On a Class of Curvature Properties of Projectively flat Finsler (α,β)-metric

Abstract: In this paper, we study a class of Finsler metric in the form $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric, $\beta = b_i y^i$ is a 1−form. We obtain a necessary and sufficient condition for F to be locally projectively flat. Further, we prove that such projectively flat Finsler metrics with the constant flag curvature must be locally Minkowskian.

Key Words: Finsler (α, β)−metrics, flag curvature, Projectively flat, Riemannian metric.

AMS Subject Classification (2010): 53C60, 53B40

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1. Introduction

In Finsler geometry, one of the fundamental problem is to study and characterize the projectively flat metrics on an open domain $U \subset R^n$. The Beltrami theorem tells us that a Riemannian metric is locally projectively flat if and only if it is of constant sectional curvature. Projectively flat metrics on U are Finsler metrics whose geodesics are straight lines. However, there are locally projectively flat Finsler metrics which are not of constant flag curvature and there are Finsler metrics of constant flag curvature which are not locally projectively flat.

It is well-known that every locally projectively flat Finsler metric is of scalar flag curvature. It is known that a Randers metric is projectively flat if and only if α is projectively flat and β is closed[1].

The main purpose of this paper is to study locally projectively flat Finsler metrics in the form $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$, with constant flag curvature.

2. PRELIMINARIES

Let M be an n-dimensional smooth manifold. We denote by TM the tangent bundle of M and by $(x, y) = (x^i, y^i)$ the local coordinates on the tangent bundle TM. A Finsler manifold (M, F) is a smooth manifold equipped with a function $F : TM \rightarrow [0,\infty)$, which has the following properties:

• Regularity: F is smooth in $TM\backslash\{0\}$;

• Positively homogeneity: $F(x, \lambda y) = \lambda F(x, y)$, for $\lambda > 0$;

• Strong convexity: the Hessian matrix of F^2, $g_{ij}(x, y) = \frac{1}{2}\left(\frac{\partial^2 F^2(x,y)}{\partial y^i \partial y^j}\right)$, is positive definite on $TM\backslash\{0\}$. We call F and the tensor g_{ij} the Finsler metric and fundamental tensor of M respectively.

For a Finser metric $F = F(x, y)$, its geodesic curves are characterized by the system of differential equations $\ddot{c}^i + 2G^i(\dot{c}) = 0$, where the local functions $G^i = G^i(x, y)$ are called the spray coefficients and given by following

$$G^i = \frac{1}{4} g^{il}\left\{\frac{\partial^2 [F^2]}{\partial x^k \partial y^l} y^k - \frac{\partial [F^2]}{\partial x^l}\right\}, \quad y \in T_x M.$$

A Finsler metric $F = F(x, y)$ on an open domain $U \subset R^n$ is said to be projectively flat, if all geodesics are straight lines. This is equivalent to $G^j = P(x, y) y^i$, where $G^i = G^i(x, y)$ are the geodesic coefficients of F, which are given by

$$G_i = \frac{1}{4} g^{il}\left\{\left[F^i_{x^m y^l} y^m - [F^2]_{x^l}\right]\right\}.$$

Lemma 2.1. [8] The geodesic coefficients G^i are related to G^i_α by

$$G_i = G^i_\alpha + \Theta\{-2Q\alpha s_0\} + J\{-2Q\alpha s_0 + r_{00}\}\frac{y^i}{\alpha} + H\{-2Q\alpha s_0 - r_{00}\}\{b^i - s\frac{y^i}{\alpha}\}, \tag{2.1}$$

where G^i_α denote the spray coefficients of α and

$$Q = \frac{\emptyset'}{\emptyset - s\emptyset'}, \quad J = \frac{\emptyset'(\emptyset - s\emptyset')}{2\emptyset((\emptyset - s\emptyset' + (b^2 - s^2)\emptyset'')}, \quad H = \frac{\emptyset''}{2\emptyset((\emptyset - s\emptyset' + (b^2 - s^2)\emptyset'')} \tag{2.2}$$

Where $s = \frac{\beta}{\alpha}, b = \|\beta_x\|_\alpha$.

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Lemma 2.2. [8] An (α,β) −metric $F = \alpha\emptyset(\frac{\beta}{\alpha})$, where $s = \frac{\beta}{\alpha}$, is projectively flat on an open subset $U \subset R^n$ if and only if

$$(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0, \quad (2.3)$$

where $y_l = a_{lj}y^j$.

For an (α,β) −metric $F = \alpha\emptyset(s)$ where $s = \frac{\beta}{\alpha}$ and $\emptyset$ =$\emptyset(s)$ is a C^∞ function on the $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is 1-form on M.

Let us define $b_{i/j}$ by

$$b_{i/j}\theta^j = db_i - b_j\theta_i^j,$$

where $\theta^i = dx^i$ and $\theta_i^j = \Gamma_{ik}^j dx^k$ denote the Levi-Civita connection form of α.

Let

$$r_{ij} = \frac{1}{2}(b_{i/j} + b_{j/i}),$$

$$s_{ij} = \frac{1}{2}(b_{i/j} - b_{j/i}).$$

Clearly, β is closed if and only if $s_{ij} = 0$. (α,β) −metric is said to be trivial if $r_{ij} = s_{ij} = 0$. put

$$r_{i0} = r_{ij}y^j, \qquad r_{00} = r_{ij}y^j y^j, \qquad r_j = b^i r_{ij},$$

$$s_{io} = s_{ij}y^j, \quad s_j = b^i s_{ij},$$

$$r_0 = r_j y^j, \; s_0 = s_j y^j.$$

3. PROJECTIVELY FLAT FINSLER (α,β) – METRIC WITH CONSTANT FLAG CURVATURE

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For an (α,β) −metric $F=\alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$, $\emptyset(s) = 1 + s + 2s^2 - \frac{s^4}{3}$, $s = \beta/\alpha$.

From the (α,β) −metric,

$$(s^2 - 1)(5s^2 - 1 - 4b^2) > 0.$$

By Lemma (2.1), we can compute

$$Q = \frac{3 + 12s - 4s^3}{3(1-2s^2 + s^4)},$$

$$H = \frac{2}{1 + 4b^2 - 5s^2}, \quad J = \frac{(1-s^2)(3 + 12s - 4s^3)}{2(1+4b^2-5s^2)(3+3s+6s^2-s^4)}.$$

By Lemma (2.2), we have the following equation

$$0 = (a_{ml}\,\alpha^2 - y_m y_l)G_\alpha^m + \frac{(3 + 12s - 4s^3)}{3(1-s^2)^2}\alpha^3 s_{l0} + \frac{2\alpha}{(1 + 4b^2 - 5s^2)}\{-2\alpha\frac{(3+ 12s-4s^3)}{3(1-s^2)^2}s_0 + r_{00}\}(b_l\alpha - sy_l). \quad (3.1)$$

Theorem 3.1. *Let* $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ *be a Finsler metric on a manifold* M. F *is projectively flat if and only if*

(1) $G_\alpha^i = \eta y^i - 2\tau\alpha^2 b^i$,

(2) $b_{i/j} = \tau[(1 + 4b^2)a_{ij} - 5b_i b_j]$,

where $\tau = \tau(x)$, $\eta = \eta_i(x)y^i$ *and* $b_{i/j}$ *denotes the coefficients of the covariant derivative of* β *with respect to* α. *In this case,*

$$G^i = (\eta + 2\tau\alpha\chi)y^i, \quad (3.2)$$

where

$$\chi = \frac{(1-s^2)(3 + 12s-4s^2)}{4(3 + 3s + 6s^2 - s^4)} - s, \quad s = \beta/\alpha. \quad (3.3)$$

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Proof: First, we rewrite (3.1) as a polynomial in y^i and α, which is linear in α. This gives

$$0 = 3(\alpha^2 - \beta^2)[(4b^2 + 1) - 5\beta^2](a_{ml}\alpha^2 - y_m y_l)G^m_\alpha$$

$$= \alpha^4(3\alpha^3 + 12\beta\alpha^2 - 4\beta^3)s_{l0}[(1 + 4b^2)\alpha^2 - 5\beta^2]$$

$$= 2\alpha^2[-2\alpha^2(3\alpha^3 + 12\beta\alpha^2 - 4\beta^3)s_0 + 3r_{00}(\alpha^2 - \beta^2)^2]$$

$$* (b_l\alpha^2 - \beta y_l). \quad (3.4)$$

The coefficients of α must be zero (note that α^{even} is a polynomial in y^i). We obtain

$$3\alpha^7 s_{l0}[(1 + 4b^2)\alpha^2 - 5\beta^2] = 12\alpha^7 s_0(b_l\alpha^2 - \beta y_l).$$

Then

$$s_{l0}[(1 + 4b^2)\alpha^2 - 5\beta^2] = 4s_0(b_l\alpha^2 - \beta y_l). \quad (3.5)$$

Contracting (3.5) with b^l yields

$$(\alpha^2 - \beta^2)s_0 = 0. \quad (3.6)$$

For $\alpha^2 - \beta^2 \neq 0$, we have

$$s_0 = 0.$$

Then it follows from (3.5) that

$$s_{l0} = 0. \quad (3.7)$$

This implies that β is closed.

Now, equation (3.4) is reduced to

$$0 = 3(\alpha^2 - \beta^2)^2[(1 + 4b^2)\alpha^2 - 5\beta^2](a_{ml}\alpha^2 - y_m y_l)G^m_\alpha$$
$$+ 4\alpha^4(3\beta\alpha^2 - \beta^3)s_{l0}[(1 + 4b^2)\alpha^2 - 5\beta^2]$$
$$+ 2\alpha^2[-8\alpha^2(3\beta\alpha^2 - \beta^3)s_0 + 3r_{00}(\alpha^2 - \beta^2)^2](b_l\alpha^2 - \beta y_l).$$

We can rewrite the above identity as follows:

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$$3(\alpha^2-\beta^2)^2\begin{Bmatrix}[(1+4b^2)\alpha^2-5\beta^2](a_{ml}\alpha^2-y_my_l)G^m_\alpha\\ +2r_{00}\alpha^2(b_l\alpha^2-y_l\beta)\end{Bmatrix}$$

$$=-4\alpha^4\beta(3\alpha^2-\beta^2)\begin{Bmatrix}s_{l0}[(1+4b^2)\alpha^2-5\beta^2]\\ -4s_0(b_l\alpha^2-\beta y_l)\end{Bmatrix}$$

Since $\alpha^4\beta(3\alpha^2-\beta^2)$ is not divisible by $(\alpha^2-\beta^2)^2$, then $\{s_{l0}[(1+4b^2)\alpha^2-5\beta^2]-4s_0(b_l\alpha^2-\beta y_l)\}$ must be divisible by $(\alpha^2-\beta^2)^2$.

This is impossible unless $s_{l0}[(1+4b^2)\alpha^2-5\beta^2]-4s_0(b_l\alpha^2-\beta y_l)=0$.

Then we conclude that β is closed.

Since $s_{ij}=0$, equation (3.4) is reduced to

$$[(1+4b^2)\alpha^2-5\beta^2](a_{ml}\alpha^2-y_my_l)G^m_\alpha+2r_{00}\alpha^2(b_l\alpha^2-y_l\beta)=0. \tag{3.8}$$

Contracting (3.8) with b^l, we get

$$[(1+4b^2)\alpha^2-5\beta^2](a_{ml}\alpha^2-y_m\beta)G^m_\alpha=-2r_{00}\alpha^2(b^2\alpha^2-\beta^2). \tag{3.9}$$

Note that the polynomial$(1+4b^2)\alpha^2-5\beta^2$ is not divisible by α^2 and $b^2\alpha^2-\beta^2$.

Thus $(b_m\alpha^2-y_m\beta)G^m_\alpha$ is divisible by $\alpha^2(b^2\alpha^2-\beta^2)$.

Therefore, there is a scalar function $\tau=\tau(x)$ such that

$$r_{00}=\tau[(1+4b^2)\alpha^2-5\beta^2]. \tag{3.10}$$

By (3.9) and (3.10), the formula for G^i can be simplified to

$$G^i=G^i_\alpha+2\tau\chi\alpha y^i+2\tau\alpha^2b^i, \tag{3.11}$$

where χ is given in (3.3).

Now, we compute G^i_α. Plugging (3.10) into (3.8) and contracting with b^l yield

$$(b_m\alpha^2-\beta y_m)G^m_\alpha+2\tau\alpha^2(b^2\alpha^2-\beta^2)=0. \tag{3.12}$$

Rewriting above equation

$$\alpha^2(G^m_\alpha b_m+2\tau b^2\alpha^2)=\beta(y_mG^m_\alpha+2\tau\alpha^2\beta). \tag{3.13}$$

Because α^2 is not divisible by β, there exists a 1−form $\eta=\eta_iy^i$ such that

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$$G_{\alpha}^{m} b_m + 2\tau b^2 \alpha^2 = \eta\beta, \tag{3.14}$$

$$G_{\alpha}^{m} y_m + 2\tau \alpha^2 \beta = \eta \alpha^2. \tag{3.15}$$

From anyone of the above two equations, we have

$$G_{\alpha}^{i} = \eta y^i - 2\tau \alpha^2 b^i. \tag{3.16}$$

This proves (1). From $d\beta = 0$ and $r_{00} = \tau[(1 + 4b^2)\alpha^2 - 5\beta^2]$, we get (2).

In this case, $$G^i = (\eta + 2\tau\alpha\chi) y^i.$$

Hence, we proved the theorem.

Lemma 3.3. *If* $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ *is projectively flat with the constant flag curvature* $K = \lambda$ *(constant), then* $\lambda = 0$.

Proof: First, we assume that F is locally projectively flat so that in a local coordinate system the spray coefficients of F are in the form (3.2).

It is known that if the spray coefficients of F are in the form $G^i = P y^i$, then F is of scalar

curvature with the flag curvature

$$K = \frac{(P^2 - P_{x^k} y^k)}{F^2}$$

Then

$$K = \frac{(\eta + 2\tau\chi\alpha)^2 - \eta_{x^k} y^k - 2\tau_{x^k} y^k \alpha\chi - 2\tau\chi'(s) s_{x^k} y^k \alpha - 2\tau\chi\alpha_{x^k} y^k}{F^2}. \tag{3.17}$$

Observe that

$$s_{x^k} y^k = \frac{r_{00}}{\alpha} + \frac{2}{\alpha^2}\{b_m \alpha - s y_m\} G_{\alpha}^{m} = \tau\alpha(1 - s^2),$$

$$\alpha_{x^k} y^k = \frac{2}{\alpha} G_{\alpha}^{m} y_m = 2(\eta - 2\tau\beta)\alpha.$$

We obtain

$$K = \frac{\eta^2 - \eta_{x^k} y^k + 4\tau^2\chi^2\alpha^2 - 2\tau_{x^k} y^k \chi\alpha - 2\tau^2\alpha^2\chi'(1 - s^2) + 8\tau^2 s\chi\alpha^2}{F^2}. \tag{3.18}$$

By (3.18), the equation $K = \lambda$ multiplied by $8\alpha^{12}F^4$, we can get

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$$A\alpha^5 + B\alpha^4 + 12\lambda\beta^{13}\alpha^3 + 24\lambda\alpha^2\beta^{14} - \lambda\beta^{16} = 0,$$

where A and B are the homogeneous polynomials in y of degree 11 and 12 respectively.

Rewriting the above equation as

$$(A\alpha^2 + 12\lambda\beta^{13})\alpha^3 + (B\alpha^4 + 24\lambda\beta^{14}\alpha^2 - \lambda\beta^{16}) = 0.$$

We must have

$$A\alpha^2 + 12\lambda\beta^{13} = 0, \quad (B\alpha^2 + 24\beta^{14}\lambda)\alpha^2 = \lambda\beta^{16}. \tag{3.19}$$

Since β^2 is not divisible by α, we conclude from the second identity in (3.19) that $\lambda = 0$.

Now, we consider the trivial case when $\tau = 0$ in (2). In this case

$$b_{i/j} = 0, \qquad G^i = G^i_\alpha = \eta y^i.$$

By Lemma 3.3, F has the zero flag curvature, thus α has the zero flag curvature, i.e., α is locally isometric to the Euclidean metric. Hence, we have proved the following

Proposition 1. Let $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ be a Finsler metric on a manifold M. Suppose that F is a locally projectively flat metric with the zero flag curvature. If $\tau = 0$, then α is a flat metric and β is parallel with respect to α. In this case, F is locally Minkowskian.

By using Lemma 3.3 and Proposition 1 we prove the following theorem.

Theorem 3.2. *Let $F = \alpha + \beta + \frac{2\beta^2}{\alpha} - \frac{\beta^4}{3\alpha^3}$ be a Finsler metric on a manifold M. Suppose that F is a locally projectively flat metric with the constant flag curvature. Then α is a flat metric and β is parallel with respect to α. In this case, F is locally Minkowskian.*

Proof: We only need to show that $\tau = 0$. Under the assumption that $K = 0$, we obtain

$$\Phi\alpha^3 + \Psi = 0, \tag{3.20}$$

where

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$$\Phi = (324\tau^2 \beta - 18\tau_0)\alpha^6 + (72(\eta^2 - \eta_0)\beta - 252\tau^2\beta^5 + 54\tau_0\beta^2)\alpha^4$$
$$+ (144(\eta^2 - \eta_0)\beta^3 - 1044\tau^2\beta^5 + 426\,\tau_0\beta^4)\alpha^2$$
$$+ (-24(\eta^2 - \eta_0) + 204\tau^2\beta^2 - 78\tau_0\beta)\beta^5,$$

$$\Psi = 27\tau^2\alpha^{10} + (36(\eta^2 - \eta_0) + 774\,\tau^2\beta^2 - 18\,\tau_0\beta)\alpha^8$$
$$+ \left((180(\eta^2 - \eta_0)\beta^2 - (1665)\beta^4\tau^2) + 330\,\tau_0\beta^3\right)\alpha^6$$
$$+ (120(\eta^2 - \eta_0)\beta^4 - 576\,\tau^2\beta^6 + 432\,\tau_0\beta^5)\alpha^4$$
$$+ (-48(\eta^2 - \eta_0)\beta^6 + 288\,\tau^2\beta^8 - 176\,\tau_0\beta^7)\alpha^2$$
$$+ (4(\eta^2 - \eta_0 - 16\,\tau^2\beta^2 + 16\tau_0\beta)\beta^8,$$

Where $\eta_0 = \eta_{x^k}y^k,\ \tau_0 = \tau_{x^k}y^k$. Note that Ψ and Φ are the homogeneous polynomials in y and $\alpha = \sqrt{a_{ij}y^iy^j}$ is in a radical form. Equation (3.20) implies that

$$\Psi = 0, \quad \Phi = 0. \tag{3.21}$$

First we start from $\Phi = 0$, that is

$$\{(54\tau^2\beta - 3\tau_0)\alpha^4 + [12(\eta^2 - \eta_0)\beta - 42\tau^2\beta^3 + 9\tau_0\beta^2]\alpha^2 + [24(\eta^2 - \eta_0)\beta^3 - 174\tau^2\beta^5 + 71\tau_0\beta^4]\}\alpha^2 = -[4(\eta^2 - \eta_0) + 34\,\tau^2\beta^2 - 13\tau_0\beta]\beta^5. \tag{3.22}$$

Since β^2 is not divisible by α, there exist scalar functions $d_i = d_i(x),\ \ i = 1,2,3$, such that

$$4(\eta^2 - \eta_0) + 34\tau^2\beta^2 - 13\tau_0\beta = d_1\alpha^2, \tag{3.23}$$

$$d_1\beta^2 + 24(\eta^2 - \eta_0) - 174\tau^2\beta^2 + 71\tau_0\beta = d_2\alpha^2, \tag{3.24}$$

$$d_2\beta^2 + 12(\eta^2 - \eta_0) - 42\tau^2\beta^2 + 9\tau_0\beta = d_3\alpha^2. \tag{3.25}$$

Then (3.22) becomes

$$54\tau^2\beta - 3\tau_0 = -\beta d_3. \tag{3.26}$$

From the above equation, we can see that τ_0 must contain the factor β. Hence there is a scalar function $d_4 = d_4(x)$ such that

$$\tau_0 = d_4\beta. \tag{3.27}$$

Plugging this into (3.26), we have

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$$\tau^2 = \frac{3d_4 - d_3}{54}. \tag{3.28}$$

Substituting (3.27) and (3.28) into (3.25) and (3.26), we have

$$d_1\beta^2 + 24(\eta^2 - \eta_0) - \frac{29}{9}\left(3d_4 - d_3\right)\beta^2 + 71d_4\beta^2 = d_2\alpha^2, \tag{3.29}$$

$$d_2\beta^2 + 12(\eta^2 - \eta_0) - \frac{7}{9}\left(3d_4 - d_3\right)\beta^2 + 9\, d_4\beta^2 = d_3\alpha^2. \tag{3.30}$$

Equation (3.30) multiplied by -2 and added to (3.29) yields

$$\left(d_1 - 2d_2 + \frac{5}{3}d_3 + 48d_4\right)\beta^4 = (d_2 - 2d_3)\alpha^2. \tag{3.31}$$

Since α^2 is not divisible by β, we have

$$d_2 = 2d_3, \qquad d_1 = \frac{7}{3}d_3 - 48d_4. \tag{3.32}$$

Inserting (3.27), (3.28) and (3.32) into (3.30), we can get

$$(\eta^2 - \eta_0) = \frac{1}{2}\{d_3\alpha^2 - \frac{25}{9}d_3\beta^2 - \frac{20}{3}d_4\beta^2\}. \tag{3.33}$$

Plugging (3.27), (3.28), (3.32) and (3.33) into (3.22), we obtain

$$(\frac{8}{3}d_3 - 48d_4)\alpha^2 = (\frac{8}{27}d_3 - \frac{80}{9})\beta^2 \tag{3.34}$$

Since β^2 is not divisible by α^2, we have

$$\frac{8}{3}d_3 - 48d_4 = 0,$$

$$\frac{8}{27}d_3 - \frac{80}{9} = 0.$$

Solving the above two equations, we get

$$d_3 = d_4 = 0,$$

so that

$$\tau^2 = \frac{3d_4 - d_3}{54} = 0.$$

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That is $\tau = 0$.

Now we need to consider $\Psi = 0$. We have

$$0 = \left(36\left(\eta^2 - \eta_0\right) + 774\tau^2\beta^2\right)\alpha^8 + \left(\left(180\left(\eta^2 - \eta_0\right)\beta^2 - (1665)\beta^4\tau^2\right) + 330\tau_0\beta^3\right)\alpha^6 + \left(120\left(\eta^2 - \eta_0\right)\beta^4 - 576\tau^2\beta^6 + 432\tau_0\beta^5\right)\alpha^4 + \left(-48\left(\eta^2 - \eta_0\right)\beta^6 + 288\tau^2\beta^8 - 176\tau_0\beta^7\right)\alpha^2 + \left(4\left(\eta^2 - \eta_0\right) - 16\tau^2\beta^2 + 16\tau_0\beta\right)\beta^8 \tag{3.35}$$

Since β^2 is not divisible by α, there exist scalar functions $c_i = c_i(x)$, $i = 1,2,3,4$ such that

$$4\left(\eta^2 - \eta_0\right) - 16\tau^2\beta^2 + 16\tau_0\beta = c_1\alpha^2, \tag{3.36}$$

$$c_1\beta^2 - 48(\eta^2 - \eta_0) + 288\tau^2\beta^2 - 176\tau_0\beta = c_2\alpha^2, \tag{3.37}$$

$$c_2\beta^2 + 120(\eta^2 - \eta_0) - 576\tau^2\beta^2 + 432\tau_0\beta = c_3\alpha^2, \tag{3.38}$$

$$c_3\beta^2 + 180\left(\eta^2 - \eta_0\right) - 1665\tau^2\beta^2 + 330\tau_0\beta = c_4\alpha^2. \tag{3.39}$$

Then (3.35) becomes

$$36(\eta^2 - \eta_0) + 774\tau^2\beta^2 = -c_4\beta^2. \tag{3.40}$$

From the above equation, we know that $\eta^2 - \eta_0$ must contain the factor β^2, so there is a scalar function $c_5 = c_5(x)$ such that

$$\eta^2 - \eta_0 = c_5\beta^2. \tag{3.41}$$

Then , from (3.40) we can get

$$\tau^2 = -\frac{36c_5 + c_4}{774}. \tag{3.42}$$

Plugging (3.41) and (3.42) into (3.26), we obtain

$$(c_3\beta + 216c_5\beta + 2c_4\beta + 330\tau_0)\beta = c_4\alpha^2. \tag{3.43}$$

Since β^2 is not divisible by α^2, we can get

$$C_4 = 0, \quad \tau_0 = -\frac{216c_5 + c_3}{330}\beta, \quad \tau^2 = -\frac{c_5}{20}. \tag{3.44}$$

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Substituting (3.41) and (3.44) into (3.25), we obtain

$$\left(c_2 - 240c_5 - \frac{9}{5}c_3\right)\beta^2 = c_3\alpha^2. \quad (3.45)$$

From the above equation, we conclude

$$c_3 = 0, \quad c_2 = 240c_5. \quad (3.46)$$

Plugging (3.41), (3.44) into (3.24), we can get

$$(c_1 + 96c_5)\beta^2 = 240c_5\alpha^2. \quad (3.47)$$

Then we have

$$c_5 = 0, \quad c_1 = 0. \quad (3.48)$$

Inserting above into the third expression of (3.44), we have $\tau^2 = 0$, so that $\tau = 0$. Hence theorem is proved completely.

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