Original Research Article

Some Relations Among Copulas

Abstract

Copulas are mathematical objects that fully capture the dependence structure among random variables and thus offer a great deal of flexibility in building multivariate stochastic models. They are widely used in market and credit models, risk aggregation, portfolio selection, insurance, and reliability theory. This study will look into the relationship between a few copulas.

Keywords: *Copula; survival copula; ordering; measure of dependency; polynomial copula; Harmonic copula; homogeneous copula.*

1. Introduction

Copulas are one-dimensional marginal distribution functions that combine or "couple" multivariate distribution functions. Copulas, on the other hand, are multivariate distribution functions with uniform one-dimensional margins on the interval (0,1). "Copulas are of interest to statisticians for two reasons," according to the first update volume of the Encyclopedia of Statistical Sciences, "the first is measures of dependence; and the second is considering a starting point for constructing families of bivariate distributions, sometimes with a view to simulation" (0, 1). Skaler theorem published the first version of copula in 1959. Over time, a variety of copulas have been presented. Joe copula, Extreme value copula, Clayton copula, Cuadras and Augé's copula, Frank copula, Gumbel-Hougaard copula, Farlie-Gumbel-Morgensten (FGM) copula, plackett copula, polynomial copula, and Archimedean copula

2. Copula Definitions and Basic Properties

A collection of random variables $(y_1,..., y_m)$ joint distribution is defined as

$$F(y_1,..., y_m) = \Pr[Y_i \le y_i; i = 1,..., m]$$

For a right continuous function to be bivariate cdf, the following conditions must be met.

$1-\lim_{y_j\to-\infty} F(y_1,y_2)=0,\ j=1,2$

$2-\lim_{y_j\to\infty} F(y_1,y_2)=1$

3- For all (a1, a2) and (a3), the rectangular inequality (b1, b2), $a_1\le b_1, a_2\le b_2$

$$F(b_1,b_2)-F(a_1,b_2)-F(b_1,b_2)+F(a_1,a_2)\ge 0.$$

If F has second derivatives, then the 2-increasing property is identical to If F has second derivatives $\partial^2 F/\partial y_1\partial y_2\ge 0$, then the 2-increasing property is equivalent to

The features of the multivariate cdf $F(y_1,...,y_m)$ are as follows:

1- $\lim_{y_j\to-\infty} F(y_1,...,y_m)=0,\ j=1,..,m$
2- $\lim_{y_j\to\infty} F(y_1,...,y_m)=1 \quad \forall j$

Consider any m-variate joint cdf $F(y_1,...,y_m)$ with univariate marginal cdfs $F_1,...,F_m$. Each marginal distribution, by definition, can take any value between 0 and 1. The Fréchet–Hoeffding lower and upper boundaries, F_L and F_U, define the combined cdf from below and above.

$$F_L(y_1,...,y_m)=\max\left[\sum_{j=1}^{m}F_j-m+1,0\right]=W$$
$$F_L(y_1,...,y_m)=\min[F_1,...,F_m]=M$$

so that

$$W=\max\left[\sum_{j=1}^{m}F_j-m+1,0\right]\le F(y_1,...,y_m)\le\min[F_1,...,F_m]=M$$

in order for m = 2, the top bound is always a cdf, whereas the lower bound is also a cdf.According to Sklar's Theorem, an m-dimensional copula (or m -copula) is a function C from the unit m-cube $[0,1]^m$ to the unit interval [0,1] that meets the following requirements:

1- for every n ≤ m and all a_n in [0,1]then $C(1,...,1,a_n,1,...,1)=a_n$;
2- if a_n = 0 for any n ≤ m then $C(a_1,...,a_m)=0$;
3- C is m-increasing.

An m-dimensional cdf whose support is contained in $[0,1]^m$ and whose one-dimensional margins are uniform on [0,1] can be defined as an m-copula.

The copula associated with F is a distribution function for an m-variate

$$F(y_1,\ldots,y_m)=C(F_1(y_1),\ldots,F_m(y_m),\theta) \qquad (1)$$

The dependence parameter θ is the major focus of estimation in many applications, where is a copula parameter known as the dependence parameter θ, which measures marginal dependence as a scaler measure of dependence.

The matching copula in (1) is unique if the margins $F_1(Y_1),\ldots,F_m(Y_m)$ are continuous; the uniqueness property is generally seen as a dependent function. Fréchet–Hoeffding bounds also apply to copulas when they are multivariate distribution functions; hence, the Fréchet–Hoeffding bounds as universal bounds for copulas, for any copula C and all $u,v \in [0,1]^m$,

$$W=\max\left(\sum_{j=1}^{m}F_j-m+1,0\right)\le C(y_1,\ldots,y_m)\le \min(F_1,\ldots,F_m)=M$$

Sklar's theorem states that if X and Y are two random variables with the same joint distribution function H and margins F and G, then for any x, y in,

$$\max(F(x)+G(y)-1,0)\le H(x,y)\le \min(F(x),G(y))$$

The Fréchet-Hoeffding bounds for joint distribution functions H with margins F and G are joint distribution functions with margins F and G. The upper bound is a copula since it is a distribution function. As a result, the upper bound is abbreviated as C_U ($y_1,\ldots,y_m$). $C_L(y_1,\ldots,y_m)$ denotes the lower bound if it is also a copula.

$$C_L(y_1,\ldots,y_m)\le C(y_1,\ldots,y_m)\le C_U(y_1,\ldots,y_m) \quad (2)$$

A copula's appealing merit should encompass the sample space between the lower and higher bounds, and θ ,the copula should approach the Fréchet–Hoeffding lower (upper) bound as it approaches the lower (upper) bound of its acceptable range.

3. Copulas for Survival

The survival function determines the likelihood of an individual existing or surviving beyond time x. (or survivor function, or reliability function)

$$\overline{F}(x)=P(X>x)=1-F(x) \qquad (3)$$

X's distribution function is denoted by F.

The margins of (X,Y) are the functions H and the joint survival function is

$$\overline{H}(x,y)=P(X>x,Y>y),$$

Where $\overline{H}(x,-\infty)$ and $\overline{H}(-\infty,y)$ are considered margins of $\overline{H}$ which involving variate survival functions $\overline{F}$ and $\overline{G}$, respectively, as a result

$$\overline{H}(x,y)=1-F(x)-G(y)+H(x,y)$$
$$=\overline{F}(x)+\overline{G}(y)-1+C(F(x),G(y))$$
$$=\overline{F}(x)+\overline{G}(y)-1+C\left(1-\overline{F}(x),1-\overline{G}(y)\right)$$

a function $\overline{C}$ from $I^2=[0,1]^2$ into $I=[0,1]$

$$\overline{C}(u,v)=u+v-1+C(1-u,1-v) \quad (4)$$

As a result $\overline{H}(x,y)=\overline{C}\left(\overline{F}(x),\overline{G}(y)\right)$, the surviving joint survival function is connected to its univariate margins in a similar way that a copula relates the joint distribution function to its margins.

$$\overline{C}(u,v)=P[U>u,V>v]=1-u-v+C(u,v)=\overline{C}(1-u,1-v) \quad (5)$$

4. Some copulas have specific properties.

- **Copulas Harmonics**

If C fulfils Laplace's equation in $(0,1)^2$ and we have continuous second-order partial dervatives on $(0,1)^2$, we get a copula called harmonic in I^2.

$$\nabla^2 C(u,v)=\frac{\partial^2}{\partial u^2}C(u,v)+\frac{\partial^2}{\partial v^2}C(u,v)=0 \qquad (6)$$

It is undeniably harmonious. Π is the sole harmonic copula, because any other harmonic copula C, C $-\Pi$ would be harmonic and equal to 0 on the boundary of I^2, and hence equal to 0 on all of I^2. Subharmonic and superharmonic copulas are closely related concepts. A copula C is subharmonic if $\nabla^2 C(u,v)\geq 0$ and superharmonic if $\nabla^2 C(u,v)\geq 0$ (for instance, the FGM copula).

$$C(u,v)=uv+\theta uv(1-u)(1-v) \qquad (7)$$

Then C_θ is subharmonic on $\theta\in[-1,0]$ and superharmonic for $\theta\in[0,1]$, respectively.

- **Copulas that are homogeneous**

If for some real number $k\geq 0$ and all u,v and λ in I, a copula C is homogeneous of degree k.

$$C(\lambda u,\lambda v)=\lambda^k C(u,v) \qquad (8)$$

The word "quasi-homogeneity" refers to an extension of homogeneity that is defined as follows: A function F is said to be quasi-homogeneous if, for any u, v,

$$\lambda\in[0,1],\ F(\lambda x,\lambda y)=\varphi^{-1}\left(f(\lambda)\varphi(F(u,v))\right)$$

for a continuous, strictly monotonic function φ from [0, 1] to $\Re$ and a function f:[0, 1] → [0, 1]. Quasi-homogeneous t-norms are characterized by ***T. Calvo* et al.** and ***R. Mesiar* et al**. (***2010***) and in terms of copulas in , where it is proved that the only homogeneous copula is the member C_θ of the Cuadras–Augé family with $\theta = 2-k$ for 1 ⩽ k ⩽ 2.

5- F-G-M copula

The F-G-M copula is the only one that has a functional polynomial quadratic in *u* and v, and it is a symmetric copula that is equal to distributed continuous random variables and exchangeability. In 1977, kotz and johson discovered the formula for the distribution function, which is given by

$$C(u,v) = uv[1+\alpha(1-u)(1-v)], \qquad \alpha \in [-1,1] \quad (9)$$

The density function is defined as follows:

$$c(u,v) = 1+\alpha(1-2u)(1-2v)$$

6- Iterating the F-G-M copula

Lin (1987) introduced a method for iterating the F-G-M distribution that began with the survival function $\bar{C}$, which Kotz and Johnson (1977) iterated as

$$\bar{C}(u,v) = (1-\alpha)(1-v)(1+\alpha\, uv) \qquad (10)$$

by substituting uv for

$$C(u,v) = uv[1+\alpha(1-u)(1-v)], \qquad \alpha \in [-1,1] \, (11)$$

we've got

$$C(u,v) = uv\left[1+\alpha(1-u)(1-v)+\beta(1-u)^2(1-v)^2\right], \qquad \alpha \in [-1,1] \quad (12)$$

The iterated F-G-M distribution was then given form by Zheng Klein (1994):

$$C(u,v) = uv + \sum_j \alpha_j \, (uv)^{1/2}\left[(1-u)(1-v)\right]^{\left(\frac{j+1}{2}\right)} \qquad (13)$$

Huang and Kotz (1999) extended the F-G-M distribution originally proposed by:

$$C(u,v) = uv\left[1+\alpha\left(1-u^p\right)\left(1-v^p\right)\right] \quad (14)$$

Its probability density function is defined as follows:

$$c(u,v) = 1+\alpha\left[1-(1+p)u^p\right]\left[1-(1+p)^{vp}\right], -\left(\max\{1,p^2\}\right)^{-2}\alpha \le p^{-1} \qquad (15)$$

Iterated F-G-M correlation coefficient

$$corr(U,V) = \frac{\alpha}{3}+\frac{\beta}{12} \qquad (16)$$

Equation for the correlation can be presented as:

$$-3(p+2)^{-2}\min\{1,p^2\}\leq\rho\leq 3\rho\big/(p+2)^2$$

The F-G-M copula's maximum correlation can be raised by increasing parameter *p*.

7- Frank copula

It's an Archimedean copula with a distribution function that's symmetric.

$$C(u,v)=\log\left[1+\frac{(\alpha^u-1)(\alpha^v-1)}{(\alpha-1)}\right] \qquad (17)$$

The probability density function is defined as follows:

$$c(u,v)=\frac{(\alpha-1)\log\alpha^{(u+v)}}{[\alpha-1+(\alpha^u-1)(\alpha^v-1)]^2}$$

The Frank generator is supplied by

$$\varphi_\alpha(t)=-\ln\left[\frac{\exp(-\alpha t)-1}{\exp(-\alpha)-1}\right],\alpha\in(-\infty,\infty)\setminus\{0\}$$

If is greater than zero, we have negative a connection, and we term our copula independent copula if is greater than zero. For $0<\alpha<1$ we got a positive association , and negative a associations if $\alpha>1$ if zero so our copula is called independent copula when $\alpha\to 1$.

For Frank copula, Kendal tau can be written as

$$\tau=1+4\left[D_1(\alpha^*)-1\right]/\alpha$$
$$=1+\frac{4}{\theta}\left(D_1(\alpha)-1\right)$$

and spearman rho comes from

$$\rho=1+12\left[D_2(\alpha*)-D_1(\alpha*)\right]/\alpha*$$

where D_1, D_2 debye function and $\alpha*=-\log(-\alpha)$

The only copula that satisfies $\hat{C}(u,v)=C(u,v)$ is Frank's copula. According to Frank (1979), both copula C and zero are associative, which means that $C[u,(v,w)]=C[(u,v),w]$.

8- Gumbel Hougaard

Gumble (1960) and Barrent (1980) stated the gumble –Barrent copula as

$$C(u,v)=u+v-1+(1-u)(1-v)\exp[-\varphi\log(1-u)\log(1-v)],\quad 0\leq\varphi\leq 1 \qquad (19)$$

Independence match to $\varphi = 0$.

Gumble (1960) and Hougaard (1986) proposed another copula as

$$C(u,v) = \exp\left\{-\left[(-\log u)^{\varphi} + (-\log v)^{\varphi}\right]^{1/\varphi}\right\}, \quad \varphi = 1 \qquad (20)$$

in this state independence matches to $\varphi = 1$.

By letting $e^{-x} = -\log u$ and $e^{-y} = -\log v$ replacing in the previous equation, the joint distribution of X and Y may be verified. The type B bivariate extreme value distribution is described by Nelson (2006).

$$H(x,y) = e^{\left[-\left(e^{-\alpha x} + e^{-\alpha y}\right)^{1/\alpha}\right]}$$

$$C(u,v) = \exp\left\{-\left[(-\log u)^{\theta} + (-\log v)^{\theta}\right]^{1/\theta}\right\}, \quad \theta \in [1,\infty) \qquad (21)$$

This family is known as the Gumble Hougaard family, and we achieve positive dependence between variables when $\theta \to \infty$. Kendal tau for this family is :

$$\tau = \frac{\alpha - 1}{\alpha}$$

there is a relation between $\log U$ and $\log V$ is $1 - \alpha^2$

copula is Survival

If we have two components failure rate by $\theta\lambda(x)$ and $\theta\lambda(y)$ then the joint survival probability $e^{-\theta[\Lambda(x)+\Lambda(y)]}$, let θ a stable distribution with Laplace transform $E\left(e^{-\theta s}\right) = e^{-s^{\gamma}}$, then the survival copula can be calculated.

$$E\left(e^{-\theta[\Lambda(x)+\Lambda(y)]}\right) = e^{-[\Lambda(x)+\Lambda(y)]^{\gamma}}$$

Assume that $\lambda(u)$ has wiebull form $\in \alpha u^{\alpha-1}$ such that

$$\Lambda(t) = \in t^{\alpha}$$

so

$$\bar{H}(x,y) = \exp\left[-\left(\varepsilon x^{\alpha} + \varepsilon y^{\alpha}\right)^{\gamma}\right], \qquad x, y > 0$$

let $\gamma = 1/\alpha$

$$\bar{H}(x,y) = \exp\left[-\left(\varepsilon x^{\alpha} + \varepsilon y^{\alpha}\right)\right]^{1/\alpha}, \qquad x, y > 0$$

$$\bar{H}(x,y) = C\left(\bar{F}(x), \bar{G}(y)\right)$$

Let $\bar{F}(x)=e^{-\varepsilon^{1/\alpha}x}, \bar{G}(x)=e^{-\varepsilon^{1/\alpha}y}$,then the survival copula of bivariate exponential exponential distribution is Gumble- Hougaard copula.

9- Bivariate Gumble Logistic Distribution

Let's say you have two random variables, *X* and *Y*, with a joint distribution function.

$$H(x,y)=\left(1+e^{-x}+e^{-y}\right)^{-1} \qquad \forall x,y\in \bar{R}$$

The lack of a parameter in a bivariate logistic distribution $F(x)=\left(1+e^{-x}\right)^{-1}$ and $G(y)=\left(1+e^{-y}\right)^{-1}$ can be addressed in a variety of methods, one of which was proposed by Ali et al. H is defined by (1978) as

$$H(x,y)=\left(1+e^{-x}+e^{-y}+(1-\theta)e^{-x-y}\right)^{-1}, \forall x,y\in \bar{R}\, and\, \theta\in[-1,1]$$

The copula can be given as

$$C_{\theta}(u,v)=\frac{uv}{1-\theta(1-u)(1-v)} \quad (22)$$

when $\theta=1$ we have Gumble's bivariate logistic distribution. This is known as the Ali-Mikhiail- Haq copula family (Hutchinson and Lai 1990), and it has an Archimedean generator $\varphi(t)=-\ln(t)$.

10- Plackett's copula

Plackett's copula is a comprehensive copula developed by Plackett (1965) as a result of his research.

$$C(u,v)=\begin{cases}\dfrac{1+(\theta-1)(u+v)-\sqrt{[1+(\theta-1)(u+v)]^{2}-4\theta(\theta-1)uv}}{2(\theta-1)}, & \theta>0\\ uv & \theta=1\end{cases} \qquad (23)$$

Its probability density function is as follows:

$$c(u,v)=\frac{\theta[(\theta-1)(u-2uv)+1]}{\left\{[1+\theta(u+v)]^{2}-4\theta(\theta-1)uv\right\}^{3/2}}$$

If the variables u and v are independent, then $\theta=1$

If $\theta=0$, the copula becomes fréchet lower-Hoeffding bound, if $\theta\to\infty$ we can get fréchet Upper-Hoeffding, and $\theta=1$ our copula is considered independent. Plackett's copula Spearman rho .

$$\rho_s = \frac{\theta+1}{\theta-1} - \frac{2\theta}{(\theta-1)^2}.\log\theta$$

There does not appear to be a known function of Kendal tau.

11- Extreme- value copulas

Let us name $C_{(n)}$ the copula of component wise maxima $X_{(n)} = \max X_i$ and $Y_{(n)} = \max Y_i$ if we have independent and identically distributed pairs of random variables with a common copula C. Theorem 3.3.1 from Nelson (2006)

$$C_{(n)}(u,v) = C^n\left(u^{1/n}, v^{1/n}\right), \quad 0 \le u,v \le 1 \qquad (24)$$

If there exists a copula C such that

$$C_*(u,v) = \lim_{n\to\infty} C^n\left(u^{1/n}, v^{1/n}\right), \qquad 0 \le u,v \le 1,$$

then C_* it is an extreme value copula.

We can get that using a mathematical way.

$$C_*(u^k, v^k) = C_*^k(u,v), \qquad k > 0$$

Gumble-Hougaard copula

$$C(u,v) = \exp\left\{-\left[(-\log u)^\alpha + (-\log v)^\alpha\right]^{1/\alpha}\right\} \qquad (25)$$

There is no other Archimedean copula that is also an extreme-value copula [Genest and Rivest(1989)].

12- Gaussian copula

Let $\Phi^{-1}(.)$ indicate the inverse of the distribution function of a standard normal random variable $\Phi(.)$. The variance-covariance matrix Σ of the Gaussian copula is defined by

$$C(u_1,...,u_p) = \Phi_\Sigma\left(\Phi^{-1}(u_1),...,\Phi^{-1}(u_p)\right) \qquad (26)$$

where Φ_Σ represents the distribution function of a p-variate normal random vector with zero means and variance-covariance matrix Σ. Fouque and Zhou provide a perturbed version of Gaussian copula (2008).

Assuming that $\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right)$ is the standard normal density, one can readily calculate

$$f(x,y) = \phi(x)\phi(y)(1 + \sin x \sin y)$$

A bivariate probability density is formed. The copula of this distribution is easy to get, as seen below.

$$C_\Phi(u,v) = uv + \int_0^u \sin(\Phi^{-1}(t))dt \int_0^v \sin(\Phi^{-1}(t))dt, \qquad (27)$$

where the right continuous inverse $\Phi^{-1}(t) = \sup\{x : \Phi(x) \le t\}$ of $t \in [0,1]$ and is the standard normal distribution function Φ. Naturally, one would worry if the resulting bivariate function is valid.

$$C_\Psi(u,v) = uv + \int_0^u \sin(\Psi^{-1}(t))dt \int_0^v \sin(\Psi^{-1}(t))dt, \ for\,(u,v) \in [0,1]^2, \qquad (28)$$

If and only if C_Ψ in (2.2) is a copula

$$\int_0^1 \sin(\Psi^{-1}(t))dt = 0 \qquad (29)$$

Denote X one random variable with distribution function Ψ, as well as the copula fully specified in (28) as a sine copula with generator. Then, the equivalence (29) is equivalent to E[sin(X)] = 0.

Assume you have a set of random variables X's with a lattice distribution $\{k\pi : k = 0, \pm 1, \pm 2, \ldots\}$

$$\sum_{k=-\infty}^{\infty} P(X = k\pi) = 1.$$

and as a result

$$\sin(\Psi^{-1}(t)) = 0$$

$$C_\Psi(u,v) = uv = C_I(u,v), \text{ for any } (u,v) \in (0,1]^2$$

Namely, C_I is a sine copula.

The normal distribution with expectation $n\pi$ and the student distribution both meet (29), and they function as copula generators in (28).

13- The sine copula's dependence indices

Whatever the generator Ψ is. Both the upper tail dependence coefficient $\lambda_U = 0$ and the lower tail dependence coefficient $\lambda_L = 0$ for every random vector with copula (28).

For each random vector $(X_1 X_2)$ with copula(28), any pair of random variables $(X_1 X_2)$ with sine copula (28) is asymptotically independent (28),

$$\rho(X_1X_2) = \frac{3}{2}\tau(X_1X_2)$$

since

$$\tau(X_1X_2) = 8\left[\int_0^1 u\sin\left(\Psi^{-1}(u)\right)du\right]^2 \leq \frac{1}{2}$$

A sine copula's Kendall's is always bounded below 2^{-1}. This also aids in determining whether the sine copula is appropriate for a given data set in practice,in fact, when

$$\Psi^{-1}(u) = \begin{cases} \frac{-\pi}{2}, & u \in \left[0, \frac{1}{2}\right] \\ \frac{\pi}{2}, & u \in \left[0, \frac{1}{2}\right] \end{cases} \qquad (30)$$

Remember that the normal or Gaussian copula $C_\alpha = H_\alpha\left(\Phi^{-1}(u), \Phi^{-1}(v)\right), \tau$ archives the maximum 2^{-1}.

$$C_\alpha = H_\alpha\left(\Phi^{-1}(u), \Phi^{-1}(v)\right)$$

The α and Φ is the distribution function of $N(0,1)$ By ***Denuit et al. (2005)***, and ***(Li, 2000)***,the normal copula C_α has the Kendall's τ and Spearman's ρ as

$$\tau = \frac{2}{\pi}\arcsin\alpha, \quad \rho = \frac{6}{\pi}\arcsin\frac{\alpha}{2}$$

where H_α is the standard bivariate normal distribution function with correlation The normal copula is not in the family of sine copulas.

It is evident that $\rho/\tau = \frac{2\arcsin\left(\alpha/2\right)}{\arcsin\alpha} \neq 3/2$. So, the normal copula is not a sine copula.

14- Gaussian (Normal) copula

The normal copula is flexible in that it allows for equal degrees of positive and negative dependence, as well as both Fréchet bounds. The usual copula looks like this:

$$C(u_1, u_2; \theta) = \Phi_G\left(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta\right),$$
$$= \int_{-\infty}^{\Phi^{-1}(u_1)}\int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi\left(1-\theta^2\right)^{1/2}} \times \left\{\frac{-\left(s^2 - 2\theta st + t^2\right)}{2\left(1-\theta^2\right)}\right\} dsdt \qquad (31)$$

where Φ is the standard normal distribution's CDF, and $\Phi_G(u_1,u_2)$ is the standard bivariate normal distribution with the correlation parameter constrained to the interval $(-1,1)$.

Any type of non-linear dependence between the elements of *y* is ruled out by the Gaussian copula. More flexible copulas that nest the normal copula as a specific example, which are derived from a multivariate distribution that nests the multivariate normal distribution, have been examined by empirical researchers.

15- Student t copulas

The Student t distribution generalises the multivariate normal distribution by adding a single more parameter, the degrees of freedom ν. Let $t_\nu^{-1}(.)$ indicate the inverse of the distribution function $t_\nu(.)$ of a Student's t random variable with degree of freedom ν. Let $G_\nu^{-1}(.)$ denote the inverse distribution function of and let $G_\nu(.)$ indicate the distribution function of $\sqrt{\nu/\chi_\nu^2}$.

$$\text{let } z_i(u_i,s)=t_{\nu_i}^{-1}(u_i)/G_{\nu_i}^{-1}(s) \; for\, i=1,\ldots,p.$$

The t copula with degrees of freedom and variance-covariance matrix has been defined by Luo and Shevchenko (2012) as

Luo and ***Shevchenko (2012)*** have defined the t copula with degrees of freedom $(\nu_1,\nu_2,\ldots,\nu_p)$ and variance-covariance matrix Σ as

$$C(u_1,u_2,\ldots,u_p)=\int_0^1 \Phi_\Sigma(z_1(u_1,s),\ldots,z_1(u_1,s))ds, \quad (32)$$

Copulas of many multivariate *t* distributions are special examples of where Φ_Σ indicates the distribution function of a *p*-variate normal random vector with zero means and variance-covariance matrix Σ (32).

In d dimensions, the typical t-copula reads

$$C_t(u;\Re,\nu)=t_{\Re,\nu}(t_\nu^{-1}(u_1),\ldots,t_\nu^{-1}(u_{d1})) \qquad (33)$$

Where the correlation matrix $\Re$ parametrizes the multivariate standard Student-t distribution $t_{\Re,\nu}$, and ν if its density is given by

$$f(x)=\frac{\Gamma\left(\frac{\nu+d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d|\Sigma|}}\left(1+\frac{(x-\mu)'\Sigma^{-1}(x-\mu)}{\nu}\right)^{-\frac{\nu+d}{2}}$$

The covariance matrix is not equal to Σ and is only defined if $\nu > 2$ the multivariate t belongs to the class of multivariate normal variance mixtures $X \overset{d}{=} \mu + \sqrt{W}Z$, and has the representation in this standard parameterization $\operatorname{cov}(X) = \frac{\nu}{\nu - 2}\Sigma$.

where W is independent of Z and $Z \approx N_d(0, \Sigma)$ satisfies $\nu/W \approx \chi_\nu^2$; equivalently W has an inverse gamma distribution $W \approx I_g(\nu/2, \nu/2)$ (1970)by ***Kelker***. This indicates that the copula of a $t_d(\nu, \mu, \Sigma)$ is the same as the copula of $t_d(\nu, 0, p)$ adistribution, where P is the dispersion matrix Σ inferred correlation matrix. As a result, the copula is unique. The copula is unique and given by

$$C_{\nu,P}^t(u) = \int_{-\infty}^{t_\nu^{-1}(u_1)} \ldots \int_{-\infty}^{t_\nu^{-1}(u_d)} \frac{\Gamma\left(\frac{\nu + d}{2}\right)}{\Gamma\left(\frac{\nu}{2}\right)\sqrt{(\pi\nu)^d |p|}} \left(1 + \frac{X' P^{-1} X}{\nu}\right)^{-\frac{\nu+d}{2}} dx \qquad (34)$$

where t_ν^{-1} signifies a standard univariate t_ν distribution's quantile function. Using the normal mixture, we create a multivariate t-distributed $X \approx t_d(\nu, 0, P)$ with the form

$$c_{\nu,P}^t(u) = \frac{f_{\nu,P}\left(t_\nu^{-1}(u_1), \ldots, t_\nu^{-1}(u_d)\right)}{\prod_{i=1}^{d} f_\nu\left(t_\nu^{-1}(u_i)\right)}, \; u \in (0,1)^d, \; (35)$$

where f_ν is the density of the univariate standard t-distribution with ν degrees of freedom and $f_{\nu,P}$ is the joint density of a $t_d(\nu, 0, P)$-) distributed random vector.

A multivariate t distribution with degrees of freedom exists when a random vector X has the t copula $c_{\nu,P}^t$ and univariate t margins with the same degree of freedom parameter ν. If we use the t copula to combine any other set of univariate distribution functions, we get multivariate dfs F, which are referred to as meta-t_ν distribution functions (see Embrechts et al. (2001) or Fang & Fang (2002)).

- **Kendall's Rank Correlation**

The measure of Kendall's Rank Correlation is determined as follows:

$$\rho_\tau(X_1, X_2) = E\left(sign\left(X_1 - \tilde{X}_1\right)\left(X_2 - \tilde{X}_2\right)\right)$$

The Kendall's tau rank correlation ρ_τ depends on the copula C Kendall's tau takes the same elegant form for the Gauss copula C_ρ^{Ga}, the t copula $C_{\nu,\rho}^t$, or the copula of essentially all relevant distributions in the elliptical class, this form being

$$\rho_\tau(X_1, X_2) = \frac{2}{\pi}\arcsin\rho$$

The two measures λ_u and λ_l coincide for the copula of an elliptically symmetric distribution like the *t*, and are denoted simply by λ for the *t* copula it is positive; Embrechts et al. (2001) calculated a simple formula, and the coefficient of tail dependence for continuously distributed random variables with the t copula $C^t_{\nu,\rho}$ is given by

$$\lambda = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right)$$

$$\lambda = 2t_{\nu+1}\left(-\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right)$$

where P has an off-diagonal element ρ.

16- Marshal and olkin

Marshal and Olkin (1967) were the first to introduce it, and it was defined as

$$C(u,v) = \begin{cases} u^{1-\alpha}v & ,u^\alpha \geq v^\beta \\ uv^{1-\beta} & if\ u^\alpha < v\beta \end{cases} ,\alpha \geq 0, \beta \leq 1$$

Complete independence match $\alpha = \beta = 1$, independency match $\alpha = \beta = 0$, independency It's possible to write it in a different formula.

$$C(u,v) = \min\left(u^{1-\alpha}, uv^{1-\beta}\right)$$

Marshal and Olkin is the surname of this family.

17- Generalized Marshal and Olkin Copula

Consider a two-component system having a CPU (central processing unit) and a co-processor, such as a two-component desktop computer. Shocks are applied to the components, which are always "fatal" to one or both of them. One of the two aeroplane engines, for example, could fail, or a large explosion could destroy both engines at the same time; or the CPU or co-processor could fail, or a power surge could destroy both at the same time. The lives of components 1 and 2 are denoted by X and Y, respectively. We shall locate the survival function, as is commonly the case when dealing with lifetimes.

$$\bar{H}(x,y) = P[X > x, Y > y]$$

Component 1's likelihood of surviving beyond time x and component 2's probability of surviving beyond time y. The shocks to the two components are considered to generate three separate poission processes, each with positive parameters λ_1, λ_2 and λ_3, depending on whether the shock kills only component 1, only component 2, or both components at the same time. These three shocks' timing and frequency Z_1, Z_2 and Z_3 of occurrence are independent exponential random variables with parameters λ_1, λ_2 and λ_3, respectively. As a result,

$$X = \min(Z_1, Z_{12}), Y = \min(Z_2, Z_{12}) \text{and hence for all } x, y \geq 0$$

$$\begin{aligned} \bar{H}(x,y) &= P[Z_1 > x]P[Z_2 > y]P[Z_{12} > \max(x,y)] \\ &= \exp[-\lambda_1 x - \lambda_2 y - \lambda_{12} \max(x,y)] \end{aligned}$$

The functions of marginal survival are

$$\bar{F}(x) = \exp[-(\lambda_1 + \lambda_{12} x)] \text{and } \bar{G}(x) = \exp[-(\lambda_2 + \lambda_{12} y)];$$

as a result, X and Y are exponential random variables with parameters $\lambda_1 + \lambda_{12}$ and $\lambda_2 + \lambda_{12}$.

Let $X = (X_1, X_2)$ a bivariate vector of distribution, For independent random $X = (X_1, X_2) = (\min\{s_1, s_3\}, \min\{s_2, s_3\})$, lifetimes $s_i \approx \bar{G}_i(x)$ with cumulative hazard function H_i, be used to find survival copula.

$$\begin{aligned} \bar{F}(x_1, x_2) &= p(X_1 > x_1, X_2 > x_2) \\ &= p(S_1 > x_1, S_2 > x_2, S_3) > \max\{x_1, x_3\} \\ &= \bar{G}_1(x_1)\bar{G}_2(x_2)\bar{G}_3(\max\{x_1, x_2\}) \end{aligned}$$

when the marginal survival function is used

$$\begin{aligned} \bar{F}_1(x_1) &= P(X_1 > x_1) = \bar{G}_1(x_1)\bar{G}_3(x_1) \\ &= \exp\{H_1(x_1) - H_3(x_1)\} \end{aligned}$$

$$\begin{aligned} \bar{F}_2(x_2) &= P(X_2 > x_2) = \bar{G}_2(x_2)\bar{G}_3(x_2) \\ &= \exp\{H_2(x_2) - H_3(x_2)\} \end{aligned}$$

where

$$\begin{aligned} \tilde{H}_1(x) &= H_1(x) + H_3(x) \\ \tilde{H}_2(x) &= H_2(x) + H_3(x) \end{aligned}$$

then

$$\bar{F}_1^{-1}(u) = \tilde{H}_1^{-1}(-\ln u), \ \bar{F}_2^{-1}(v) = \tilde{H}_2^{-1}(-\ln v)$$

The X surviving copula is called $\hat{C}_x(u,v)$, and for this reason that (u,v) we have $\bar{F}_1^{-1}(u) > \bar{F}_2^{-1}(v)$

$$\begin{aligned}\ln \hat{C}_x(u,v) &= \ln \bar{F}\left(\bar{F}_1^{-1}(u), \bar{F}_2^{-1}(v)\right)\\ &= -H_1\left(\bar{F}_1^{-1}(u)\right) - H_2\left(\bar{F}_2^{-1}(v)\right) - H_3\left(\bar{F}_1^{-1}(u)\right)\\ &= -\tilde{H}_1\left(\bar{F}_1^{-1}(u)\right) - H_2\left(\bar{F}_2^{-1}(v)\right)\\ &= \ln u - H_2\left(\tilde{H}_2^{-1}(-\ln v)\right)\\ &= \ln u + \ln v + H_3\left(\tilde{H}_2^{-1}(-\ln v)\right)\end{aligned}$$

Similarly, for (u,v) such $F_1^{-1}(u) \le F_2^{-1}(v)$, we have

$$\ln \hat{C}_x(u,v) = \ln u + \ln v + H_3\left(\tilde{H}_1^{-1}(-\ln u)\right)$$

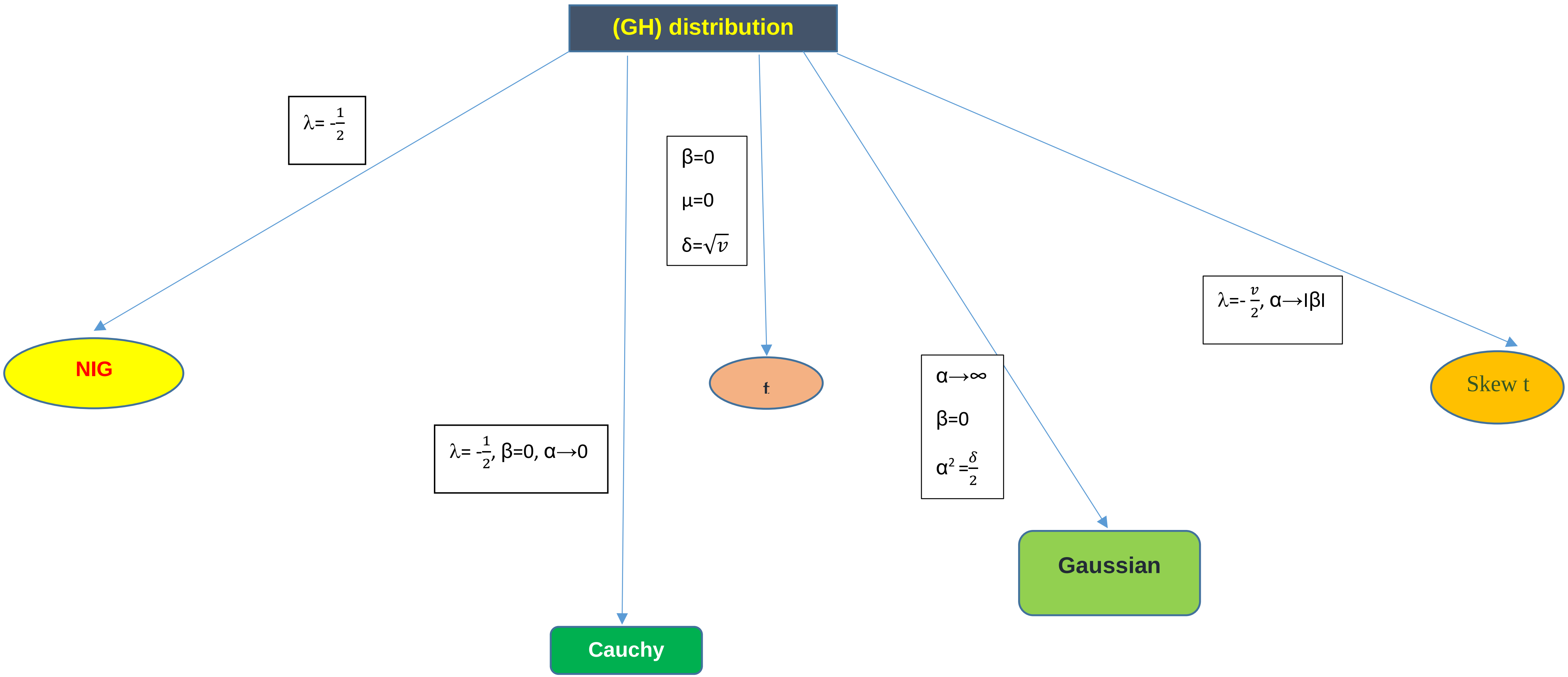
thus

$$\hat{C}_x(u,v) = \begin{cases} uv\exp\left\{H_3\left(\tilde{H}_1^{-1}(-\ln u)\right)\right\}, & \tilde{H}_1^{-1}(-\ln u) \le \tilde{H}_2^{-1}(-\ln v) \\ uv\exp\left\{H_3\left(\tilde{H}_{21}^{-1}(-\ln v)\right)\right\}, & \tilde{H}_1^{-1}(-\ln u) > \tilde{H}_2^{-1}(-\ln v) \end{cases}$$

The Generalized Marshal Olkin (GMO)survival copula, the tail independence lower tail $\lambda_l = 0$, upper tail $\lambda_u = \min(\alpha, \beta)$, is any copula that takes the previous form.

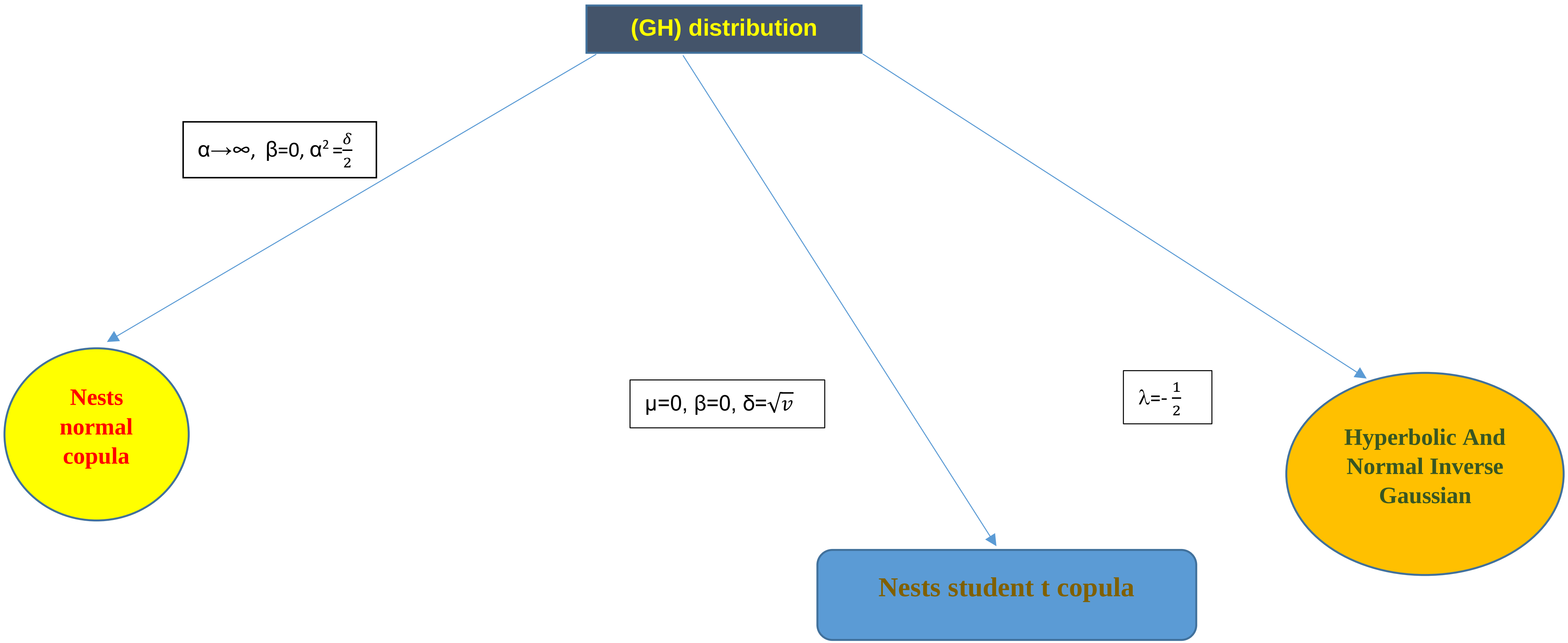
family	**generator** $\phi(t)$	**copula with two variables** $C_\alpha(u,v)$	**Inverse Generator (Laplace Transform)** $\tau(s) = \phi^{-1}(s)$
Independence	$-\ln t$	uv	$\exp(-s)$
Clayton(1978), cook-johnson(1981), okas (1982)	$t^{-n} - 1$	$\left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-1/\alpha}$	$(1+s)^{-1/\alpha}$
Gumbel(1960), hougaard(1986)	$-\ln t^{\alpha}$	$\exp\left\{-\left[(-\ln u)^{\alpha} + (-\ln v)^{\alpha}\right]^{1/\alpha}\right\}$	$\exp\left(-s^{-1/\alpha}\right)$
Frank(1979)	$-\ln \dfrac{e^{\alpha t} - 1}{e^{\alpha} - 1}$	$\dfrac{1}{\alpha}\ln\left(1 + \dfrac{(e^{\alpha u} - 1)(e^{\alpha v} - 1)}{e^{\alpha} - 1}\right)$	$\alpha^{-1}\ln\left\|1 + e^{s}(e^{\alpha} - 1)\right\|$

Table (1) Archimedean Copulas and Their Generators and Their Inverse



Figure(1)

Fig. 1. GH distribution



Figure(2) GH distribution 2

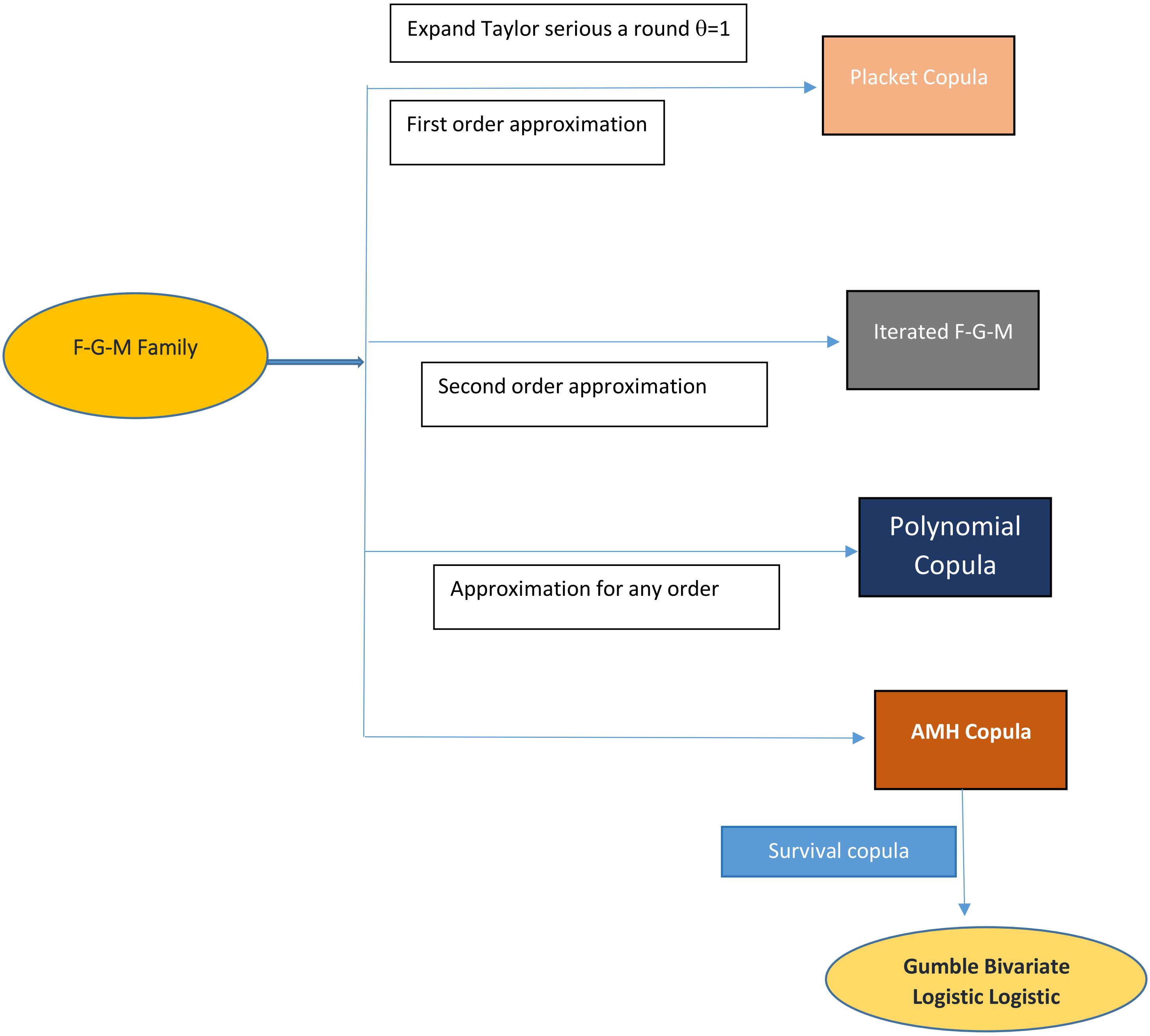


Figure (3)

F-G-M family distribution

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