta + \beta\frac{D}{\alpha}z\right) - rw \tag{18}$$

Or

$$0 = \frac{\sigma^2}{2}z^2\frac{d^2w}{dz^2} + \frac{dw}{dz}z\left(\sigma^2(\beta+1) - r + \frac{D}{\alpha}z\right) + w\left(\frac{\sigma^2}{2}\beta(\beta+1) - r(\beta+1) + \beta\frac{D}{\alpha}z\right) \tag{19}$$

To cancel by z take β so that $\frac{1}{2}\sigma^2\beta = r$ and let $\frac{D/\alpha}{\sigma^2/2} = -1$. Therefore $\beta = \frac{-2D}{\sigma^2}$. We then obtain

$$0 = z\frac{d^2w}{dz^2} + \frac{dw}{dz}(\beta + 2 - z) - w\beta \tag{20}$$

Equation (20) is a form of confluent hyper-geometric equation. It is then solved to obtain a recurrence formula.

Solution at $z = 0$: let

$$P_0(z) = -\beta$$
$$P_1(z) = (\beta + 2) - z \tag{21}$$
$$P_2(z) = z$$

Then, $P_2(0) = 0$. Hence, $z = 0$ is a singular point. Let us start with $z = 0$. To see if it is regular, we study the following limits:

$$\lim_{z\to z_0}\frac{(z-z_0)P_1(z)}{P_2(z)} = \lim_{z\to 0}\frac{(z-0)\{(\beta+2)-z\}}{z} = \beta+2 \quad (22)$$

$$\lim_{z\to z_0}\frac{(z-z_0)^2P_0(z)}{P_2(z)} = \lim_{z\to z_0}\frac{(z-z_0)^2(-\beta)}{z}$$

$$= \lim_{z\to 0}\frac{z^2(-\beta)}{z} = 0 \quad (23)$$

Hence, both limits exist and $z = 0$ is a regular singular point.

Therefore we assume the solution of the form

$$w = \sum_{n=0}^{\infty} a_n z^{n+r} \quad (24)$$

with $a_0 \neq 0$.

Hence,

$$w' = \sum_{n=0}^{\infty} a_n (n+r)\, z^{n+r-1} \quad (25)$$

$$w'' = \sum_{n=0}^{\infty} a_n (n+r)(n+r-1)\, z^{n+r-2} \quad (26)$$

Substituting equations (24), (25) and (26) into equation (20) gives

$$z\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)\, z^{n+r-2}$$
$$+(\beta+2)\sum_{n=0}^{\infty} a_n(n+r)\, z^{n+r-1} \quad (27)$$
$$-z\sum_{n=0}^{\infty} a_n(n+r)z^{n+r-1}$$
$$-\beta\sum_{n=0}^{\infty} a_n z^{n+r} = 0$$

Simplifying equation (27) gives

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)\, z^{n+r-1}$$
$$+(\beta+2)\sum_{n=0}^{\infty} a_n(n+r)\, z^{n+r-1} \quad (28)$$
$$-\sum_{n=0}^{\infty} a_n(n+r)z^{n+r}$$
$$-\beta\sum_{n=0}^{\infty} a_n z^{n+r} = 0$$

In order to simplify equation (28), we need all powers of z to be the same, that is, equal to $n+r-1$, the smallest power of z. Hence, we switch the indices as follows.

$$\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)\, z^{n+r-1}$$
$$+(\beta+2)\sum_{n=0}^{\infty} a_n(n+r)\, z^{n+r-1} \quad (29)$$

$$-\sum_{n=1}^{\infty} a_{n-1}(n+r-1)z^{n+r-1}$$

$$-\beta \sum_{n=1}^{\infty} a_{n-1} z^{n+r-1} = 0$$

Isolating the first terms of the sums starting from 0 gives

$$\begin{aligned} & a_0\{r(r-1)+(\beta+2)r\}z^{r-1} \\ & +\sum_{n=1}^{\infty} a_n(n+r)(n+r-1)\, z^{n+r-1} \\ & +(\beta+2)\sum_{n=1}^{\infty} a_n(n+r)\, z^{n+r-1} \\ & -\sum_{n=1}^{\infty} a_{n-1}(n+r-1)z^{n+r-1} \\ & -\beta \sum_{n=1}^{\infty} a_{n-1} z^{n+r-1} = 0 \end{aligned} \tag{30}$$

From the linear independence of all powers of z, that is, of the functions $1, z, z^2$ and so forth, the coefficients of z^r vanish for all r. Hence, from the first term, we have

$$a_0\{r(r-1)+(\beta+2)r\} \tag{31}$$

which is the indicial equation.

Since $a_0 \neq 0$, we have

$$r(r-1)+(\beta+2)r = 0 \tag{32}$$

Hence, the solutions of the above indicial equation are given below:

$$r^2 - r + r\beta + 2r = 0$$

$$r(r+1+\beta) = 0$$

$$\Rightarrow r = 0 \; or \; r = -(1+\beta)$$

That is

$$r_1 = 0\,;\; r_2 = -(1+\beta) \tag{33}$$

Also, from the rest of the terms, we have

$$(n+r)\{(n+r-1)+(\beta+2)\}a_n = \{(n+r-1)+\beta\}a_{n-1} \tag{34}$$

$$a_n = \frac{\beta+(n+r-1)}{(n+r)\{(\beta+2)+(n+r-1)\}}\, a_{n-1} \tag{35}$$

for $n \geq 1$.

For $r = 0$,

$$a_n = \frac{\beta+(n-1)}{n\{(\beta+2)+(n-1)\}}\, a_{n-1}, n \geq 1 \tag{36}$$

Equation (36) is the required recurrence formula.

4.0 Conclusion

Where the consideration of non-Gaussian behaviour is relevant, fractal geometry helps in financial modelling analysis. This is most relevant when price movements do not fully follow mild assumption and where we determine that there is long-term dependence or market memory. Hence, a more accurate non-Gaussian model of price movements can open the way for a new, more reliable type of financial theory, one that takes account of both mild and wild markets. By plotting different values of n and β we expect to identify patterns characterizing the stock price index returns preceding significant market drawdowns. Identification of such a pattern would contribute to the empirical evidence of the possible predictability of market crashes. Such patterns preceding market crashes can be caused by a slow build-up of long-range time dependencies reflecting interactions among traders. Hausdorff measure in this case is to determine which subsets of R^{n+1} where R^n is the n-dimensional Euclidean space (size of the market) are of zero heat capacity that is where there is no market signal and hence market crash with respect to the heat equation. With Hausdorff measure and the heat equation via packing dimension $h(s) - \bar{v}$ and $\frac{h(s)}{s^n}$ there is no market signal as it tends to zero hence, the market is likely to crash at that point indicating shortfall on the wealth investment.

5.0 References

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