ANALYTICITY OF THE RESOLVENT FUNCTION

Abstract

Analytic dependence on a complex parameter appears at many places in the study of differential and integral equations. The display of analyticity in the solution of the Fredholm equation of the second kind is an early signal of the important role which analyticity was destined to play in spectral theory. The definition of the resolvent set is very explicit, this makes it seem plausible that the resolvent is a well behaved function. Let T be a closed linear operator in a complex Banach space X. In this paper we show that the resolvent set of T is an open subset of the complex plane and the resolvent function of T is analytic. Moreover, we show that if T is a bounded linear operator, the resolvent function of T is analytic at infinity, its value at infinity being 0 (where 0 is the bounded linear operator 0 in X). Consequently, we also show that if T is bounded in X then the spectrum of T is non-void.

Subject Classification: xxxxxx

Keywords: resolvent set, resolvent function, analyticity, spectrum.

1 Introduction

The display of analyticity in the solution of the Fredholm equation of the second kind is an early signal of the important role which analyticity was destined to play in spectral theory. The utility of analytic functions and the calculus of residues in connection with expansion series of spectral functions has been known since the work of Cauchy. The uses of analyticity in operator theory were originally apparent in a classical context which required no recognition of analyticity in any sense other than what was entailed in speaking of complex valued analytic functions of a complex variable. This was so because the differential and integral operators under consideration were acting on numerically valued functions. For example, in the solution of the Fredholm's integral function, the function $D(s, t : \lambda)$ is a numerical function of s, t and λ [9] A significant progression has been made towards a conscious recognition of the concept of an analytic function with values which are not merely complex numbers but are, instead, elements of a function space, or space of an operator, or even abstract.

Definition 1. *Let X be a complex normed linear space and T a linear operator in X. If $\lambda \in \rho(T)$, we represent the continuous inverse $(\lambda I - T)^{-1}$ by $R(\lambda : T)$ and call it the resolvent (at λ) of T. The domain of the resolvent function $R(.; T)$ is $\rho(T)$. (Obviously, its range is in $B(X)$).*

If X is a Banach space and T is closed, then for each $\lambda \in \rho(T)$, $R(\lambda : T) \in B(X)$ and thus for each $\lambda \in \rho(T)$ the operator $R(\lambda : T)$ determines a one-to-one correspondence between $\mathfrak{D}_T$ and $R_{\lambda I - T} = X$. However, in the case of an arbitrary linear operator T, the domain of $R(. : T)$ is still $\rho(T)$, however the values $R(\lambda : T)$ for each $\lambda \in \rho(T)$ are bounded linear operators whose domains are dense in X; it does not follow that $R(\lambda : T) \in B(X)$ for $\lambda \in \rho(T)$ in the general situation. We wish to emphasize that our interest ahead lies mainly in situations where X is strongly complete(this of course means $H = \mathbb{K}$, a Hilbert space in our context). For a closed T, the resolvent set $\rho(T)$ is an open subset of $\mathbb{C}$ (see proposition 3 ahead). Thus the resolvent function $R(.; T)$ is a particular case of a function Q whose domain is an open subset of the complex plane and whose values are elements of $B(X)$, that is, Q is an operator-valued function and $\mathfrak{D}_Q$ is an open subset of $\mathbb{C}$. Before defining analyticity concepts in the case of such functions, we prove some useful results connected with the resolvent function. Most of the definitions and theorems can be found in [1], [3]

1.1 The resolvent function

Proposition 1. *Let T be a closed linear operator in a complex Banach space X. If $\lambda, \mu \in \rho(T)$, then*

$$R(\lambda : T) - R(\mu : T) = (\mu - \lambda)R(\lambda : T)R(\mu : T) \tag{1.1}$$

$$R(\lambda : T)R(\mu : T) = R(\mu : T)R(\lambda : T) \tag{1.2}$$

Proof. $R(\lambda : T) = R(\lambda : T)(\mu I - T)R(\mu : T)$
$=R(\lambda : T)\{(\mu - \lambda)I + (\lambda I - T)\}R(\mu : T)$
$=(\mu - \lambda)R(\lambda : T)R(\mu : T) + R(\mu : T)$
Although $\mathfrak{D}_T$ need not coincide with X, the distributive laws for operator multiplication hold in the situation considered here, as may be checked directly; also
$R(\lambda : T)(\lambda I - T) = I_{D_T}$, $(\lambda I - T)R(\lambda : T) = I_X$.
By interchanging λ and μ in 1.1, we get
$R(\mu; T) - R(\lambda; T) = (\lambda - \mu)R(\mu; T)R(\lambda; T)$
Comparing this relation with 1.1, we get at once relation 1.2. □

The commutativity relation 1.2 is a special case of the following general result:

Proposition 2. *Let X be a complex Banach space, $T \in B(X)$ and S be a closed linear operator in X.*

(i) If $T \leftrightarrow S$, then $T \leftrightarrow R(\lambda : S)$ for each $\lambda \in \rho(S)$

(ii) If for some $\lambda_o \in \rho(S)$, we have $T \leftrightarrow R(\lambda_o; S)$, then $T \leftrightarrow S$.

Proof. (i)Let $T \leftrightarrow S$. Then $Tx \in \mathfrak{D}_S$ and $STx = TSx$ for all $x \in \mathfrak{D}_S$. Let $\lambda \in \rho(S)$. Then (keeping in mind that $Tx \in \mathfrak{D}_S$ for all $x \in \mathfrak{D}_S$)
$(\lambda I - S)Tx = \lambda Tx - STx = \lambda Tx - TSx = T(\lambda I - S)x \forall\, x \in \mathfrak{D}_S$.
This gives on pre-multiplication by the operator $R(\lambda; S)$(which belongs to $B(X)$ since $\lambda \in \rho(S)$, S is closed and X strongly complete)

$$R(\lambda; S)(\lambda I - S)Tx = R(\lambda; S)T(\lambda I - S)x \forall\, x \in \mathfrak{D}_S \tag{1.3}$$

Since $R(\lambda; S)$ determines a one-to-one correspondence between its domain X and its range $\mathfrak{D}_S$, we can put $X = R(\lambda; S)y$ for exactly one $y \in X$, also the $x's$ exhaust $\mathfrak{D}_S$ as $y's$ exhaust X. Thus 1.3 implies
$TR(\lambda; S)y = R(\lambda; S)Ty$ forall $y \in X$
since $R(\lambda; S)(\lambda I - S) = I_{\mathfrak{D}_S}$ and $(\lambda I - S)x = y$. But this means that
$T \leftrightarrow R(\lambda; S)$ for each $\lambda \in \rho(S)$.
(ii)Suppose λ_o is some element of $\rho(S)$ such that $T \leftrightarrow R(\lambda_o; S)$.
Then
$TR(\lambda_o; S)(\lambda_o I - S)X = R(\lambda_o; S)T(\lambda_o I - S)x \,\forall\, x \in \mathfrak{D}_S$
that is,

$$Tx = R(\lambda_o; S)T(\lambda_o I - S)x \,\forall\, x \in \mathfrak{D}_S \tag{1.4}$$

This means that $Tx \in \mathfrak{D}_S$ since the element $R(\lambda_o; S)T(\lambda_o I - T)x$ belongs to $\mathfrak{D}_S$ (since $R(\lambda_o; S)$ maps onto $\mathfrak{D}_S$). Then, on pre-multiplying both sides of 1.4 by $(\lambda_o I - S)$ we get

$$(\lambda_o I - S)Tx = T(\lambda_o I - S)x \,\forall x \in \mathfrak{D}_S \tag{1.5}$$

Relation 1.5 yields $STx = TSx$ for all $x \in \mathfrak{D}_S$ and this with the condition $Tx \in \mathfrak{D}_S$ for all $x \in \mathfrak{D}_S$ yields $T \leftrightarrow S$ □

Remark 1. *Proposition 2 establishes that in order for a $T \in B(X)$ to commute with a closed operator S it is necessary that T commute with the resolvent $R(\lambda_o; S)$ for each $\lambda \in \rho(S)$ and it is sufficient that T commute with $R(\lambda_o; S)$ for at least one $\lambda_o \in \rho(S)$*

As a converse to proposition 1 we have:

Proposition 3. *Let X be a Complex Banach space and let $\{R_\lambda\}$ be a family of bounded linear operators with domain X defined at every point λ in a subset Ω of $\mathbb{C}$ such that*

(i) $R_\lambda - R_\mu = (\mu - \lambda) R_\lambda R_\mu$ *for each pair* $\lambda, \mu \in \Omega$

(ii) $R_\lambda y = \overline{0} \Rightarrow y = \overline{0}$ *for at least one* $\lambda \in \Omega$
Then there exists a unique closed linear operator T with resolvent $R(;T)$ such that $R(\lambda; T) = R_\lambda$ for each $\lambda \in \Omega$.

Proof. By hypothesis, there is at least one value of λ, say $\lambda = \mu$ such that $R_\mu y = \overline{0}$ implies $y = \overline{0}$. If λ is an arbitrary point of Ω and y is an element for which $R_\lambda y = \overline{0}$, then using (i), we have
$R_\mu y = R_\lambda y + (\lambda - \mu) R_\mu R_\lambda y = \overline{0}.$
Hence $y = \overline{0}$. The correspondence between the domain X of R_λ and its range $\mathfrak{R}_{R_\lambda}$ is therefore one-to-one. By inverting the correspondence and using the inverse in an appropriate manner we can define an operator T in X in the following manner. Let $y \in \mathfrak{R}_{R_\lambda}$, then $y = R_\lambda x$ for some $x \in X$. We assign to y the unique element $x - \lambda_y = x - \lambda R_\lambda x$. We shall prove the correspondence
$y \mapsto x - \lambda R_\lambda x$
is independent of λ. To do this, it is necessary to show that if $\lambda, \mu \in \Omega$ and $x \in X$ are given, there exists an element $x' \in X$ such that $R_\lambda x = R_\mu x'$ and $x - \lambda R_\lambda x = x' - \mu R_\mu x'$. It is evident that, if one element x' exists such that $R_\lambda x = R_\mu x'$, it must be unique. The element x' is defined by $x' = x + (\mu - \lambda) R_\lambda x$.
Then
$R_\mu x' = R_\mu x + (\mu - \lambda) R_\mu R_\lambda x = R_\lambda x$ by (i)
and also
$x' - \mu R_\mu x' = x' - \mu R_\lambda x = x + (\mu - \lambda) R_\lambda x - \mu R_\lambda x = x - R_\lambda x.$
We now take $\mathfrak{D}_T$ to be the range $\mathfrak{R}_{R_\lambda}$ and the operator T defined on $\mathfrak{D}_T$ which takes the element $y = R_\lambda x$ of $\mathfrak{D}_T$ to the element $x - \lambda R_\lambda x$. Note that $\mathfrak{D}_T$ is a linear subspace of X(for it is the range of $R_\lambda \in B(X)$) and T is linear. The linear subspace $\mathfrak{D}_T, \mathfrak{R}_{R_\lambda}$ and the linear operator T are independent of λ as we have seen.
If $\lambda \in \Omega$, we have $\lambda y - Ty = x, y = R_\lambda x$; so $\lambda I - T$ and R_λ are inverses of each other. Since $R_\lambda \in B(X)$, it is closed.; the same is true for $\lambda I - T$. Hence T is closed. It is now evident that $\rho(T) \supseteq \Omega$ and that $R(\lambda; T) = R_\lambda$ for every $\lambda \in \Omega$. It is also clear that T is uniquely determined by R_λ □

The functional equation 1.1 satisfied by the resolvent, when it exists suggests strongly that the resolvent must depend "analytically" on the complex variable λ in a sense to be explained below. First we prove

Proposition 4. *Let T be a closed linear operator in Banach space X. Let $\lambda_o \in \rho(T)$. Then for all λ satisfying $|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1}$. The inverse map $(\lambda I - T)^{-1}$ exists and is bounded*

Proof. Since $\lambda_o \in \rho(T)$, it follows that $(\lambda_o I - T)^{-1} \in B(X)$, that is, $(\lambda_o;T) \in B(X)$. If $x \in \mathfrak{D}_T$, we can write $x = R(\lambda_o;T)(\lambda_o I - T)x$;
hence

$\|x\| \leq \|R(\lambda_o;T)\|\|(\lambda_o I - T)x\|$

Thus $\|(\lambda_o I - T)x\| \geq \|R(\lambda_o;T)\|^{-1}\|x\|$ for all $x \in \mathfrak{D}_T$
Suppose $\lambda \in \mathbb{C}$ and

$$|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1} \tag{1.6}$$

then

$\|(\lambda I - T)x\| = \|(\lambda_o I - T)x - (\lambda_o - \lambda)Ix\| \geq \|(\lambda_o I - T)x\| - |\lambda - \lambda_o|\|x\|$
$\geq (\|R(\lambda_o;T)\|^{-1} - |\lambda - \lambda_o|)\|x\| \, \forall x \in \mathfrak{D}_T$.

This shows that $(\lambda I - T)x = \overline{0}$ implies $x = \overline{0}$ (since $(\lambda I - T)$ is bounded from below) and thus $(\lambda I - T)^{-1}$ exists for such λ
Furthermore

$\|(\lambda I - T)^{-1}\| \leq \|R(\lambda_o;T)\|\{1 - |\lambda - \lambda_o|\|R(\lambda_o;T)\|\}^{-1}$ □

Remark 2. *It is important to note that we may not conclude $\lambda \in \rho(T)$ for $|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1}$ since we have not yet proved that $\mathfrak{R}_{\lambda I - T}$ is dense in X. We shall now show that all such λ do belong to $\rho(T)$ and that $R(\lambda;T)$ can be expressed in a "Taylor series" expansion about λ_o which converges in norm to $R(\lambda;T)$ just like analytic functions of a complex variable. This property is referred to by saying that $R(\lambda;T)$ is a holomorphic function (of λ) in each component (maximally connected subset) of $\rho(T)$.*

Proposition 5. *Let T be a closed linear operator in a complex Banach space X and $\lambda_o \in \rho(T)$. Then all points $\lambda \in \mathbb{C}$ satisfying $|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1}$ belong to $\rho(T)$ and for these values of λ we have*

$R(\lambda;T) - \sum_{n=0}^{\infty}(\lambda_o - \lambda)^n R(\lambda_o;T)^{n+1}$

where the convergence on the right hand side is in the uniform sense, that is, in the norm of $B(X)$. In other words, $\rho(T)$ is an open subset of the complex plane and in each component (the maximally connected subsets) of $\rho(T), R(\lambda;T)$ is a holomorphic function of λ.

Proof. We know that $R(\lambda;T) \in B(X)$ for each $\lambda \in \rho(T)$. If $\lambda_o \in \rho(T)$ and λ is such that $|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1}$ then

$\sum_{n=0}^{\infty} |\lambda_o - \lambda|^n \|R(\lambda_o;T)^{n+1}\| \leq \sum_{n=0}^{\infty} |\lambda_o - \lambda|^n \|R(\lambda_o;T)\|^{n+1} < \infty$;

(we have a geometric series of common ratio less than 1 on the right side). Hence the operator sums S_m defined by

$S_m = \sum_{n=0}^{m}(\lambda_o - \lambda)^n R(\lambda_o;T)^{n+1} \; m = 0, 1, 2, ...$

(which belongs to $B(X)$) converges uniformly to an operator S
If $x \in X$ and S_{x-y}, then we have $y = s - lim_{m\to\infty} S_m x$; note that
$S_m x \in \mathfrak{D}_T = \mathfrak{D}_{\lambda I - T}$ for each $x \in X \ldots (*)$
(Note $S_m \to S$ in the norm of $B(X)$ implies $S_m \xrightarrow{s} S$)
we must now show that S is the inverse of $\lambda I - T$, that is, $\lambda I - T$ is invertible and $(\lambda I - T)^{-1} = S$. When $x \in \mathfrak{D}_T = \mathfrak{D}_{\lambda I - T}$,
$S(\lambda I - T)x = S - lim_{m\to\infty} S_m(\lambda I - T)x = S - lim_{m\to\infty} \sum_{n=0}^{m} (\lambda_o - \lambda)^n R(\lambda_o; T)^{n+1}(\lambda I - T)$
$= S - lim_{m\to\infty}[\sum_{n=0}^{m} (\lambda_o - \lambda)^n R(\lambda_o; T)^{n+1}\{(\lambda_o I - T)x - (\lambda_o - \lambda)x\}]$
$= S - lim_{m\to\infty}[\sum_{n=0}^{\infty} (\lambda_o - \lambda)^n R(\lambda_o; T)^n x - \sum_{n=0}^{m} (\lambda_o - \lambda)^{n+1} R(\lambda_o; T)^{n+1} x]$
$= S - lim_{m\to\infty}[x - (\lambda_o - \lambda)^{m+1} R(\lambda_o; T)^{m+1} x] = x$
Since $|\lambda_o - \lambda| < \|R(\lambda_o; T)\|^{-1}$.
On the other hand, if $x \in X$, putting $Sx = y = S - lim_{m\to\infty} y_m$, where $y_m = S_m x$ (see 1.6 above) and noting that $y_m \in \mathfrak{D}_{\lambda I - T}$, we have $(\lambda I - T)Sx = (\lambda I - T)(S - lim_{m\to\infty} S_m x) = (\lambda I - T)(S - lim_{m\to\infty} y_m)$
We first compute $(\lambda I - T)y_m$ and then employ the hypothesis that T is closed:
$(\lambda I - T)y_m = (\lambda I - T)[\sum_{n=0}^{m} (\lambda_o - \lambda)^n R(\lambda_o; T)^{n+1} x]$
$= \sum_{n=0}^{m} (\lambda_o - \lambda)^n [(\lambda_o I - T) - (\lambda_o - \lambda)I] R(\lambda_o; T)^{n+1} x$
$= \sum_{n=0}^{m} (\lambda_o - \lambda)^n [R(\lambda_o; T)^n x] = \sum_{n=0}^{m} (\lambda_o - \lambda)^{n+1} R(\lambda_o; T)^{n+1} x$
$= x - (\lambda_o - \lambda)^{n+1} R(\lambda_o; T)^{m+1} x \xrightarrow{S} x$ as $m \to \infty$
Since $|\lambda_o - \lambda| < \|R(\lambda_o; T)\|^{-1}$.
Since $y_m \xrightarrow{s} Sx, (\lambda I - T)y \xrightarrow{s} x$ and $\lambda I - T$ (T is closed by hypothesis). We obtain that $Sx \in \mathfrak{D}_{\lambda I - T}$ and $(\lambda I - T)Sx = x$. Then from the relations between $\lambda I - T$ and S. We see $\lambda I - T$ is invertible and $(\lambda I - T)^{-1} = S$. Thus $\lambda \in \rho(T)$ and we can identify S with $R(\lambda; T)$ when $|\lambda_o - \lambda| < \|R(\lambda_o; T)\|^{-1}$. It thus follows that the resolvent set $\rho(T)$ is either ϕ or open, consequently $\sigma(T)$, the spectrum of T, is closed. □

Suppose X is a Banach space and $T \in B(X)$. Then we saw earlier that for $|\lambda| \geq \|T\|, \lambda I - T$ is invertible and $(\lambda I - T)^{-1} = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$. Thus $\{\lambda \in \mathbb{C} : |\lambda| > \|T\|\} \subset \rho(T)$ also for such λ we have
$\|(\lambda I - T)x\| \geq (|\lambda| - \|T\|)\|x\|$;
so we obtain $\|R(\lambda; T)\| \leq (|\lambda| - \|T\|)^{-1}$ for λ satisfying $|\lambda| > \|T\|$. Moreover, $\sigma(T)$ is bounded since $\sigma(T) \subseteq \{\lambda : |\lambda| \leq \|T\|\}$
Thus $\sigma(T)$ is compact for $T \in B(X)$. We summarize all these results in the next proposition 6.

Proposition 6. *Let X be a complete Banach space and $T \in B(X)$. Then for all $\lambda \in \mathbb{C}$ satisfying $|\lambda| > \|T\|$ we have $\lambda \in \rho(T)$ and for these λ we have $R(\lambda; T) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n$, the convergence of the series on the right being in the norm of $B(X)$. Furthermore, $\|R(\lambda; T)\| < (|\lambda| - \|T\|)^{-1}$ for all such λ. Also $\sigma(T)$ is a compact subset of $\mathbb{C}$*

Corollary 1.1. *If $U \in B(H)$ (where H is a complex Hilbert space) is unitary, then $\sigma(U)$ is a closed subset of the unit circle $\{\lambda : |\lambda| = 1\}$ in $\mathbb{C}$.*

Proof. Since U is unitary, $\|U\| = 1$. If λ satisfies $|\lambda| > 1$, then by the above proposition, $\lambda \in \rho(U)$. To consider the case when $|\lambda| < 1$, we observe that $\lambda = 0 \in \rho(U)$ and $\|U^{-1}\| = 1$. Applying Proposition 5 we find that λ satisfying $|\lambda| < 1$ also belong to $\rho(U)$. □

Proposition 7. *Let T be a closed linear operator in a complex Banach space X. Then the resolvent $R(;T) : \rho(T) \to B(X)$ is a continuous function that is, for any $\lambda_o \in \rho(T)$ and any sequence (λ_n) of elements of $\rho(T)$ such that $\lambda_n \to \lambda_o$ as $n \to \infty$, we have*

$\|R(\lambda_n;T) - R(\lambda_o;T)\| \to 0.$

If $\sigma(T)$ is non-void, then for every $\lambda \in \rho(T)$, we have

$\|R(\lambda;T)\| \geq \{dist(\lambda, \sigma(T))\}^{-1}$

For every sequence (λ_n) of elements from $\rho(T)$ such that $\lambda_n \to \lambda$, where $\lambda \in \sigma(T)$, we therefore have

$\|R(\lambda_n;T)\| \to \infty$

Proof. Let $\lambda_o, \lambda \in \rho(T)$ such that $|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1}$. Then by Proposition 5 we have

$\|R(\lambda;T) - R(\lambda_o;T)\| \leq \sum_{n=1}^{\infty} |\lambda_o - \lambda|^n \|R(\lambda_o;T)\|^{n+1} < \frac{|\lambda_o - \lambda| \|R(\lambda_o;T)\|^2}{1 - |\lambda_o - \lambda| \|R(\lambda_o;T)\|}$

since the series involved in the middle is a geometric series of common ratio $|\lambda_o - \lambda| \|R(\lambda_o;T)\| < 1$. Now

$\frac{|\lambda_o - \lambda| \|R(\lambda_o;T)\|^2}{1 - |\lambda_o - \lambda| \|R(\lambda_o;T)\|} = \frac{\|R(\lambda_o;T)\|}{\frac{1}{|\lambda_o - \lambda| \|R(\lambda_o;T)\|} - 1}$

and can be made as close to 0 as we please by choosing a λ sufficiently close to λ_o. Hence the continuity at λ_o follows for any $\lambda_o \in \rho(T)$.

If $\lambda \in \rho(T)$ then by Proposition 5 the point λ' also belongs to $\rho(T)$ if $|\lambda' - \lambda| < \|R(\lambda;T)\|^{-1}$. Consequently, $|\lambda' - \lambda| \geq \|R(\lambda;T)\|^{-1}$ for all $\lambda' \in \sigma(T)$ and thus

$\|R(\lambda;T)\|^{-1} \leq inf\{|\lambda' - \lambda| : \lambda' \in \sigma(T)\} = dist(\lambda, \sigma(T)).$ □

1.2 Analyticity of the resolvent

Definition 2. *Let G be a non void open subset of $\mathbb{C}$, X a complete Banach space and φ a map on G into X. φ is said to be **analytic** on G if $\forall\, \lambda_o \in G$ there is a real $r > 0$ and a sequence $(Z_n)_{n=0}^{\infty}$ of elements of X such that $\forall\, \lambda \in G$ satisfying $|\lambda - \lambda_o| < r$ we have that*

$\varphi(\lambda) = \sum_{n=0}^{\infty} (\lambda - \lambda_o)^n Z_n.$

The convergence of the series on the right hand side should be interpreted in terms of the norm in X.

We say that φ is analytic at infinity if φ is defined in a neighborhood of infinity. That is $\forall\, \lambda$ satisfying $|\lambda| > N$ where N is some positive number and the function ψ defined by

$\psi(\lambda) = \varphi(\frac{1}{\lambda})$ $\forall |\lambda| < \frac{1}{N}$ is analytic in the neighborhood $|\lambda| < \frac{1}{N}$ and then the value of φ at ∞ is taken to be $\psi(0)$.

In most situations of operator theory we replace X by $B(X)$, the Banach space of all bounded linear operators on X.

Another notion of analyticity which may be referred to as **weak analyticity** is used quite often. In this case, we say that $\varphi : G \to B(X)$ is **weakly analytic** if the numerical function $F : G \to \mathbb{C}$ defined by

$F(\lambda) = f(\varphi(\lambda)x)$

is analytic in the usual sense $\forall x \in X$ and $f \in X^*$.

In the case of a Hilbert space H, we may say (using Riesz representation theorem, [2]) that φ is weakly analytic if $\forall\ x, y \in H$, a numerical function $F : G \to \mathbb{C}$ defined by

$F(\lambda) = <\varphi(\lambda)x, y>$

is analytic in the usual sense.

From here now turn to the analyticity property of the resolvent.

Proposition 8. *Let T be a closed linear operator in a complex Banach space X with a nonvoid resolvent set $\rho(T)$. Then for any $x \in X$ and $f \in X^*$, the functions*

(i) $R(.;T) : \rho(T) \to B(X)\ \lambda \mapsto R(\lambda;T)$

(ii) $R(.;T)x : \rho(T) \to X\ \lambda \mapsto R(\lambda;T)x$

(iii) $f(R(.;T)x) : \rho(T) \to \mathbb{C}\ \lambda \mapsto f(R(\lambda;T)x)$

are holomorphic

Proof. Since $\rho(T)$ is open, it suffices to show that each of the functions in $(i), (ii)$ and (iii) has a power series expansion about each point in $\rho(T)$. Let $\lambda_o \in \rho(T)$ and $\lambda \in \mathbb{C}$ such that $|\lambda - \lambda_o| < r = \|R(\lambda_o;T)\|^{-1}$. By proposition 5, we have

$$R(\lambda;T) = \sum_{n=0}^{\infty} (\lambda_o - \lambda)^n R(\lambda_o;T)^{n+1} \tag{1.7}$$

in the sense of norm convergence in $B(X)$; thus (i) is established.

Since uniform convergence implies strong convergence in $B(X)$, we obtain

$R(\lambda;T) = \sum_{n=0}^{\infty} (\lambda_o - \lambda)^n R(\lambda_o;T)^{n+1}x$

in the sense of norm convergence in X.

It is also immediate from 1.7 that for each pair $(x, f) \in X \times X^*$

$f(R(\lambda;T)x) = \sum_{n=0}^{\infty} (\lambda_o - \lambda)^n f(R(\lambda_o;T)^{n+1}x)$

for all λ satisfying $|\lambda - \lambda_o| < r$. Thus $R(.;T)$ is weakly holomorphic on $\rho(T)$ □

Remark 3. *In proposition 8, we do not investigate analyticity at the point at infinity in the extended complex plane.*

In the case of a bounded linear operator we have:

Proposition 9. *Let $T \in B(X)$, where X is a complex Banach space. The resolvent function $R(.;T)$ is analytic at infinity and $R(\infty;T) = 0$, the zero operator.*

Proof. Since $T \in B(X)$, we can select a $\lambda \in \mathbb{C}$ such that $\|\lambda T\| < 1$. Then, $(I - \lambda T)^{-1} \in B(X)$ and has the series expansion

$$(I - \lambda T)^{-1} = 1 + \lambda T + \lambda^2 T^2 + \lambda^3 T^3 + ...$$

where the series on the right side converges in the norm of $B(X)$. If $\lambda \neq 0$ in addition, then

$$\lambda(I - \lambda T)^{-1} = (\frac{1}{\lambda} I - T)^{-1} = R(\frac{1}{\lambda}; T) = \lambda(I + \lambda T + \lambda^2 T^2 + ...)$$

Set

$$\psi(\lambda) = R(\frac{1}{\lambda}; T)$$

Thus for $\lambda \neq 0$ and $|\lambda| < \frac{1}{\|T\|}$, we have

$$\psi(\lambda) = \lambda(I + \lambda T + \lambda^2 T^2 + ...)$$

which is a power series in λ. As the series in the parenthesis converges when λ is small, the factor λ in front of it guarantees that $\psi(0) = 0$, the zero operator. Thus $R(\infty; T) = 0$ □

We are now in a position to show that if $T \in B(X)$, where X is a complex Banach space, then $\sigma(T)$ is always nonvoid. The result is non trivial even in finite dimensional case where the spectrum just consists of eigenvalues. As linear operators in the finite-dimensional case are representable by finite matrices, it is enough to show that every finite matrix has an eigenvalue, which is equivalent to showing that the characteristic polynomial of every finite matrix has at least one zero(that is, root). This requires us to show that every polynomial equation (with complex coefficients)has at least one (complex) root. Thus the finite-dimensional case of the investigation about the nonvoidness of the spectra is as deep as the fundamental theorem of algebra, [3] whose proof is usually based on the theory of complex analytic functions. The theory of such functions enters the study of operators in every case even when the dimension of the space is infinite!.

Proposition 10. *Let $T \in B(X)$, where X is a complex Banach space. Then $\sigma(T)$ is nonvoid.*

Proof. Assume the contrary, that is,$\sigma(T) = \phi$. Then all the points of $\mathbb{C}$ belong to $\rho(T)$. Also the function $F : \mathbb{C} \to \mathbb{C}$ defined by

$$F(\lambda) = f(R(\lambda; T)x)$$

for any fixed $x \in X$ and $f \in X^*$ is holomorphic on $\mathbb{C}$, that is, an entire function. Since $R(\infty; T) = 0$ at infinity(see proposition 9), the function F is bounded in a neighbourhood of ∞ and therefore in the whole plane. Hence, by Liouville's theorem [3], of classical function theory it follows that $F(\lambda)$ is constant, that is $f(R(\lambda; T)x)$ is constant. Since $R(\infty; T) = 0$, it follows that $f(R(\lambda; T)x) = 0$ identically in x, f and λ. Hence, by a consequence of the Hahn-Banach theorem [3], we have $R(\lambda; T)x = 0$ for all $\lambda \in \mathbb{C}$ and all $x \in X$. [If $X = H$, a complex Hilbert space, we would have $< R(\lambda; T)x, y >= 0$ for all $x \in H$ and $\lambda \in \mathbb{C}$. Hence, putting $y = x$, we get $< R(\lambda; T)x, x >= 0$ for all $x \in H$ and for all $\lambda \in \mathbb{C}$. Thus $R(\lambda; T) = 0$ for all $\lambda \in \mathbb{C}$]. This is clearly absurd. Hence the assumption of void spectrum is unacceptable □

Remark 4. *(i) We could also prove as below without explicitly referring to proposition 9. In proposition 6 we showed that if $|\lambda| > \|T\|$, then $\|R(\lambda; T)\| \leq (|\lambda| - \|T\|)^{-1}$.*

This shows that
$lim_{|\lambda|\to\infty}\|R(\lambda;T)\| = 0$
For any fixed $f \in X^*$ *and any* $x \in X$, *we have*
$|F(\lambda)| = |f(R(\lambda;T)x)| \leq \|f\|\|R(\lambda;T)\|\|x\|$;
hence, for each fixed $(x, f) \in X \times X^*$, *we have*

$$lim_{|\lambda|\to\infty} f(R(\lambda;T)x) = 0 \tag{1.8}$$

If $\sigma(T) = \varphi$ *that is,* $\rho(T) = \mathbb{C}$, *then* F *is holomorphic on* $\mathbb{C}$, *that is, is an entire function. Moreover, 1.8, shows that each* F *is bounded.*
Hence(Liouville's Theorem [3]), F *is identically 0 on* $\mathbb{C}$*,etc.*

(ii) *The conclusion in the above proposition 10 is not valid in real Banach spaces. Note that in these instances* $\sigma(T)$ *must be a subset of* $\mathbb{R}$. *Otherwise, we could not meaningfully discuss the transformation* $\lambda I - T$. *As an example, consider the linear operator* T *on the Hilbert space* $\mathbb{R}^2$ *determined by the matrix*

$$M = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{1.9}$$

with respect to the standard base in $\mathbb{R}^2$. *It is easily verified that* $\lambda I - T$ *is not injective if and only if the determinant of the matrix* $\lambda I - M$ *vanishes at* λ *(that is,* λ *is a root of* $\lambda^2 + 1 = 0$*). This shows that* $\sigma(T) = \phi$. *However* $\sigma(T) \neq \phi$ *for any self-adjoint operator in a Hilbert space over* R.

1.3 Conclusion

The uses of analyticity is operator theory were originally apparent in a classical context which required no recognition of analyticity in any sense other than what was entailed in speaking of complex-valued analytic functions of a complex variable. This was so because the differential and integral operators under consideration were acting on numerically valued functions. For $|\lambda - \lambda_o| < \|R(\lambda_o;T)\|^{-1}$ we have shown that all such λ do belong to the resolvent set of T and that $R(\lambda;T)$ can be expressed in a "Taylor series" expansion about λ_o which converges in norm to $R(\lambda;T)$ just like analytic functions of a complex variable. We have thus shown that if T is a closed linear operator in a complex Banach space X with a non-void resolvent set $\rho(T)$ then $R(\lambda;T)$ is analytic. Moreover if T is a bounded linear operator in X then $R(\lambda;T)$ is analytic at ∞ and $R(\infty;T) = 0$ (where 0 is the zero operator in X). Consequently we have shown that if T is a bounded operator in X then the spectrum of T is non-void.

References

[1] Conway, J. B. (2000). *A course in operator theory.* American Mathematical Soc..

[2] Mangatiana A. Robdera (2015): *On the Riesz Representation Theorem and Integral Operators*, Quaestiones Mathematicae, DOI: 10.2989/16073606.2015.1091045

[3] Simon, B. (2015). *Operator theory* (Vol. 4). American Mathematical Soc..

[4] Panigrahi, B. L., Srivastava, P. D. (2012). *Spectrum and fine spectrum of generalized second order forward difference operator* Δ^2_{uvw} *on sequence space* l_1. Demonstratio Mathematica, 45(3), 593-609.

[5] Srivastava, P. D., Kumar, S. (2012). *Fine Spectrum of the Generalized Difference Operator* Δ_v *on Sequence Space* l_1. Thai Journal of Mathematics, 8(2), 221-233.

[6] Zhang, Z. H. E. N. G. H. E. (2013). *Resolvent set of Schrödinger operators and uniform hyperbolicity.* arXiv preprint arXiv:1305.4226.

[7] Abdmouleh, F., Charfi, S., Jeribi, A. (2012). *On a characterization of the essential spectra of the sum and the product of two operators.* Journal of Mathematical Analysis and Applications, 386(1), 83-90.

[8] Chen, C., Li, M. (2010),*On fractional resolvent operator functions.* In Semigroup Forum (Vol. 80, No. 1, pp. 121-142). Springer-Verlag.

[9] Taylor, A. (1971). *Notes on the History of the Uses of Analyticity in Operator Theory.* The American Mathematical Monthly, 78(4), 331-342. doi:10.2307/2316894

[10] Mukhtarov, O. S., Olğar, H., Aydemir, K. (2015), *Resolvent operator and spectrum of new type boundary value problems.* Filomat, 29(7), 1671-1680.