

Exact solutions of sixth and fifth degree equations

Abstract

The real problematic with algebraic polynomial equations is how to exactly solve any sixth and fifth degree polynomial equations. In this study, we give a new absolute method that presents a new decomposition to exactly solve a sixth degree polynomial equation, while the corresponding fifth degree equation can be easily transformed into a sixth degree equation of this kind (sixth degree equation solvable by this method), then the sextic equation (sixth degree equation) obtained will be solved by applying the principles of this method; moreover, the solutions of the quintic equation (fifth degree equation) will be easily deduced.

Keywords: Exact solving; Sixth and fifth degree polynomial equations; New absolute method.

1. Introduction

The general polynomial equation is impossibility solved in radicals when their degree is higher than the fourth [1]; this proposition (lemma) is confirmed by the work of Abel in 1826, it is published under the Abel's impossibility theorem. Afterwards, the available work of Galois in 1832 increasingly supports Abel's theorem by demonstrating through group theory. Other works show the existence of algebraic solutions of some general polynomial equations of degree greater than or equal to five [2]. Moreover, the use of symbolic coefficients gives possibility to solve this kind of equations, such as the solving of the general quintic equation (fifth degree equation) is done by bring radicals, whereas the general sextic equation (sixth degree equation) is solvable under the framework of Kampe de Fariet functions [3].

In this work, we give an absolute exact method which solves a general polynomial equation of the sixth degree where its coefficients obey certain conditions. The process of solving consists to decompose the sextic equation into two cubic equations (third degree equations), and then it becomes possible to find all the six solutions (roots) of the general sextic equation. The decomposition involves breaking down the sextic equation by adding terms simultaneously in the two expressions of the

equation in order to find the square of the cubic equation in the first expression, the second expression exhibits a quartic equation (fourth degree equation), it must be equal to a square of second degree equation (second degree equation whose discriminant (Δ) is equal to zero) by changing the variable x^2 to y ; therefore, the transformed sextic equation contains the square of a cubic equation in the right term and the square of a quadratic equation (second degree equation) in the left term, and the problem will be easily solved. Ultimately, the sextic equation is transformed into two different cubic equations which can usually be solved by Cardan's method [4-6].

Furthermore, the quintic equation is one that can be turned to the sextic equation, by multiplying by the term $(x-\alpha)$, then the five roots of the corresponding quintic equation will be found by solving the sextic equation that their coefficients obey the conditions of this method.

2. Theory

2.1. Solving of sextic equations

The general sextic equation takes the general form:

$$x^6 + ax^5 + bx^4 + cx^3 + dx^2 + ex + f = 0 \quad (1)$$

Where, $a, b, c, d, e,$ and f are real coefficients of the general sextic equation.

Then, we must transform this equation (1) into a sextic equation without the fifth degree term, the transformation is done by changing x by $x = (X - \frac{a}{6})$. The equation (1) becomes:

$$X^6 - AX^4 - BX^3 - CX^2 - DX - E = 0 \quad (2)$$

After, the equation (2) becomes:

$$X^6 = AX^4 + BX^3 + CX^2 + DX + E \quad (3)$$

The main idea of this problem is to decompose this equation into two different equations by forming the square of a cubic equation in the right-hand side, which must be equal to $(X^3 + a_1X + a_2)^2$, and the square of quadratic equation in the left-hand side. In equation (3), it is obvious to add the following term $(2a_1X^4 + 2a_2X^3 + a_1^2X^2 + 2a_1a_2X + a_2^2)$ to both sides of the equation (3), in order to obtain the expression of the square of the cubic equation in the right-hand side of equation (3).

$$X^6 + 2a_1X^4 + 2a_2X^3 + a_1^2X^2 + 2a_1a_2X + a_2^2 = AX^4 + BX^3 + CX^2 + DX + E + (2a_1X^4 + 2a_2X^3 + a_1^2X^2 + 2a_1a_2X + a_2^2) \quad (4)$$

Then, equation (4) becomes:

$$(X^3 + a_1X + a_2)^2 = (A+2a_1)X^4 + (B+2a_2)X^3 + (C + a_1^2)X^2 + (D + 2a_1a_2)X + (E + a_2^2) \quad (5)$$

To have a quadratic equation in the left-hand side of equation (5), we must put the coefficients of the terms X^3 and X equal to zero, and change the variable X^2 to Y .

$$\begin{cases} B + 2a_2 = 0 \\ D + 2a_1a_2 = 0 \end{cases} \quad (6)$$

After calculation, a_1 and a_2 are equal to:

$$\begin{cases} a_1 = D/B \\ a_2 = -B/2 \end{cases} \quad (7)$$

The equation (5) becomes:

$$(X^3 + a_1X + a_2)^2 = (A + 2\frac{D^2}{B^2})Y^2 + (C + \frac{D^2}{B^2})Y + (E + \frac{B^2}{4}) \quad (8)$$

To have a square of a polynomial of degree one in the expression on the left-hand side of equation (8), we must impose the discriminant of the following equation $((A + 2\frac{D^2}{B^2})Y^2 + (C + \frac{D^2}{B^2})Y + (E + \frac{B^2}{4}) = 0)$ equal to zero. Therefore,

$$\Delta = (C + \frac{D^2}{B^2})^2 - 4(A + 2\frac{D^2}{B^2})(E + \frac{B^2}{4}) = 0 \quad (9)$$

So, it is underlined that each sextic equation of type $X^6 - AX^4 - BX^3 - CX^2 - DX - E = 0$, whose coefficients obey to the relation (9), will be solved using the formalism of this method.

The equation (8) becomes:

$$(X^3 + a_1X + a_2)^2 = (A + 2\frac{D^2}{B^2})(Y + Y_0)^2 \quad (10)$$

$$\text{With } Y_0 = \frac{1}{2} \frac{(C + \frac{D^2}{B^2})}{(A + 2\frac{D^2}{B^2})}.$$

After transformation, equation (10) becomes:

$$\begin{cases} X^3 + a_1X + a_2 = +\sqrt{A + 2\frac{D^2}{B^2}}(X^2 + Y_0) \\ X^3 + a_1X + a_2 = -\sqrt{A + 2\frac{D^2}{B^2}}(X^2 + Y_0) \end{cases} \quad (11)$$

Each of these two cubic-type equations mentioned in equation (11) will be solved by applying Cardan's method [4-6]. After solving these last two cubic equations, we will have three roots for each cubic equation, in the end we will obtain the six roots of the sextic equation considered.

2.2. Solving of quintic equations

The general quintic equation takes the general form:

$$x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0 \quad (12)$$

To solve this equation by this method is it necessary to make it in a polynomial equation of degree six by multiplying by the factor $(x - \alpha)$.

The equation (12) becomes:

$$(x^5 + ax^4 + bx^3 + cx^2 + dx + e)(x - \alpha) = [x^6 + (a - \alpha)x^5 + (b - \alpha a)x^4 + (c - \alpha b)x^3 + d - \alpha cx^2 + (e - \alpha dx - \alpha e) = 0 \quad (13)$$

After, we set the coefficient of the term x^5 equal to zero for $\alpha = a$, the equation (13) becomes:

$$x^6 + (b-a^2)x^4 + (c-ab)x^3 + (d-ac)x^2 + (e-ad)x - ae = 0 \quad (14)$$

In order to minimize the number of characters, it is necessary to set the following coefficients:

$$\begin{cases} A = a^2 - b \\ B = ab - c \\ C = ac - d \\ D = ad - e \\ E = ae \end{cases} \quad (15)$$

Therefore, the equation (14) is transformed according to this form:

$$x^6 - Ax^4 - Bx^3 - Cx^2 - Dx - E = 0 \quad (16)$$

According to the statement of this method, the equation (15) will be solved if and only if its coefficients obey the following relation:

$$\left(C + \frac{D^2}{B^2}\right)^2 - 4\left(A + 2\frac{D^2}{B^2}\right)\left(E + \frac{B^2}{4}\right) = 0 \quad (17)$$

3. Example

Let the following sextic equation:

$$x^6 + x^4 - 4x^3 - x^2 - 4x + 3 = 0$$

The equation becomes:

$$x^6 = -x^4 + 4x^3 + x^2 + 4x - 3$$

By adding terms in both sides of the equation, the equation becomes again:

$$(x^3 + a_1x + a_2)^2 = (2a_1 - 1)x^4 + (2a_2 + 4)x^3 + (a_1^2 + 1)x^2 + (2a_1a_2 + 4)x + (a_2^2 - 3)$$

In the left-hand side of the previous equation, we force the coefficients of x^3 and x to be equal to zero.

Therefore, we will have:

$$\begin{cases} 2a_2 + 4 = 0 \\ 2a_1a_2 + 4 = 0 \end{cases} \Leftrightarrow \begin{cases} a_1 = +1 \\ a_2 = -2 \end{cases}$$

And the sextic equation becomes:

$$(x^3 + x - 2)^2 = x^4 + 2x^2 + 1$$

The discriminant of the following equation ($x^4 + 2x^2 + 1 = 0$) must be equal to zero.

Therefore: $\Delta = 2^2 - 4 \times (1) \times (1) = 0$.

Here, $x^4 + 2x^2 + 1 = (x^2 + 1)^2$

At the end we will have:

$$\begin{cases} x^3 + x - 2 = (x^2 + 1) \\ x^3 + x - 2 = -(x^2 + 1) \end{cases}$$

After,

$$\begin{cases} x^3 - x^2 + x - 3 = 0 \\ x^3 + x^2 + x - 1 = 0 \end{cases}$$

We will then determine the solutions of the two cubic equations using Cardan's method [4-6].

The solutions for the two equations are:

- The $(x^3 - x^2 + x - 3 = 0)$ equation:
 $x_1 \approx 1.5747431$
 x_2 and x_3 are complex solutions.
- The $(x^3 + x^2 + x - 1 = 0)$ equation:
 $x_4 \approx 0.543689$
 x_5 and x_6 are complex solutions.

4. Conclusions

In this study, we have given a method that exactly solves both sixth degree and fifth degree equations, where their coefficients satisfy conditions, the main conclusions of this study are summarized in the following points:

1. The sextic equation of the type $X^6 - AX^4 - BX^3 - CX^2 - DX - E = 0$ is solved by this method if and only if the coefficients satisfy the following criterion: $\left(C + \frac{D^2}{B^2}\right)^2 = 4\left(A + 2D2B2E + B24\right)$.
2. Each fifth degree equation (quintic equation) $(x^5 + ax^4 + bx^3 + cx^2 + dx + e = 0)$ turns into a sixth degree equation by multiplying by $(x - \alpha)$, the corresponding sextic equation becomes $(x^6 + (b - a^2)x^4 + (c - ab)x^3 + (d - ac)x^2 + (e - ad)x - ae = 0)$, the considered quintic equation can be solved by this absolute method if its coefficients obey the following constraint:

$$\left(C + \frac{D^2}{B^2}\right)^2 - 4\left(A + 2\frac{D^2}{B^2}\right)\left(E + \frac{B^2}{4}\right) = 0, \text{ where } \begin{cases} A = a^2 - b \\ B = ab - c \\ C = ac - d \\ D = ad - e \\ E = ae \end{cases}$$

References

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