Original Research Article

ON THE CONVERGENCE AND STABILITY OF FINITE DIFFERENCE METHOD FOR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Abstract.
In this paper, we verify the convergence and stability of implicit (modified) finite difference scheme. Knowing fully that consistency and stability are very important criteria for convergence, we have prove the stability of the modified implicit scheme using the von Newmann method and also verify the convergence by comparing the numerical solution with the exact solution. The results shows that the schemes converges even as the step size is refined.

Keyword: Partial differential equation, Finite difference method, Implicit scheme, Stability, Modified Implicit scheme, Parabolic Equations.

AMS 2010 Subject Classification: $35A20$, $35A35$, $35B35$, $35K05$

1. Introduction

One of the most important aspect of Mathematics that is used to model many physical problems in other field like chemistry, engineering, physics among others is partial differential equations. Solving the model partial differential equations can be done by using analytical method. It is interesting to note that not all partial differential equation that can be solve analytically, hence the need for numerical methods. Numerical method is a way of computing the solution of a differential equations to its exact solutions. There are different types of numerical methods, they are; finite difference methods, finite element methods, mesh method, spectral method among others. In this work we shall concentrate on finite different methods for finding the solution of a partial differential equations by discretizing the domain into finite number of regions and compute the solution at the mesh points of the domain.

Different numerical experts and researchers in Mathematics and related fields have used the finite difference methods a lot. [1] Compared the exact solution of parabolic equations with its numerical solution using modified Crank-Nicolson scheme. [3] Investigated the stability of Modified Crank-Nicolson scheme using von-Newmann method. They show that the scheme is consistent, convergent and stable. [4] compared modified Crank-Nicolson scheme with the classical Crank-Nicolson scheme. [5] modified the simple explicit scheme and prove that it is much more stable than the simple explicit case, enabling larger time steps to be used. [6] established an improved θ method to improve the θ-iterated Crank-Nicolson scheme to second order accuracy. [7] Modified the Crank-Nicolson scheme to get a 3-level implicit finite difference scheme similar to the Crank-Nicolson scheme. The method utilizes on extra grid point at the lower level and the result is shown to be more accurate than the Crank-Nicolson scheme. There are lot of comprehensive texts on this area of research, and such text include [8, 9, 10, 11, 12 and 13]

In this work, We propose a modified implicit finite difference scheme and show that it is unconditionally stable and convergent by investigating it stability using von-Newmann method. The convergence is tested for using a numerical example, we compare the numerical solution with the exact solutions, refined the mesh size and compared with the exact solution.

2. Problem Definition and Methodology

We consider a parabolic second order linear partial differential equation of the form

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \varphi}{\partial x^2} \tag{1}$$

with initial condition

$$\varphi(x,0) = f(x),\ \ a < x < b \tag{2}$$

and boundary conditions

$$\varphi(a,t) = z_1,\ \ \varphi(b,t) = z_2,\ \ 0 \leq t \leq d \tag{3}$$

Equation (1) - (3) is referred to as one dimensional heat equation and it is generally called initial boundary value problem.
For the equations (1) - (3) above, the following partial derivatives are required;

$$\frac{\partial \varphi}{\partial x} = \frac{\varphi_{i+1,j} - \varphi_{i,j}}{h} + O(h) \ \ \textit{forward difference approximation}$$

$$\frac{\partial \varphi}{\partial x} = \frac{\varphi_{i+1,j} - \varphi_{i-1,j}}{2h} + O(h^2) \quad \textit{central difference approximation}$$

$$\frac{\partial \varphi}{\partial t} = \frac{\varphi_{i,j} - \varphi_{i,j-1}}{k} + O(h) \ \ \textit{backward difference approximation}$$

$$\frac{\partial^2 \varphi}{\partial x^2} = \frac{\varphi_{i+1,j} - 2\varphi_{i,j} + \varphi_{i-1,j}}{h^2} + O(h^2)$$

There are different types of finite difference schemes for solving equation (1) -(3) above, and they are explicit scheme, implicit scheme and Crank-Nicolson scheme. In this work, our focus is on the Implicit scheme, its derivation and the modification which is as follows:

2.1 Derivation of the Implicit scheme

The implicit scheme for the heat equation (1) is derived as follows; we replace the time derivative and the second partial derivative with the following finite approximations $\frac{\varphi_{i,j+1}-\varphi_{i,j}}{k}$ and $\frac{\varphi_{i-1,j+1}-2\varphi_{i,j+1}+\varphi_{i+1,j+1}}{h^2}$ respectively, then equation (1) becomes

$$\frac{\varphi_{i,j+1}-\varphi_{i,j}}{k}=\frac{\varphi_{i-1,j+1}-2\varphi_{i,j+1}+\varphi_{i+1,j+1}}{h^2}$$

$$\varphi_{i,j+1}-\varphi_{i,j}=\frac{k}{h^2}\varphi_{i-1,j+1}-2\varphi_{i,j+1}+\varphi_{i+1,j+1}$$

$$-\varphi_{i,j}=\frac{k}{h^2}\varphi_{i-1,j+1}-\varphi_{i,j+1}-\frac{2k}{h^2}\varphi_{i,j+1}+\frac{k}{h^2}\varphi_{i+1,j+1}$$

$$\varphi_{i,j}=-\frac{k}{h^2}\varphi_{i-1,j+1}+\varphi_{i,j+1}+\frac{2k}{h^2}\varphi_{i,j+1}-\frac{k}{h^2}\varphi_{i+1,j+1}$$

$$\varphi_{i,j}=-\frac{k}{h^2}\varphi_{i-1,j+1}+\left(1+\frac{2k}{h^2}\right)\varphi_{i,j+1}-\frac{k}{h^2}\varphi_{i+1,j+1}$$

which is the same as

$$\varphi_{i,j}=-r\left(\varphi_{i-1,j+1}+\varphi_{i+1,j+1}\right)+\left(1+2r\right)\varphi_{i,j+1} \tag{4}$$

In equation (4), we let $r=\frac{k}{h^2}$ and equation (4) is the Implicit scheme.

2.2 Derivation of Modified Implicit scheme

The modified implicit scheme is derived by replacing the time derivative and the second partial derivative of equation (1) with the finite approximations $\frac{\varphi_{i,j}-\varphi_{i,j-1}}{k}$ and $\frac{\varphi_{i-1,j}-2\varphi_{i,j}+\varphi_{i+1,j}}{h^2}$ respectively, then equation (1) becomes

$$\frac{\varphi_{i,j}-\varphi_{i,j-1}}{k}=\frac{\varphi_{i-1,j}-2\varphi_{i,j}+\varphi_{i+1,j}}{h^2}$$

$$\varphi_{i,j}-\varphi_{i,j-1}=\frac{k}{h^2}\varphi_{i-1,j}-2\varphi_{i,j}+\varphi_{i+1,j}$$

$$-\varphi_{i,j-1}=\frac{k}{h^2}\varphi_{i-1,j}-\varphi_{i,j}-\frac{2k}{h^2}\varphi_{i,j}+\frac{k}{h^2}\varphi_{i+1,j}$$

$$-\varphi_{i,j-1}=\frac{k}{h^2}\varphi_{i-1,j}+\varphi_{i,j}+\frac{2k}{h^2}\varphi_{i,j}-\frac{k}{h^2}\varphi_{i+1,j}$$

$$\varphi_{i,j}=-\frac{k}{h^2}\varphi_{i-1,j}+\left(1+\frac{2k}{h^2}\right)\varphi_{i,j}-\frac{k}{h^2}\varphi_{i+1,j}$$

which is the same as

$$\varphi_{i,j-1}=-r\left(\varphi_{i-1,j}+\varphi_{i+1,j}\right)+\left(1+2r\right)\varphi_{i,j} \tag{5}$$

In equation (5), we let $r = \frac{k}{h^2}$ and equation (5) is the Modified Implicit scheme. The equation (5) can be written in matrix form $A\varphi = b$ defined as follows:

$$\begin{bmatrix} 1+2r & -r & 0 & \dots & 0 \\ -r & 1+2r & -r & \dots & 0 \\ 0 & -r & 1+2r & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & -r \\ 0 & 0 & 0 & -r & 1+2r \end{bmatrix} \begin{bmatrix} \varphi_{1,j-1} \\ \varphi_{2,j-1} \\ \varphi_{3,j-1} \\ \vdots \\ \varphi_{n,j-1} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_n \end{bmatrix} \tag{6}$$

2.3 Stability of Modified Implicit scheme by von-Newmann method

The stability of the parabolic partial differential equation (1) - (3) is investigated using the finite difference approximation (5) stated below:

$$\varphi_{i,j-1} = -r\left(\varphi_{i-1,j} + \varphi_{i+1,j}\right) + (1+2r)\,\varphi_{i,j}$$

Let the solution of the finite difference approximation be given in separable form as stated below

$$\epsilon_{i,j} = \varepsilon^{\gamma ih}\varepsilon^{z\beta jk} = \varepsilon^{\gamma ih + z\beta jk}$$

where $\gamma = \gamma(\beta)$ is complex, define $\xi = \varepsilon^{\gamma h}$ which is the amplification factor, then we have

$$= \xi^i \varepsilon^{z\beta jk} \tag{7}$$

substituting equation (7) into (5) we have

$$(1+2r)\,\xi^i\varepsilon^{z\beta jk} - r\left(\xi^i\varepsilon^{z\beta(j-1)k} + \xi^i\varepsilon^{z\beta(j+1)k}\right) = \xi^{i-1}\varepsilon^{z\beta jk}$$

which gives

$$\xi^i\varepsilon^{z\beta jk}\left[(1+2r) - r(\varepsilon^{-z\beta k} + \varepsilon^{z\beta k})\right] = \xi^i\varepsilon^{z\beta k}\xi^{-1}$$

$$\xi^{-1} = (1+2r) - r(\varepsilon^{-z\beta k} + \varepsilon^{z\beta k}) \tag{8}$$

from trigonometry identity we have that

$$2\cos\beta k = \varepsilon^{-z\beta k} + \varepsilon^{z\beta k}$$

and

$$1 - \cos\beta k = 2\sin^2\left(\frac{\beta k}{2}\right)$$

substituting into equation (8) we have

$$\xi^{-1} = (1+2r) - r(2\cos\beta k) = 1 + 2r(1-\cos\beta k)$$

$$\xi^{-1} = \left[1 + 4r\sin^2\left(\frac{\beta k}{2}\right)\right]$$

$$\xi = \frac{1}{\left[1 + 4r\sin^2\left(\frac{\beta k}{2}\right)\right]} \tag{9}$$

from equation (9), it is apparent that $|\xi| \leq 1$ for all values of r, and therefore, the modified implicit approximation is unconditionally stable.

3.0 Numerical examples

For the purpose of convergence we present the following numerical examples, here we consider the fact that: a finite difference approximation is said to be convergent if

$$\varphi_{i,j} = ||\bar{\varphi_{i,j}} - \varphi_{i,j}|| \to 0 \ as\ h, k \to 0$$

Where $\bar{\varphi_{i,j}}$ is the exact solution, $\varphi_{i,j}$ is the numerical approximation and $\varphi_{i,j}$ is the error. This is demonstrated and represented in the tables below.

Example 1:

Consider the following parabolic partial differential equation:

$$\frac{\partial \varphi}{\partial t} - \frac{\partial^2 \varphi}{\partial x^2} = 0, \quad 0 < x < 1 \tag{10}$$

with boundary conditions

$$\varphi(0,t) = \varphi(1,t) = 0, \ 0 < t \tag{11}$$

and initial condition

$$\varphi(x,0) = \sin(\pi x), \ 0 \leq x \leq 1 \tag{12}$$

In this numerical example, the step size $h = 0.1, \ r = 0.05$. The exact solution of the problem (10) - (12) is given by $e^{-\pi^2 t}\sin(\pi x)$.

Solution

solving problems (10) together with the initial and boundary condition, using equation (5) we get the following tri-diagonal matrix for $1 \leq i \leq 9$ at $j = 1$, we get a tridiagonal matrix which is represented below;

$$\begin{bmatrix} 1.1 & -0.05 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.05 & 1.1 & -0.05 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.05 & 1.1 & -0.05 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.05 & 1.1 & -0.05 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.05 & 1.1 & -0.05 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.05 & 1.1 & -0.05 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.05 & 1.1 & -0.05 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.05 & 1.1 & -0.05 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.05 & 1.1 \end{bmatrix} \begin{bmatrix} \varphi_{1,1} \\ \varphi_{2,1} \\ \varphi_{3,1} \\ \varphi_{4,1} \\ \varphi_{5,1} \\ \varphi_{6,1} \\ \varphi_{7,1} \\ \varphi_{8,1} \\ \varphi_{9,1} \end{bmatrix} = \begin{bmatrix} 0.3090 \\ 0.5878 \\ 0.8090 \\ 0.9511 \\ 1.0000 \\ 0.9511 \\ 0.8090 \\ 0.5878 \\ 0.3090 \end{bmatrix}$$

the results of the next steps $1 \leq i \leq 10$, and $2 \leq j \leq 9$ is given in the table1.

Table 1: table of results at $k = 0.0005,\ r = 0.05$ and $h = 0.1$

t	x	j	$\varphi_{1,\ j}$	$\varphi_{2,\ j}$	$\varphi_{3,\ j}$	$\varphi_{4,\ j}$	$\varphi_{5,\ j}$	$\varphi_{6,\ j}$	$\varphi_{7,\ j}$	$\varphi_{8,\ j}$	$\varphi_{9,\ j}$
0.0005	0.1	1	0.3075	0.3060	0.3045	0.3030	0.3015	0.3000	0.2986	0.2972	0.2958
0.001	0.2	2	0.5895	0.5821	0.5793	0.5765	0.5737	0.5709	0.5681	0.5653	0.5625
0.0015	0.3	3	0.8051	0.8012	0.7973	0.7934	0.7895	0.7857	0.7819	0.7781	0.7743
0.002	0.4	4	0.9465	0.9419	0.9373	0.9327	0.9282	0.9237	0.9192	0.9147	0.9102
0.0025	0.5	5	0.9951	0.9903	0.9855	0.9807	0.9759	0.9712	0.9665	0.9618	0.9571
0.003	0.6	6	0.9465	0.9419	0.9373	0.9327	0.9282	0.9237	0.9192	0.9147	0.9102
0.0035	0.7	7	0.8051	0.8012	0.7973	0.7934	0.7895	0.7857	0.7819	0.7781	0.7743
0.004	0.8	8	0.5849	0.5821	0.5793	0.5765	0.5737	0.5709	0.5681	0.5653	0.5625
0.0045	0.9	9	0.3075	0.3060	0.3045	0.3030	0.3015	0.3000	0.2986	0.2972	0.2958

Table 2: comparison with exact solution at $x = 0.5$ with different values of t

t	$modified\ Implicit\ scheme$	$exact\ solutions$	$errors$	$percentage\ error$
0.0025	0.9951	0.9903	3×10^{-4}	0.03
0.003	0.9465	0.9419	4×10^{-4}	0.04
0.0035	0.8051	0.8012	5×10^{-4}	0.05
0.004	0.5849	0.5821	5×10^{-4}	0.05
0.0045	0.3075	0.3060	5×10^{-4}	0.05

the percentage error is the difference of the solutions expressed as a percentage of the exact solution of the partial differential equation.

Example 2.

We consider the same parabolic partial differential equation (10) - (12) with a refined mesh size as given below:
$h = 0.05,\ r = 0.05$ and $k = 0.000125$ solving using the modified scheme we

obtained the following tri-diagonal matrix for the refined mesh size.

$$\begin{bmatrix} 1.1 & -0.05 & 0 & 0 & \cdots & \cdots & 0 \\ -0.05 & 1.1 & -0.05 & \ddots & \ddots & \cdots & 0 \\ 0 & -0.05 & 1.1 & -0.05 & \ddots & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \cdots & 0 \\ 0 & 0 & 0 & 0 & \ddots & \cdot & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \ddots & -0.05 & 1.1 \end{bmatrix} \begin{bmatrix} \varphi_{1,1} \\ \varphi_{2,1} \\ \varphi_{3,1} \\ \varphi_{4,1} \\ \varphi_{5,1} \\ \vdots \\ \vdots \\ \varphi_{18,1} \\ \varphi_{19,1} \end{bmatrix} = \begin{bmatrix} 0.1564 \\ 0.3090 \\ 0.4540 \\ 0.5878 \\ 0.7071 \\ 0.8090 \\ 0.8910 \\ 0.9511 \\ 0.9877 \\ 1.0000 \\ \vdots \\ \vdots \\ 0.3090 \\ 0.1564 \end{bmatrix}$$

solving the above tri-diagonal matrix and comparing the results with the exact solution at $x = 0.5$ we have the following results

Table 3: comparison of refined mesh size ($h = 0.05,\ r = 0.05$) and $k = 0.000125$ with exact solutions

t	$modified\ Implicit\ scheme$	$exact\ solution$	$error$	$percentage\ error$
0.001	0.9904	0.9902	2×10^{-4}	0.02
0.001125	0.9892	0.9890	2×10^{-4}	0.02
0.00125	0.9880	0.9877	3×10^{-4}	0.03
0.001375	0.9868	0.9865	3×10^{-4}	0.03
0.0015	0.9856	0.9853	3×10^{-4}	0.03

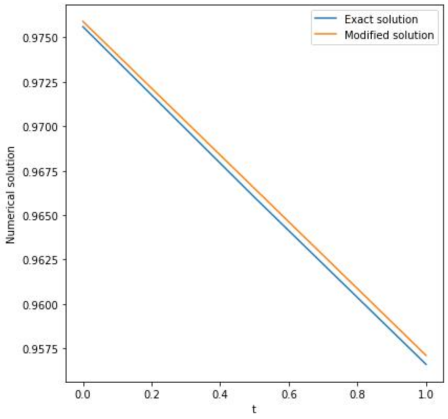

Figure 1: comparison graph of the exact solution and numerical solution at x = 0.5

Table 2, is the comparison of the numerical solutions (modified implicit scheme) with the exact solutions at $h = 0.1,\ r = 0.05$, we have compared the two solutions at $x = 0.5$ for different values of t. In table 3, we have also compare the results of the refined mesh size $h = 0.05,\ r = 0.05$ using the modified implicit scheme with the exact solution and the percentage errors are obtained. we compared the results at $x = 0.5$ for different values of t.
figure 1 above is the comparison graph of the exact solution and the numerical solution at $x = 0.5$ which shows clearly that the scheme is good and efficient as the solutions is very close to exact solutions. Also, figure 2, is the solution curve at $t = 0.0025$ before refinement while figure 3, is the solution curve after refinement which shows that the refined solution is more finer and it also implies that the refined solution converges very fast.

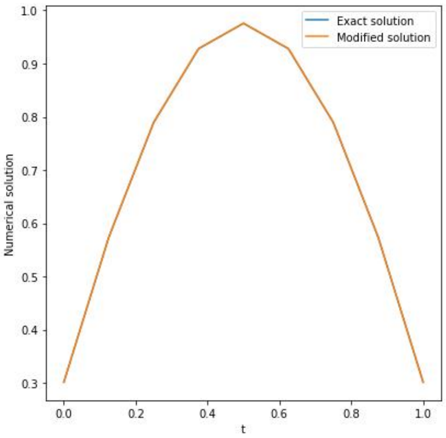

Figure 2: numerical solution graph at t = 0.0025

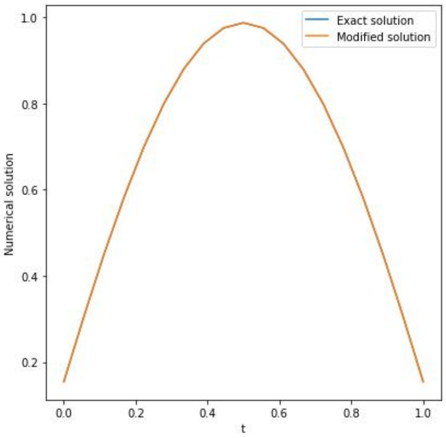

Figure 3: numerical solution graph at t = 0.000125

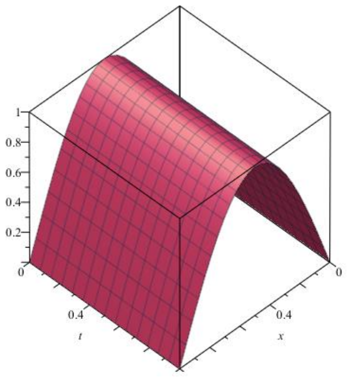

Figure 4: 3D graph of the solution

4. Discussion

Table 1, 2 and 3 shows that the modified implicit scheme is good and efficient for solving one dimensional heat equations. It shows that the method performs well, is consistent and agree with the analytical solutions. The method gives a better results in terms of accuracy and requires the solution of tri-diagonal system at every time level.

5. Conclusion

From our results analysis, it is observed that our method gives a good approximates solutions and converges faster compared to the implicit scheme. Also, the percentage error of our solution is less than that of other methods in literature. Considering our results from tables 2 and 3 it is observed that our scheme is stable and table 3 shows that our method converges as the mesh size tends to zero, which proves convergent.

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