Original Research Article

QUADRATIC POLYNOMIAL VELOCITY PROFILE IN LAMINAR BOUNDARY LAYER THICKNESS

Abstract

Motion of fluid elements can be described by the Navier-Stokes equations. They arise from the use of Law of motion to a fluid. In this investigation, two dimensional Navier-Stokes have been engaged. Then they are applied to an incompressible viscous fluid movement down an inclined plane with net flow. These leads to examining the effects to the velocity of the motion at various angles of inclination and finding the boundary layer thickness. Viscous laminar incompressible fluid flow also flow on an inclined position which makes it necessary to investigate the flow on an inclined plane. Results that have been achieved are of the flow over horizontal flat plate. Solution that has been obtained involves a flat photographic film being pulled up by a processing bath by rollers at an angle θ to the horizontal. Quadratic polynomial function approximate velocity profile has been obtained under initial boundary layer conditions. This velocity profile has been used in momentum integral equation for flow over an inclined plane to get the boundary layer thickness. Boundary layer thickness is one of the parameters that is used to obtain the flow velocity down inclined plane.

Keywords: Quadratic polynomial function; Boundary layer thickness; viscous fluid; Velocity profile; incompressible flow

1 Introduction

Navier-Stokes equations were originally derived in the 1840s (5) on the basis of conservation laws and first-order approximations. But if one assumed sufficient randomness in microscopic molecular processes they could also be derived from molecular dynamics, as done in the early 1900s(5). The aerodynamic boundary layer was first defined by Ludwig Prandtl (6) in a paper presented on August 12, 1904 at the third International Congress of Mathematicians in Heidelberg, Germany. It allows aerodynamicists to simplify the equations of fluid flow by dividing the flow field into two areas: one inside the boundary layer, where viscosity is dominant and the majority of the drag experienced by a body immersed in a fluid is created, and one outside the boundary layer where viscosity can be neglected without significant effects on the solution. This allows a closed-form solution for the flow in both areas, which is a significant simplification over the solution of the full Navier-Stokes equations. The majority of the heat transfer to and from a body also takes place within the boundary layer, again allowing the equations to be simplified in the flow field outside the boundary layer. Mei (2)wrote in his lecture notes on fluid dynamics that with a general pressure gradient, the boundary layer equations can be solved by a variety of modern numerical means like finite element method. Sonin(3)derived

Navier-Stokes Equations of horizontal flows but could not derive flows on an inclined position where it is influenced by force of gravity. According to a coating experiment (4) involving a flat photographic film, being pulled up from a processing bath by rollers with a steady velocity U_∞ at an angle θ to the horizontal. As the film leaves the bath, it entrains some liquid. We find that velocity at different angles are not given in order to find the thickness of the liquid that is required. However, the thickness is determined by a steady velocity moving up without net flows and not downwards with net flows. The boundary layer thickness approximation of a flow down an inclined plate with net flow using the momentum integral approach requires more investigation. This is done in this paper.

2 Mathematical Formulation

2.1 Flow down an inclined plane

Consider the motion of fluid which is caused by a component of a gravity force parallel to the inclined non-porous solid plane surface flowing downwards, where the following assumptions are made; the fluid is Newtonian, laminar, incompressible and unidirectional. We will have to consider continuity equation and momentum equation which describes the steady flow in the form

$$\nabla.(\rho \mathbf{v}) = 0 \tag{2.1}$$

$$\frac{D\mathbf{v}}{Dt} = \mathbf{g} - \frac{1}{\rho}\nabla P + \nu \nabla^2 \mathbf{v} \tag{2.2}$$

where $\mathbf{v}$ is fluid velocity with components $[u, v, w]$ in the $x-$, $y-$ and $z-$ directions respectively. ρ is fluid density, P is the pressure, $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity, $\mathbf{g}$ is force due to gravity and $\frac{D\mathbf{v}}{Dt}$ defines the material derivative operator. Since the flow is in one direction, that is $x-$direction (we orient our coordinate system such that that the $x-$axis is parallel to the inclined plane and the $y-$axis is transverse to this plane), then $v = w = 0$ and equations [(2.1)-(2.2)] reduce to

$$\frac{\partial u}{\partial x} = 0 \tag{2.3}$$

$$g_x - \frac{1}{\rho}\frac{\partial P}{\partial x} + \frac{\mu}{\rho}\frac{\partial^2 u}{\partial y^2} = 0 \tag{2.4}$$

$$g_y - \frac{1}{\rho}\frac{\partial P}{\partial y} = 0 \tag{2.5}$$

$$g_z - \frac{1}{\rho}\frac{\partial P}{\partial z} = 0 \tag{2.6}$$

in which $[g_x, g_y, g_z]$ are components of gravity in the $x-$, $y-$ and $z-$ directions respectively. since $g_y = g_z = 0$ and assume that the flow is entirely driven by the gravity alone, we have

$$0 = 0 + \mathbf{g}\sin\theta + \nu\frac{\partial^2 u}{\partial y^2} \tag{2.7}$$

The above equation models the following fluid: Considering a viscous liquid flowing while in contact with only one solid surface, the fluid motion is being caused by a component of the gravity force parallel to the solid surface. A plane surface inclined above the horizontal by an angle θ and covered with a liquid layer of constant thickness δ that flows parallel to the plane in the downhill direction. The upper surface of the fluid ($y = \delta$) is in contact with the air, in which the pressure is constant (P=P_a at $y = \delta$, where P_a is the atmospheric pressure) and which exerts a negligible shear stress on the liquid surface ($\tau_{xy} = 0$ at $y = \delta$). On re-arranging the model equation and integrating twice with respect to y we find

$$u = -\frac{\mathbf{g}(\sin\theta)y^2}{2\nu} + c_1 y + c_2 \tag{2.8}$$

where c_1 and c_2 are constants of integration. Applying the boundary conditions that u=0 at y=0, and $\tau_{xy} = \mu \frac{\partial u}{\partial y} = 0$ at $y = \delta$, we obtain the velocity distribution $u(y)$ as

$$u = \frac{\mathbf{g} \sin\theta}{\nu} \left(\delta y - \frac{y^2}{2} \right) \tag{2.9}$$

2.2 Momentum integral equation

The method of momentum integral equation is employed in determining the boundary layer thickness, δ. Momentum integral equation given by(2):

$$\frac{d}{dx} \int_0^\delta \left(uU_\infty - u^2 \right) dy + \frac{dU_\infty}{dx} \int_0^\delta \left(U_\infty - u \right) dy = \frac{\tau_w}{\rho} \tag{2.10}$$

where U_∞ is the uniform velocity outside the boundary layer and τ_w is the wall shear stress. In the present form of equation(2.10), can represent both laminar and turbulent flows, since no assumption has yet been made for the shear stress, τ_w.

3 Method of Solution

3.1 Solution of the momentum integral equation.

The steps involved in solving equation (2.10) are:

1. choosing a velocity profile that satisfies all the essential and some additional boundary conditions,
2. evaluating the integrals and reducing the left hand side to a differential expression on δ,
3. postulating the law of shear stress for τ_w, depending on the flow regime, for laminar flow $\tau_w = \mu \left(\frac{\partial u}{\partial y} \right)_{y=0}$ by Newton's law of shear stress and
4. solving the differential equation for δ.

3.2 Estimate of Boundary Layer Velocity Profile

The essential conditions to be satisfied by the boundary layer velocity profile are: :

1. $y = 0, u = 0, v = 0$ no slip on the wall.
2. $y = \delta, u = U_\infty$ free stream velocity at the edge of boundary layer.
3. $y = \delta, \mu \frac{\partial u}{\partial y} = 0$ no shear stress at the edge of the boundary layer.

It is then proposed that the boundary layer velocity profile can be written in terms of a quadratic polynomial in y

$$u = A + By + Cy^2 \tag{3.1}$$

Where A, B, and C are real constants. We are to obtain the velocity profile approximation u which is required to find the boundary layer thickness δ of flow with net flow, then use it to find the effect of velocity by varying angles of inclination between $0 < \theta < \frac{\pi}{2}$.

Differentiating equation (3.1), partially with respect to y,

$$\frac{\partial u}{\partial y} = B + 2Cy \tag{3.2}$$

Using the conditions 1, 2 and 3, we obtain velocity profile as;

$$\frac{u}{U_\infty} = 2\frac{y}{\delta} - \left(\frac{y}{\delta}\right)^2 \tag{3.3}$$

3.3 Boundary layer thickness

We now find the expressions to approximate the values of boundary layer.
Since U_∞ is not varying, it is independent of x, then $\frac{dU_\infty}{dx} = 0$.
Equation (2.10) can be reduced to

$$\frac{d}{dx}\int_0^\delta \frac{u}{U_\infty}\left(1 - \frac{u}{U_\infty}\right)dy = \frac{\tau_w}{U_\infty^2 \rho} \tag{3.4}$$

Using the velocity profile u in (3.3) where $Re = \frac{U_\infty x}{\nu}$, we obtain;

$$\delta = \left(\frac{30}{Re}\right)^{\frac{1}{2}} x \tag{3.5}$$

3.4 Calculation of velocity along Inclined Plane

The expressions for the velocity down inclined plane at various angles is obtained by substituting the expressions of δ obtained in section 2.4 above into equation (2.9).

The velocity u down inclined plane can be approximated by substituting equation (3.5) into equation (2.9) to give,

$$u = \frac{g\sin\theta}{\nu}\left[\left(\frac{30}{Re}\right)^{\frac{1}{2}} xy - \frac{y^2}{2}\right] \tag{3.6}$$

4 Results

Taking water as an illustration at temperature $5^0 C$, then $\nu = 1.519 \times 10^{-6} m^2/s$(7), $\mathbf{g} = 9.8m/s^2$, $\rho = 1000kg/m^3$, $U_\infty = 1m/s$
Now the Reynolds number(Re) for laminar flow is between 0 and 2300, and $Re = \frac{U_\infty x}{\nu}$. We find $x = \frac{\nu Re}{U_\infty}$ such that Re falls between 0 and 2300. x is the distance from the point where the fluid from the main stream meets the inclined plane, and along the surface of the inclined plane. We now compare results for u when δ used was obtained by momentum integral approach at various angles of inclination between $0 < \theta < \frac{\pi}{2}$. δ, and is the boundary layer thickness approximation from the quadratic function velocity profile substituted into momentum integral equation. u is the velocity down inclined plane when δ, is substituted into equation (2.9).
The results are obtained when the data below is used.
$x = 0.0005m, Re = 0.3292 \times 10^3$, δ=7.0711×10^{-5},
$x = 0.0010m, Re = 0.6583 \times 10^3$, δ= 1.4142×10^{-4},
$x = 0.0015m, Re = 0.9875 \times 10^3$, δ=2.1213×10^{-4},
$x = 0.0020m, Re = 1.3167 \times 10^3$, δ=2.8284×10^{-4},
$x = 0.0025m, Re = 1.6458 \times 10^3$, δ=3.5355×10^{-4},
$x = 0.0030m, Re = 1.9750 \times 10^3$, δ= 4.2426×10^{-4},
The results obtained above can be represented in the table below.

Table 1: Velocity and the angles of inclination

θ	0	$\frac{\pi}{12}$	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{5\pi}{12}$
$u(m/s)(x = 0.0005m)$	0	0.0042	0.0081	0.0114	0.0140	0.0156
$u(m/s)(x = 0.0010m)$	0	0.0167	0.0323	0.0456	0.0559	0.0623
$u(m/s)(x = 0.0015m)$	0	0.0376	0.0726	0.1026	0.1257	0.1402
$u(m/s)(x = 0.0020m)$	0	0.0668	0.1290	0.1825	0.2235	0.2493
$u(m/s)(x = 0.0025m)$	0	0.1044	0.2016	0.2851	0.3492	0.3895
$u(m/s)(x = 0.0030m)$	0	0.1503	0.2903	0.4106	0.5029	0.5609

5 Discussion

Momentum integral approach has been used to obtain the boundary layer thickness, δ. Quadratic polynomial approximation of the velocity profile in boundary layer was obtained. This approximation velocity profile is used in the momentum integral equation to generate the boundary layer thickness.The results are generated with the help of MatLab. In Table 1 , x increases from zero the point where Re is 2300, the boundary layer thickness δ, and u, also increases. This is because of molecular interactions that generate viscous forces.

Atthe surface, the flow has zero relative speed, because the fluid seems to stick to the surface of the inclined plane due to adhesive forces. The fluid transfers momentum to the adjacent layers through the action of viscosity. This leads to increase in velocity and the boundary layer thickness as x increases.This is after taking into consideration various angles of inclination between $0 < \theta < \frac{\pi}{2}$. For laminar flow, it is expected that the velocity do not go beyond $1m/s$ i.e main stream velocity but approach it. It is preferable to use the quadratic polynomial velocity profile because it utilizes more boundary conditions.

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