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Original Research Article

Generalized Exponential Poisson Geometric distribution using Marshall-Olkin Technique

Abstract

A new family of continuous distribution is generated by using Marshall Olkin and T-X methods; it is probability function has been generated also distribution function. reliability function and hazard function. A special case is proposed called geometric Poisson exponential distribution (GEPGD). some of it is properties are investigated; parameters estimation is obtained by maximum likelihood method, least square method, weighted least square method and Cramér- Von- Mises. the simulation study for GEPGD parameters. Finally. real life application for studying the importance and the flexibility of the proposed distributions.

Keywords: Marshall Olkin distribution, generalized exponential power series. exponential distribution. Poisson distribution. geometric distribution.

Introduction

The fundamental type of distribution in reliability analysis is a lifetime distribution. This model expressed the lifetime of a component or a system. Many lifetime distributions are related to extreme values, as the system stops working when the first component breaks, as in a series connection, or the system stops working when the last component breaks, as in a parallel connection. A lot of lifetime distributions were improved by adding parameters to be more flexible in reliability applications. One of its main uses is in reliability theory, where the Marshall–Olkin copula models the dependence between random variables subjected to external shock. There are many methods of adding parameters one of the most important methods of adding parameters was introduced by Marshall and Olkin (1997). There are many distributions were obtained by adding parameter using Marshall- Olkin method as Marshall-Olkin generalized Weibull distribution was introduced by Jose and Alice (2001), Ghitany et.al

(2005) introduced Marshall–Olkin extended Weibull distribution and its application to censored data. Alice and Jose (2005) introduced a Marshall-Olkin logistic distribution. Marshall–Olkin extended Lomax distribution and its application to censored data was obtained Ghitany et.al (2007). Jayakumar and Thomas (2008) introduced on a generalization to Marshall-Olkin scheme and its application to Burr type XII distribution. The Marshall–Olkin Fréchet distribution was introduced by Krishna *et.al* (2013). Alzaatreh *et.al (*2013) introduced a method for generating families of continuous distributions namely the T-X family of distributions, which has a connection with the hazard functions and each generated distribution is considered as a weighted hazard function of the random variable X.

This paper has been organized as follows: Section 2 introduces model formulation which includes the probability density function for the new distribution also it is distribution function and survival function. the behavior of the density function, in addition to some basic properties and the expressions for moments. some reliability measures represented in survival and hazard rate besides study the behavior of hazard function. Section 3 represented various estimation methods for the parameters. Simulation and application are discussed in section 4 and 5.

2. Generalized Exponential Poisson Geometric distribution (GEPGD)

If a system contains of N units, which they are independently of each other at any time. Assumed that the r.vs $X_1, X_2, \ldots, X_N$ represent the units lifetimes having Generalized exponential Poisson distribution (GEPD) which introduced by Mahmoudi and Jafari (2012) as a special case of Generalized exponential–power series. Because of property of the geometric extreme stability as a result of using the geometric property (Marshall Olkin 1997). It motivates us to consider the number of components N having a geometric distribution and using Marshall Olkin technique, on a system contains N components connected on series, the PDF and t of the GEPGD is introduced as:

$$f(x;\beta,\theta,p,\alpha) = \frac{\alpha p\beta\theta\,(e^{\theta}-1)\,(1-e^{-\beta x})^{\alpha-1}\,e^{-\beta x+\theta\left(1-e^{-\beta x}\right)^{\alpha}}}{\left((e^{\theta}-1)-(1-p)(e^{\theta}-e^{\theta(1-e^{-\beta x})\alpha})\right)^{2}},\;\; x,\beta,\theta,p,\alpha > 0 \quad (2.1)$$

the CDF of the GEPGD is obtained as

$$F(x;\beta,\theta,p,\alpha) = \frac{e^{\theta\left(1-e^{-\beta x}\right)^{\alpha}}-1}{(e^{\theta}-1)-(1-p)\left(e^{\theta}-e^{\theta\left(1-e^{-\beta x}\right)^{\alpha}}\right)}, \quad x,\beta,\theta,p,\alpha > 0 \qquad (2.2)$$

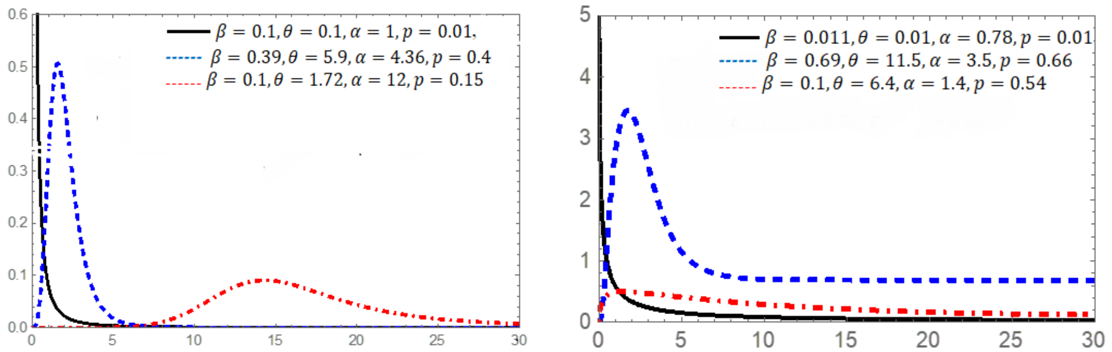
The survival and hazard functions can be obtained as following:

$$\bar{F}(x;\beta,\theta,p,\alpha) = S(x) = \frac{p\left(e^{\theta}-e^{\theta\left(1-e^{-\beta x}\right)^{\alpha}}\right)}{(e^{\theta}-1)-(1-p)\left(e^{\theta}-e^{\theta\left(1-e^{-\beta x}\right)^{\alpha}}\right)}$$

$$h(x;\beta,\theta,p,\alpha) = \frac{\alpha\beta\theta\,(e^{\theta}-1)\,\left(1-e^{-\beta x}\right)^{\alpha-1}\,e^{-\beta x+\theta\left(1-e^{-\beta x}\right)^{\alpha}}}{\left\{(e^{\theta}-1)-(1-p)\left(e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}\right)\right\}\left\{e^{\theta}-e^{\theta(1-e^{-\beta x})^{\alpha}}\right\}}$$

Fig 1

The behavior of the PDF and the $h(x)$ of the GEPGD at different values of α, p, β, θ are showed in the below figure.



- Table (1): Numerical value of mean and variance at different value of parameters

Parameters value		$\theta = 0.1$		$\theta = 0.5$		$\theta = 1.5$	
		μ_1	σ^2	μ_1	σ^2	μ_1	σ^2
• at $\alpha = 0.1$							
$\beta = 0.1$	$p = 0.1$	0.1662	2.301	0.1355	1.872	0.0779	1.060
	$p = 0.5$	0.7824	10.826	0.6442	8.902	0.3774	5.143
	$p = 1.5$	2.084	28.629	1.745	23.993	1.059	14.421
$\beta = 0.5$	$p = 0.1$	0.0332	0.0920	0.0271	0.0749	0.0156	0.0424
	$p = 0.5$	0.1565	0.4330	0.1288	0.3561	0.0755	0.2058
	$p = 1.5$	0.4168	1.145	0.3489	0.9597	0.2117	0.577
$\beta = 1.5$	$p = 0.1$	0.0111	0.0102	0.0090	0.0083	0.0052	0.005
	$p = 0.5$	0.0522	0.0481	0.0429	0.0396	0.0252	0.0229
	$p = 1.5$	0.1389	0.1272	0.1163	0.1066	0.0706	0.0641
at $\alpha = 0.5$							
$\beta = 0.1$	$p = 0.1$	1.047	11.476	0.8804	9.516	0.5639	5.693
	$p = 0.5$	3.728	43.026	3.220	36.856	2.177	23.581
	$p = 1.5$	7.621	87.582	6.738	77.514	4.792	53.508

$\beta = 0.5$	$p = 0.1$	0.2095	0.4590	0.1761	0.3806	0.1128	0.2277
	$p = 0.5$	0.7456	1.721	0.644	1.474	0.4353	0.9432
	$p = 1.5$	1.524	3.503	1.348	3.101	0.9584	2.140
$\beta = 1.5$	$p = 0.1$	0.0698	0.0510	0.0587	0.0423	0.0376	0.0253
	$p = 0.5$	0.2485	0.1912	0.2147	0.1638	0.1451	0.1048
	$p = 1.5$	0.5081	0.3893	0.4492	0.3445	0.3195	0.2378
at $\alpha = 1.5$							
$\beta = 0.1$	$p = 0.1$	3.941	29.137	3.527	25.108	2.696	16.743
	$p = 0.5$	9.129	80.183	8.285	71.432	6.465	50.871
	$p = 1.5$	14.897	134.085	13.663	122.752	10.904	93.039
$\beta = 0.5$	$p = 0.1$	0.7883	1.165	0.7053	1.004	0.5392	0.6697
	$p = 0.5$	1.826	3.207	1.657	2.857	1.293	2.035
	$p = 1.5$	2.973	5.363	2.733	4.910	2.181	3.722
$\beta = 1.5$	$p = 0.1$	0.2628	0.1295	0.2351	0.1116	0.1797	0.0744
	$p = 0.5$	0.6086	0.3564	0.5523	0.3175	0.4309	0.2261
	$p = 1.5$	0.9911	0.5959	0.9109	0.5456	0.7269	0.4135

3. The Estimation Methods

In this section some methods of parameter estimation will be discussed for the new distribution GEPGD.

(i) Maximum likelihood Estimation

One if the most importance methods of estimation is the Maximum Likelihood (MLE). The log–likelihood function is given by

$$ln\,\ell(x;\beta,\theta,p,\alpha) = n\,ln[\alpha p\beta\theta] + nLog[e^{\theta}-1] + (\alpha-1)\sum_{i=1}^{n} ln[1-e^{-\beta x_i}] - 2\sum_{i=1}^{n} ln[e^{\theta(1-e^{-\beta x_i})^{\alpha}}(e^{\theta}-p) + e^{\theta}(p-1)] + \sum_{i=1}^{n}(-\beta x_i + \theta + \theta(1-e^{-\beta x_i})^{\alpha})$$

The partial derivatives of the log-likelihood for the parameters $\beta, \theta, p\ and\ \alpha$ can be expressed by the following equations

$$\frac{\partial Ln\ell(x;\beta,\theta,p,\alpha)}{\partial\beta} = \frac{n}{\beta} + (-1+\alpha)\sum_{i=1}^{n}\frac{e^{-\beta x_i}x_i}{1-e^{-\beta x_i}} + \sum_{i=1}^{n} e^{-\beta x_i}\left(1-e^{-\beta x_i}\right)^{-1+\alpha}\alpha\theta x_i - 2\sum_{i=1}^{n}\frac{e^{(1-e^{-\beta x_i})^{\alpha}\lambda-\beta x_i}(1-e^{-\beta x_i})^{-1+\alpha}(e^{\theta}-p)\alpha\lambda x_i}{e^{(1-e^{-\beta x_i})^{\alpha}\lambda}(e^{\theta}-p)+e^{\theta}(-1+p)}$$

$$\frac{\partial Ln\ell(x;\beta,\theta,p,\alpha)}{\partial\alpha}=\frac{n}{\alpha}+\sum_{i=1}^{n}Log[1-e^{-\beta x_i}]+\sum_{i=1}^{n}(1-e^{-\beta x_i})^{\alpha}\theta Log[1-e^{-\beta x_i}]-$$

$$2\sum_{i=1}^{n}\frac{e^{(1-e^{-\beta x_i})^{\alpha}\lambda}(1-e^{-\beta x_i})^{\alpha}(e^{\theta}-p)\lambda Log[1-e^{-\beta x_i}]}{e^{(1-e^{-\beta x_i})^{\alpha}\lambda}(e^{\theta}-p)+e^{\theta}(-1+p)}$$

$$\frac{\partial Ln\ell(x;\beta,\theta,p,\alpha)}{\partial\theta}=n(1+\frac{1}{-1+e^{\theta}}+\frac{1}{\theta})+\sum_{i=1}^{n}(1+(1-e^{-\beta x_i})^{\alpha})-$$

$$2\sum_{i=1}^{n}\frac{e^{\theta+(1-e^{-\beta x_i})^{\alpha}\lambda}+e^{\theta}(-1+p)}{e^{(1-e^{-\beta x_i})^{\alpha}\lambda}(e^{\theta}-p)+e^{\theta}(-1+p)}$$

$$\frac{\partial Ln\ell(x;\beta,\theta,p,\alpha)}{\partial p}=\frac{n}{p}-2\sum_{i=1}^{n}\frac{e^{\theta}-e^{(1-e^{-\beta x_i})^{\alpha}\lambda}}{e^{(1-e^{-\beta x_i})^{\alpha}\lambda}(e^{\theta}-p)+e^{\theta}(-1+p)}$$

The MLEs can be obtained by solving the derivatives of the $ln\,\ell(x;\beta,\theta,p,\alpha)$ to zero. For the GEPGD the solution of the nonlinear equations has no closed form, also some numerical methods are needed for the solution.

(ii) Method of Least Square Estimation

The LSE and WLSE were proposed by Swain, Venkatraman & Wilson (1988) to estimate the parameters of Beta distributions.

The least square estimators can be obtained by

$$\sum_{j=1}^{n}\left[F(X_j)-\frac{j}{n+1}\right]^2$$

Suppose $F(X_j)$ denotes the cdf of the ordered r.v, $X_1<X_2<\cdots<X_n$, since $\{X_1,X_2,\dots,X_n\}$ is a random sample of size n from a cdf, the LSE of β,θ,p,α can be expressed by

$$\sum_{j=1}^{n}\left[\frac{e^{\theta}(-1+e^{(1-e^{-x(j)^{\beta}})^{\alpha}\theta})}{e^{(1+(1-e^{-x(j)^{\beta}})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x(j)^{\beta}})^{\alpha}\theta}p}-\frac{j}{n+1}\right]^2$$

$$\frac{\partial LSE(x;\beta,\theta,p,\alpha)}{\partial\beta}=\sum_{j=1}^{n}2\left(-\frac{j}{1+n}+\right.$$

$$\left.\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{\theta})(1-e^{-\beta x_j})^{\alpha}p\alpha\theta x_j}{(-1+e^{\beta x_j})(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial LSE(x;\beta,\theta,p,\alpha)}{\partial\theta}=\sum_{j=1}^{n}2\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(-\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{(1-e^{-\beta x_j})^{\alpha}\theta}+(1-e^{-\beta x_j})^{\alpha}-e^{\theta}(1-e^{-\beta x_j})^{\alpha})p}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial LSE(x;\beta,\theta,p,\alpha)}{\partial\alpha}=\sum_{j=1}^{n}2\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{\theta})(1-e^{-\beta x_j})^{\alpha}p\theta\,\mathrm{Log}[1-e^{-\beta x_j}]}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial LSE(x;\beta,\theta,p,\alpha)}{\partial P}=\sum_{j=1}^{n}2\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(-\frac{e^{\theta}(e^{\theta}-e^{(1-e^{-\beta x_j})^{\alpha}\theta})(-1+e^{(1-e^{-\beta x_j})^{\alpha}\theta})}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

The unknown parameters estimators of the WLSE can be written as:

$$\sum_{j=1}^{n}\omega_j\left[F(X_j)-\frac{j}{n+1}\right]^2$$

The weights ω_j are equal to $\frac{1}{V(X_j)}=\frac{(n+1)^2(n+2)}{j(n-j+1)}$, also the weighted least square estimators of β,θ,p,α can be obtained by:

$$\sum_{j=1}^{n}\frac{(n+1)^2(n+2)}{j(n-j+1)}\left[\frac{e^{\theta}(-1+e^{(1-e^{-x(j)^{\beta}})^{\alpha}\theta})}{e^{(1+(1-e^{-x(j)^{\beta}})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x(j)^{\beta}})^{\alpha}\theta}p}-\frac{j}{n+1}\right]^2$$

$$\frac{\partial WLSE(x;\beta,\theta,p,\alpha)}{\partial\beta}=\sum_{j=1}^{n}2\frac{(n+1)^2+(n+2)}{j(n-j+1)}\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{\theta})(1-e^{-\beta x_j})^{\alpha}p\alpha\theta x_j}{(-1+e^{\beta x_j})(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial WLSE(x;\beta,\theta,p,\alpha)}{\partial\theta}=\sum_{j=1}^{n}2\frac{(n+1)^2+(n+2)}{j(n-j+1)}\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(-\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{(1-e^{-\beta x_j})^{\alpha}\theta}+(1-e^{-\beta x_j})^{\alpha}-e^{\theta}(1-e^{-\beta x_j})^{\alpha})p}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial WLSE(x;\beta,\theta,p,\alpha)}{\partial\alpha} = \sum_{j=1}^{n} 2\,\frac{(n+1)^2+(n+2)}{j(n-j+1)}\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{\theta})(1-e^{-\beta x_j})^{\alpha}p\theta\mathrm{Log}[1-e^{-\beta x_j}]}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial WLSE(x;\beta,\theta,p,\alpha)}{\partial P} = \sum_{j=1}^{n} 2\,\frac{(n+1)^2+(n+2)}{j(n-j+1)}\left(-\frac{j}{1+n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(-\frac{e^{\theta}(e^{\theta}-e^{(1-e^{-\beta x_j})^{\alpha}\theta})(-1+e^{(1-e^{-\beta x_j})^{\alpha}\theta})}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

(iii) Method of Cramér- Von- Mises

Another method of estimation parameter is Cramér-von-Mises estimators of $\beta,\theta,p\ and\ \alpha$ can be obtained by:

$$C(x;\beta,\theta,p,\alpha) = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}/\beta,\theta,p,\alpha) - \frac{2i-1}{2n}\right]^2$$

the Cramér-von-Mises estimators of β,θ,p,α can be obtained by:

$$C(x;\beta,\theta,p,\alpha) = \frac{1}{12n} + \sum_{i=1}^{n}\left[\frac{e^{\theta}(-1+e^{(1-e^{-x_{(j)}\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_{(j)}\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_{(j)}\beta})^{\alpha}\theta}p} - \frac{2i-1}{2n}\right]^2$$

$$\frac{\partial C(x;\beta,\theta,p,\alpha)}{\partial\beta} = \sum_{j=1}^{n} 2\left(-\frac{-1+2i}{2n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{\theta})(1-e^{-\beta x_j})^{\alpha}p\alpha\theta x_j}{(-1+e^{\beta x_j})(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial C(x;\beta,\theta,p,\alpha)}{\partial\theta} = \sum_{j=1}^{n} 2\left(-\frac{-1+2i}{2n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(-\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{(1-e^{-\beta x_j})^{\alpha}\theta}+(1-e^{-\beta x_j})^{\alpha}-e^{\theta}(1-e^{-\beta x_j})^{\alpha})p}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial C(x;\beta,\theta,p,\alpha)}{\partial\alpha} = \sum_{j=1}^{n} 2\left(-\frac{-1+2i}{2n}+\frac{e^{\theta}(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta})}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(\frac{e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}(-1+e^{\theta})(1-e^{-\beta x_j})^{\alpha}p\theta\mathrm{Log}[1-e^{-\beta x_j}]}{(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p)^2}\right)$$

$$\frac{\partial C(x;\beta,\theta,p,\alpha)}{\partial P}=\sum_{j=1}^{n}2\left(-\frac{-1+2i}{2n}+\frac{e^{\theta}\left(-1+e^{(1-e^{-x_j\beta})^{\alpha}\theta}\right)}{e^{(1+(1-e^{-x_j\beta})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-x_j\beta})^{\alpha}\theta}p}\right)\left(-\frac{e^{\theta}\left(e^{\theta}-e^{(1-e^{-\beta x_j})^{\alpha}\theta}\right)\left(-1+e^{(1-e^{-\beta x_j})^{\alpha}\theta}\right)}{\left(e^{(1+(1-e^{-\beta x_j})^{\alpha})\theta}+e^{\theta}(-1+p)-e^{(1-e^{-\beta x_j})^{\alpha}\theta}p\right)^{2}}\right)$$

4. Simulation study

A simulation Scheme for the GEPGD will be made by generating 5000 samples for the parameters β, θ, p *and* α at different sample sizes. The simulation nodes were chosen at different values at β, θ, p *and* α.

- **Simulation study for GEPGD using MLE**

Table (2): The MSE and the Bias of the estimates using MLE.

N	$\beta = 0.1$		$\theta = 1.5$		$p = 0.15$		$\alpha = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	0.00021	0.0065	1.216	0.5922	0.2073	0.0579	0.1375	0.0688
40	0.00014	0.0064	0.8201	0.5813	0.0096	0.0330	0.0806	0.0276
60	0.00013	0.0062	0.7254	0.5410	0.0062	0.0286	0.0657	0.0133
80	0.00009	0.0059	0.5379	0.5350	0.0037	0.0252	0.0484	0.0131
100	0.00008	0.0050	0.4710	0.5331	0.0010	0.0245	0.0415	0.0122
N	$\beta = 2$		$\theta = 1.5$		$p = 0.5$		$\alpha = 1$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	1.709	0.5229	1.928	0.2353	1.378	0.3987	0.0513	0.0219
40	0.7893	0.2558	1.841	0.2004	1.251	0.3534	0.0263	0.0159
60	0.4601	0.1092	1.778	0.1927	1.045	0.3113	0.0202	0.0149
80	0.4355	0.1079	1.583	0.1412	1.023	0.2975	0.0166	0.0050
100	0.3781	0.0589	1.330	0.1144	0.9164	0.2838	0.0145	0.0029
N	$\beta = 2.5$		$\theta = 0.75$		$p = 0.7$		$\alpha = 1.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	0.6813	0.1009	1.958	0.5184	1.813	0.3781	0.1847	0.0706
40	0.3264	0.0507	1.841	0.4972	1.402	0.3491	0.0793	0.0084
60	0.3003	0.0443	1.474	0.4198	1.139	0.3195	0.0557	0.0021
80	0.2564	0.0056	1.300	0.3451	1.051	0.3075	0.0417	0.0015
100	0.2461	0.0009	1.244	0.2996	1.004	0.3049	0.0378	0.0002

- **Simulation study for GEPGD using Least square method.**

Table (3) : The MSE and the Bias of the estimates using LSE.

N	$\beta = 0.1$		$\theta = 1.5$		$p = 0.15$		$\alpha = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	0.003	0.027	0.097	0.092	0.086	0.073	0.890	0.408
40	0.003	0.018	0.081	0.062	0.076	0.068	0.886	0.389
60	0.002	0.013	0.058	0.058	0.068	0.060	0.681	0.357
80	0.001	0.011	0.036	0.028	0.051	0.052	0.505	0.308
100	0.0001	0.009	0.027	0.013	0.013	0.045	0.416	0.289
N	$\beta = 2$		$\theta = 1.5$		$p = 0.5$		$\alpha = 1$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	0.072	0.3207	5.251	0.8472	8.975	0.9367	0.1233	0.0515
40	0.069	0.3301	5.013	0.8707	7.127	0.7568	0.0607	0.0372
60	0.062	0.3831	4.613	0.8669	5.810	0.6354	0.0490	0.0201
80	0.045	0.3623	3.875	0.7816	5.118	0.5611	0.0413	0.0179
100	0.015	0.3892	3.849	0.7934	4.774	0.5352	0.0379	0.0019
N	$\beta = 2.5$		$\theta = 0.75$		$p = 0.7$		$\alpha = 1.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	1.870	0.611	4.553	0.771	5.838	0.456	0.279	0.029
40	1.483	0.578	4.061	0.717	4.258	0.315	0.119	0.019
60	1.304	0.573	4.332	0.712	3.691	0.287	0.086	0.002
80	1.116	0.537	4.392	0.704	3.342	0.256	0.073	0.001
100	0.997	0.524	4.427	0.617	4.305	0.237	0.070	0.001

- **Simulation study for GEPGD using Weight Least square method.**

Table (4) : The MSE and the Bias of the estimates using WLSE.

N	$\beta = 0.1$		$\theta = 1.5$		$p = 0.15$		$\alpha = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	0.003	0.031	5.979	0.9561	5.056	0.640	1.788	0.377
40	0.003	0.021	5.371	0.8011	4.932	0.610	0.755	0341
60	0.002	0.015	4.708	0.7067	3.207	0.470	0.532	0.307
80	0.002	0.016	3.891	0.6165	2.527	0.366	0.402	0.279
100	0.001	0.013	3.854	0.6145	2.145	0.284	0.345	0.251
N	$\beta = 2$		$\theta = 1.5$		$p = 0.5$		$\alpha = 1$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	2.528	0.4514	6.499	0.9885	7.141	0.9026	0.1139	0.0276
40	1.784	0.3991	5.637	0.9260	6.709	0.7117	0.0602	0.0214
60	1.364	0.3613	5.370	0.8808	5.259	0.5897	0.0468	0.0176
80	1.101	0.3570	4.126	0.8236	4.025	0.4579	0.0358	0.0154
100	0.9049	0.3504	4.291	0.7890	3.015	0.3664	0.0302	0.0135
N	$\beta = 2.5$		$\theta = 0.75$		$p = 0.7$		$\alpha = 1.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	1.873	0.655	5.452	0.918	6.293	0.614	0.871	0.430
40	0.003	0.020	5.371	0.800	4.932	0.610	0.755	0.343
60	0.002	0.015	4.707	0.406	3.206	0.470	0.532	0.307
80	0.002	0.014	3.891	0.161	2.527	0.366	0.402	0.279
100	0.001	0.013	3.261	0.036	2.221	0.312	0.384	0.234

- **Simulation study for GEPGD using Cramér- Von- Mises method**

Table (5): The MSE and the Bias of the estimates using Cramér- Von- Mise.

N	$\beta = 0.1$		$\theta = 1.5$		$p = 0.15$		$\alpha = 3.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	0.0029	0.0120	8.498	0.871	3.688	0.5546	0.9399	0.0674
40	0.0284	0.0088	7.639	0.703	3.356	0.5398	0.8491	0.0528
60	0.0026	0.0038	7.175	0.619	3.039	0.5142	0.6357	0.0203
80	0.0023	0.0032	6.287	0.337	2.838	0.4678	0.4745	0.0189
100	0.0023	0.0031	5.791	0.243	2.551	0.4215	0.3967	0.0185
N	$\beta = 2$		$\theta = 1.5$		$p = 0.5$		$\alpha = 1$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	3.345	0.2275	7.196	0.9307	7.542	0.8327	0.1507	0.0764
40	2.226	0.2187	5.505	0.7598	5.408	0.6149	0.0640	0.0279
60	1.6799	0.2184	4.061	0.6720	4.994	0.5475	0.0496	0.0101
80	1.417	0.1296	3.517	0.5895	3.585	0.4209	0.0401	0.0056
100	1.160	0.0508	3.374	0.5700	3.930	0.3422	0.0360	0.0008
N	$\beta = 2.5$		$\theta = 0.75$		$p = 0.7$		$\alpha = 1.5$	
	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias
20	1.767	0.4484	3.934	0.6282	3.912	0.3040	0.4164	0.1910
40	1.327	0.4271	3.775	0.5567	3.782	0.2694	0.1385	0.1053
60	1.118	0.4231	3.763	0.5565	3.569	0.2581	0.0940	0.0525
80	0.9535	0.3855	3.402	0.4103	3.128	0.2293	0.0752	0.0430
100	0.8547	0.3381	3.149	0.3835	2.328	0.1341	0.0694	0.0185

As a result, the MSE and the Bias values are decreases while the sample size increases which indicates of the unbiasness and efficiency properties of the GEPGD parameters using different estimation methods.

From the simulation the MLE give the best results of biases and MSEs. LSE is the second good estimator, followed by the Weighted Least Square estimators. we can argue that the MLE, least squares estimators and the WLSE are among the best performing estimators for GEPGD.

5. Real Lifetime Applications

Three different engineering and medical applications were applied to test the goodness of fit and the flexibility of GEPGD compared with other well-known lifetime distributions. The MLE was used to estimate the parameters. The goodness of fit criteria

(AIC, BIC, AICC, HQIC, and CVM), besides AD and K-S with p-value (K-S) were calculated for the fitted distributions. The results are discussed as follows

(a) Application 1: Stress-rupture Lifetime Data

The stress-rupture life of Kevlar 49/ epoxy strands was measured to 101 points since were exposure to constant sustained pressure at the 90% stress level until all, had failed, so that the following data with exact times of failure had been collected by (Andrews and Herzberg, 1985; Barlow et al., 1984), the failure time in hours are.
0.01,0.01,0.02,0.02,0.02,0.03,0.03,0.004,0.05,0.06,0.07,0.07,0.08,0.09,0.09,0.10,0.10, 0.11,0.11,0.12,0.13,0.18,0.19,0.2,0.23,0.24,0.24,0.29,0.34,0.35,0.36,0.38,0.40,0.42,0.52,0.54,0.56,0.60,0.6,0.63,0.65,0.67,0.68,0.72,0.72,0.72,0.73,0.79,0.8,0.8,0.83,0.85,0.9,0.92,0.99,1,1.01,1.02,1.03,1.05,1.1,1.11,1.15,1.18,1.2,1.29,1.31,1.33,1.34,1.4,1.45,1.5,1.51,1.53,1.54,1.54,1.55,1.58,1.6,1.63,1.64,1.8,1.8,1.81,2.02,2.05,2.14,2.17,2.33,3.03,3.03,3.34,4.20,4.69,7.8.

Table (6) : The parameter estimates, $-$ log and K-S for application 1

Distribution	MLEs	-log	K-S	P-value
GEPGD	$\hat{\beta}=0.049$ $\hat{\theta}=16.117$ $\hat{p}=3.595$ $\hat{\alpha}=0.695$	94.9216	0.0743	0.6704
Weibull	$\hat{\alpha}=0.818$ $\hat{\alpha}=0.904$ $\hat{\gamma}=0.004$	95.4474	0.1196	0.1318
Gamma (2P)	$\hat{\alpha}=0.79$ $\hat{\alpha}=1.29$	95.6855	0.09252	0.3676
Generalized Gamma (4p)	$\hat{k}=1.160$ $\hat{\alpha}=0.575$ $\hat{\alpha}=1.768$ $\hat{\gamma}=0.004$	95.9233	0.1021	0.2750
Burr (4P)	$\hat{k}=9.616$	96.4289	0.0997	0.3009

	$\hat{\alpha} = 0.904$ $\hat{\alpha} = 10.867$ $\hat{\gamma} = 0.004$			
Pearson (4p)	$\widehat{\alpha_1} = 0.791$ $\widehat{\alpha_2} = 10.892$ $\hat{\alpha} = 11.752$ $\hat{\gamma} = 0.004$	96.2907	0.1344	0.0645

Table (7): Goodness of fit criteria for Application 1

Distribution	AIC	BIC	AICC	HQIC	AD	CVM
GEPGD	197.843	208.059	198.288	201.971	0.7042	0.1107
Weibull	198.636	208.732	198.900	201.733	1.2891	0.2755
Gamma (2P)	204.315	209.423	204.445	206.379	0.8405	0.1532
Generalized Gamma (4p)	199.847	210.062	200.294	203.291	0.9661	0.1901
Burr (4P)	200.858	209.917	200.146	203.819	1.1794	0.2087
Pearson (4p)	200.581	210.797	201.026	204.709	1.7109	0.3837

The GEPGD showed that is more flexible and more fitting comparing with other distributions, since has the smallest statistic criteria and the smallest log likelihood, hence the above values of measures will lead to the pdf of the GEPGD has its own shape and may be difficult to replace by any other known distribution.

(b) Application 2: Device Operational Lifetime Data

Studying the performance of a component over time can help to estimate the device's operational lifetime. Changes in voltage and temperature will influence on the lifetime of individual components. The following data represents the failure in time for 83 commercial grade ceramic capacitors. A capacitor is a two-terminal electrical component that stores potential energy in an electric field, which is widely used as parts of electrical circuits in many common electrical devices. The failure times of capacitors are:

0.0055, 0.0058, 0.0094, 0.0127, 0.0200, 0.0232, 0.0337, 0.0356, 0.0410, 0.0520, 0.0819, 0.0833, 0.0908, 0.1255, 0.1287, 0.1423, 0.1457, 0.1798, 0.2365, 0.2626, 0.2862, 0.4019, 0.4317, 0.4621, 0.5119, 0.5168, 0.6059, 0.6121, 0.6335, 0.6817, 0.8308, 0.8717, 0.8793, 0.9316, 1.0570, 1.1927, 1.2481, 1.2548, 1.2726, 1.2791, 1.3303, 1.3335, 1.4645, 1.5280, 1.6029, 1.6481, 1.7448, 1.7999, 1.8700, 1.9411, 1.9434, 1.9951, 2.0470, 2.0644, 2.1199, 2.1519, 2.2110, 2.2737, 2.3793, 2.5791, 2.6068, 2.7118, 2.8781, 3.0026, 3.0069, 3.0385, 3.0679, 3.1364, 3.1446, 3.6826, 3.6882, 3.8219, 4.0604, 4.1546, 4.2332, 4.4034, 4.6027, 4.8590, 5.4514, 5.6610, 5.8404, 6.8148, 9.3283.

Table (8): The parameter estimates, $-\log$ and K-S for application 2

Distribution	MLEs	-log	K-S	P-value
GEPGD	$\hat{\beta} = 0.573$ $\hat{\theta} = 0.0001$ $\hat{p} = 2.836$ $\hat{\alpha} = 0.461$	127.529	0.0670	0.8501
Weibull	$\hat{\alpha} = 0.767$ $\hat{\alpha} = 1.630$ $\hat{\gamma} = 0.005$	129.133	0.1222	0.1670
MOGE	$\hat{\alpha} = 1.119$ $\hat{\theta} = 0.356$ $\hat{\beta} = 0.595$	129.683	0.0821	0.6293
Burr (4P)	$\hat{k} = 6.435$ $\hat{\alpha} = 0.829$ $\hat{\alpha} = 12.043$ $\hat{\gamma} = 0.005$	131.872	0.1639	0.0231
Pearson (4p)	$\widehat{\alpha_1} = 0.608$ $\widehat{\alpha_2} = 4.916$ $\hat{\alpha} = 12.295$ $\hat{\gamma} = 0.005$	132.132	0.15609	0.0350
MOEED	$\hat{\alpha} = 0.468$ $\hat{\theta} = 0.738$	133.425	0.1175	0.1866
Gamma (2P)	$\hat{\alpha} = 1.016$ $\hat{\alpha} = 1.812$	134.121	0.1313	0.1045

Table (9): Goodness of fit criteria for Application 2

Distribution	WT	AIC	BIC	AICC	HQIC	AD	CVM
GEPGD	0.0555	263.057	272.732	263.569	266.944	0.3917	0.0556
Weibull	0.2084	264.266	271.523	264.570	267.128	1.462	0.2442
MOGE	0.1039	265.366	272.623	265.670	268.282	0.7435	0.1039
Burr (4P)/'	0.3467	271.744	281.419	272.256	275.631	2.6692	0.5153
Pearson (4p)	0.2500	272.265	281.94	272.778	276.152	1.1031	0.1699
MOEED	0.2263	270.850	275.687	270.999	272.793	2.2713	0.2263
Gamma (2P)	0.1539	272.240	277.078	272.391	274.184	1.0023	0.1841

The GEPGD showed that is more flexible and more fitting comparing with other distributions, since has the smallest statistic criteria and the smallest log likelihood, hence the above values of measures will lead to the pdf of the GEPGD has its own shape and may be difficult to replace by any other known distribution.

(c) Application 3: Bladder Cancer Data

The following data represented the remission times (in months) for cancer Patients, a random sample of 128 bladder cancer patients were collected by Lee and Wang (2003). The data are:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 3.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 1.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.4, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10,1.46,4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69.

Table (10) The parameter estimates,– log and K-S for application 3

Distribution	**MLEs**	**-log**	**K-S**	**P-value**
GEPGD	$\hat{\beta} = 0.067$ $\hat{\theta} = 0.0002$ $\hat{p} = 0.1881$ $\hat{\alpha} = 1.655$	407.795	0.0340	0.9984
Pearson (4p)	$\widehat{\alpha_1} = 2.7857$ $\widehat{\alpha_2} = 2.2819$ $\hat{\alpha} = 4.9677$ $\hat{\gamma} = 0.3492$	408.557	0.0360	0.9963
Burr (4P)	$\hat{k} = 1.1489$ $\hat{\alpha} = 1.622$ $\hat{\alpha} = 6.7468$ $\hat{\gamma} = 0.0755$	410.008	0.0322	0.9953
MOEED	$\hat{\alpha} = 0.1066$ $\hat{\theta} = 0.9668$	412.182	0.0819	0.3776
Weibull	$\hat{\alpha} = 0.767$ $\hat{\alpha} = 1.630$ $\hat{\gamma} = 0.005$	419.769	0.1187	0.0525

Table (11): Goodness of fit criteria for Application 3

.

Distribution	**AIC**	**BIC**	**AICC**	**HQIC**	**AD**
GEPGD	823.589	834.997	823.914	828.224	0.1909
Pearson (4p)	825.114	836.522	825.439	829.749	0.2191
Burr (4P)	828.016	839.425	828.342	832.562	0.2781
MOEED	828.461	834.069	828.461	830.682	1.1922
Weibull	845.537	854.093	845.731	849.014	3.8194

The results of the above shows that the GEPGD is more flexible and more fitting comparing with other distributions, since has the smallest statistic criteria and the smallest log likelihood, hence the above values of measures will lead to the pdf of the GEPGD has its own shape and may be difficult to replace by any other known distribution.

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