Short Research Article

Almost Periodic Solution of a Class of Generalized Sine-Gordone Equation

ABSTRACT

The aim of this paper is to prove the existence of non-negative almost periodic solutions of a kind of generalized Sine-Gordon equation. The main method used is the maximum principle of telegraph equation established by Mawhin, Ortega and Robles-Pérez. The main technique used is Banach fixed point theorem in functional analysis. The conclusion is that when the coefficients and nonlinear terms of the equation meet certain conditions, the generalized equation has a unique non-negative almost periodic solution. Generalized the results of Mawhin, Ortega and Robles-Pérez.

Keywords: Sine-Gordon equation, almost periodic solution, maximum principle, Banach fixed point theorem

1. INTRODUCTION

In this paper, we discuss the existence of almost periodic solutions for a class of generalized Sine-Gordon equations by using the maximum principle of bounded weak solutions of telegraph equations established by Mawhin and Ortega. The telegraph equation is an important mathematical and physical equation, and its form is:

$$Lu+\lambda u=u_{tt}-\Delta_x u+cu_t+\lambda u=f(t,x),$$

where $c>0,\lambda\in\mathbb{R}$. In [1, 2], Mawhin and Ortega studied a nonlinear form of the telegraph equation, namely Sine-Gordon equations:

$$u_{tt}-\Delta_x u+cu_t+\lambda\sin u=f(t,x),$$

which $u\in\mathcal{D}'(\mathbb{R}\times\mathbb{T}^n)$, and its almost periodic solution has been studied. The result is as follows:

If $c>0,\lambda\in(0,c^2/4],f\in AP(\mathbb{R}\times\mathbb{T}^n),\parallel f\parallel_{L^\infty}<\lambda$, then the above equation has a solution that satisfies

$$\parallel u\parallel_{L^\infty}<\frac{\pi}{2},$$

which is unique and almost periodic. Almost periodic solution is a generalization of periodic solution. In some practical problems, considering almost periodic solution of differential equation is more practical than considering periodic solution of differential equation. So it is of great significance to discuss almost periodic solutions of differential equations. You can refer to [4, 5, 6] for more information on almost periodic functions. Inspired by their work in [1, 2, 3], this paper discusses a broader class of generalized Sine-Gordon equations:

$$u_{tt}-\Delta_x u+cu_t+g(\varphi(u))=f(t,x),$$

where

$$g(x)=\sum_{k=1}^{m}a_k x^{2k-1},\varphi(u)=\lambda\sin u-\mu\cos u.$$

We will prove the existence of non-negative almost periodic solutions to this equation under certain conditions.

2. PREPARATION

There are some basic lemmas about the maximum principle to prepare for proving the main theorem in this paper. The spatial dimension n in this paper refers to $n = 1, 2, 3$.

Definition 1. Make $c > 0, f \in L^{\infty}(\mathbb{R} \times \mathbb{T}^3)$, where $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Consider the equation

$$Lu + \lambda u = u_{tt} - \Delta_x u + cu_t + \lambda u = f(t, x), \tag{2.1}$$

if the bounded solution $u(t, x)$ of this equation in $\mathbb{R} \times \mathbb{R}^3$ satisfies

$$\begin{aligned} u(t, x_1, x_2, x_3) &= u(t, x_1 + 2\pi, x_2, x_3) = u(t, x_1, x_2 + 2\pi, x_3) \\ &= u(t, x_1, x_2, x_3 + 2\pi), \end{aligned}$$

we call that u is a solution of equation (2.1) in $L^{\infty}(\mathbb{R} \times \mathbb{T}^3)$. The definition of bounded solution in $L^{\infty}(\mathbb{R} \times \mathbb{T}^2)$ and in $L^{\infty}(\mathbb{R} \times \mathbb{T})$ are similar.

Note: to call u the bounded solution of equation (2.1) in $\mathbb{R} \times \mathbb{R}^n$ means that

$$\int_{\mathbb{R}\times\mathbb{T}^n} (L^*\phi + \lambda\phi)u = \int_{\mathbb{R}\times\mathbb{T}^n} f\phi$$

is true for all $\phi \in \mathcal{D}(\mathbb{R} \times \mathbb{T}^n)$, where $L^*\phi = \phi_{tt} - \Delta_x\phi - c\phi_t$.

Definition 2. Given a function $f: \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ and a vector $\alpha = (t_0, x_0) \in \mathbb{R} \times \mathbb{T}^n$, the translate $T_\alpha f$ is defined as

$$(T_\alpha f)(t, x) = f(t + t_0, x + x_0).$$

Definition 3. If a function $f: \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}$ is continuous and from every sequence α_n in $\mathbb{R} \times \mathbb{T}^n$ it is possible to extract a subsequence α_k such that $T_{\alpha_k} f$ has a uniform limit. It means that for any $\epsilon > 0$, there is a $N = N(\epsilon) > 0$ such that $\left|\left(T_{\alpha_k} f\right)(t, x) - g(t, x)\right| < \epsilon$ for any $k > N$ and any $(t, x) \in \mathbb{R} \times \mathbb{T}^n$. Then we say that f is almost periodic, and write

$$f \in AP(\mathbb{R} \times \mathbb{T}^n).$$

It can be proved that $AP(\mathbb{R} \times \mathbb{T}^n)$ becomes Banach space with L^{∞} norm, and embedded into $L^{\infty}(\mathbb{R} \times \mathbb{T}^n) \cap C(\mathbb{R} \times \mathbb{T}^n)$. In order to prove the main result of this paper, we need to use several lemmas about the maximum principle.

Lemma 1 ([1, 2, 5]). For each $\lambda \in (0, c^2/4]$ and $f \in L^{\infty}(\mathbb{R} \times \mathbb{T}^n)$, equation (2.1) has a unique bounded solution $u(t, x)$. And if $f(t, x) \geq 0$ is true almost everywhere in $\mathbb{R} \times \mathbb{T}^n$, then $u(t, x) \geq 0$ for almost all $(t, x) \in \mathbb{R} \times \mathbb{T}^n$.

Lemma 2 ([1, 2, 5]). Assume that $\lambda \in (0, c^2/4]$ and $f \in L^{\infty}(\mathbb{R} \times \mathbb{T}^n)$, the bounded solution of equation (2.1) satisfies the following estimate:

$$\| u \|_{L^\infty} \leq \frac{1}{\lambda} \| f \|_{L^\infty}.$$

Lemma 3 ([1, 2]). Assume that $\lambda \in (0, c^2/4]$ and $f \in AP(\mathbb{R} \times \mathbb{T}^n)$, then equation (2.1) has a unique solution u in $AP(\mathbb{R} \times \mathbb{T}^n)$.

Let's state the main result of this paper.

Theorem. Consider the equation

$$u_{tt} - \Delta_x u + cu_t + g(\varphi(u)) = f(t, x), \tag{2.2}$$

in which

$$g(x) = \sum_{k=1}^{m} a_k x^{2k-1}, a_1 > 0, a_2, \cdots, a_m \geq 0, c > 0,$$

$$\varphi(u) = \lambda \sin u - \mu \cos u, \lambda > 0, \mu \geq 0,$$

$$\sum_{k=1}^{m} (2k-1) a_k (\lambda^2 + \mu^2)^{k-1/2} \leq \frac{c^2}{4}, \sum_{k=2}^{m} (2k-1)^2 a_k (\lambda^2 + \mu^2)^{k-1} \leq a_1,$$

$$\mu \sum_{k=1}^{m} (2k-1) a_k \lambda^{2k-2} < \lambda \sum_{k=1}^{m} (2k-1) a_k \mu^{2k-2},$$

and $f(t, x) \geq 0$ almost everywhere, $f \in AP(\mathbb{R} \times \mathbb{T}^n)$, $\| f \|_{L^\infty} < g(\lambda)$. Then equation (2.2) has a unique non-negative solution in $AP(\mathbb{R} \times \mathbb{T}^n)$ satisfying:

$$\| u \|_{L^\infty} < \frac{\pi}{2}.$$

3. PROOF OF THE THEOREM

Proof. Fix constants A and θ satisfying

$$\| f \|_{L^\infty} \leq A < g(\lambda), \theta \in (0, \pi/2), g(\varphi(\theta)) > A, -g(\varphi(-\theta)) > A,$$

and $\lambda g'(u) \geq \varphi'(\theta) g'(\varphi(\theta))$. By continuity, let θ go to $(\pi/2)^-$, we know that such θ exists.

Let $\Omega = \{u \in AP(\mathbb{R} \times \mathbb{T}^n) |\, u \geq 0, \| u \|_{L^\infty} \leq \theta\}$, it is easy to show that Ω is a complete metric space given the metric $d(u_1, u_2) = \| u_1 - u_2 \|_{L^\infty}$. For $u \in \Omega$, we consider the equation

$$v_{tt} - \Delta_x v + c v_t + \frac{c^2}{4} v = \frac{c^2}{4} u - g(\varphi(u)) + f(t,x). \tag{2.3}$$

Let $q(t,x) = \frac{c^2}{4} u - g(\varphi(u)$, from the definition and properties of almost periodic functions, we know that

$$q(t,x) \in AP(\mathbb{R} \times \mathbb{T}^n),$$

so $q + f \in AP(\mathbb{R} \times \mathbb{T}^n)$. According to lemma 3, equation (2.3) has a unique solution $v \in AP(\mathbb{R} \times \mathbb{T}^n)$, and this determines a mapping

$$F: \Omega \to AP(\mathbb{R} \times \mathbb{T}^n), \qquad u \mapsto v.$$

Obviously, the fixed point of F is the non-negative almost periodic solution of equation (2.2) which satisfies $\| u \|_{L^\infty} < \pi/2$. So let's prove that F is a compressed mapping and maps Ω into Ω.

Since

$$q = \frac{c^2}{4} u - \sum_{k=1}^{m} a_k \varphi(u)^{2k-1},$$

we have

$$q'(u) = \frac{c^2}{4} - \sum_{k=1}^{m} (2k-1) a_k \varphi(u)^{2k-2} \varphi'(u).$$

Because $|\varphi(u)|, |\varphi'(u)| \le \sqrt{\lambda^2 + \mu^2}$, it can be seen $q'(u) \ge 0$, q is an increasing function of u, then

$$q + f \le \frac{c^2}{4} \theta - g(\varphi(\theta)) + A < \frac{c^2}{4} \theta.$$

And $\omega \equiv \theta$ is a solution of equation $\omega_{tt} - \Delta_x \omega + c\omega_t + \frac{c^2}{4} \omega = \frac{c^2}{4} \theta$. By comparing this equation with equation (2.3), we know from lemma 1 that $v \le \theta$. Similarly, since

$$q + f \ge q(0) + 0 = -g(\varphi(0)) = -g(-\mu) \ge 0,$$

we can see that $v \ge 0$ from lemma 1. That is $v \in \Omega$.

Next, we must prove that F is a compressed mapping.

Let $u_1, u_2 \in \Omega$ and $v_1 = F(u_1), v_2 = F(u_2)$, then $d = v_1 - v_2$ is the solution to the following equation:

$$d_{tt} - \Delta_x d + c d_t + \frac{c^2}{4} d = \hat{q}(t,x),$$

where $\hat{q}(t,x) = \frac{c^2}{4}(u_1 - u_2) - g(\varphi(u_1)) + g(\varphi(u_2))$. It is known from the mean value theorem of differential that there exists ξ between u_1 and u_2 such that

$$\hat{q} = (u_1 - u_2)\left[\frac{c^2}{4} - \varphi'(\xi) g'(\varphi(\xi))\right].$$

Let

$$h(\xi) = \frac{c^2}{4} - \varphi'(\xi) g'(\varphi(\xi)) = \frac{c^2}{4} - \varphi'(\xi) \sum_{k=1}^{m} (2k-1) a_k \varphi(\xi)^{2k-2},$$

then

$$h'(\xi) = \varphi(\xi)\left[a_1 - \sum_{k=2}^{m} (2k-1) a_k \varphi(\xi)^{2k-4} (\varphi(\xi)\varphi''(\xi) + (2k-2)\varphi'(\xi)^2)\right].$$

The equation in square brackets above is non-negative based on $|\varphi''(\xi)| \le \sqrt{\lambda^2 + \mu^2}$ and the conditions given. It is easy to see that $\varphi(\xi)$ is increasing in $(0, \pi/2)$. Because $\varphi(0) \le 0, \varphi(\pi/2) > 0$, we can see that $h(\xi)$ decreases and then increases in $(0, \pi/2)$. By condition obviously there is $h(\xi) \ge 0$, let's estimate delta $\| h(\xi) \|_{L^\infty}$.

Since $0 \le \xi \le \theta$, we have

$$\sup h(\xi) = \max\{h(0), h(\theta)\} = h(\theta)$$
$$= \frac{c^2}{4} - \varphi'(\theta) \sum_{k=1}^{m} (2k-1) a_k \varphi(\theta)^{2k-2} < \frac{c^2}{4}.$$

Lemma 2 shows that

$$\| v_1 - v_2 \|_{L^\infty} = \| d \|_{L^\infty} \le \frac{4}{c^2} \| \hat{q} \|_{L^\infty} \le \frac{4h(\theta)}{c^2} \| u_1 - u_2 \|_{L^\infty}.$$

Since $\frac{4}{c^2} h(\theta)$ is a constant less than 1, so F is a compressed mapping. Therefore, according to Banach fixed point theorem,

$$F: \Omega \to \Omega$$

has a unique fixed point u, and thus u is the non-negative almost periodic solution satisfying equation (2.2). Suppose that equation (2.2) has another almost periodic solution $\tilde{u}$ that satisfies $\| \tilde{u} \|_{L^\infty} < \pi/2$. Adjust θ so that $\| \tilde{u} \|_{L^\infty} < \theta < \pi/2$. It follows that $\tilde{u} \in \Omega$, so $\tilde{u}$ is also the fixed point of F. Further, we know $\tilde{u} = u$ from the uniqueness of fixed points, which means the solution is unique.

4. CONCLUSION

In this paper, the existence result of almost periodic solution of Sine-Gordon equation is generalized, for the reason that equation (2.2) becomes Sine-Gordon equation if $\mu = 0$ and $g(x) = x$. The addition of $\mu cosu$ to the Sine-Gordon equation looks interesting, and this aspect of work remains to be studied.

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