Review Article

ECO-EPIDEMILOGICAL MODEL ANALYSIS OF PERY-PREDATOR SYSTEM

ABSTRACT. In this paper disease transmission dynamics of prey-predator system is Analysed. For this purpose, a prey-predator system of species comprising of three compartments is constructed.. These species are Predator, Susceptible Prey and Infected Prey. The predation functional response is considered to follow modified holling type II functional response and a non-linear dynamical system of ODE is constructed. The existence, Positivity, and boundedness of solutions to the governing equations have been studed. Stability analysis of all possible equilibrium points of the model has been carried out by imposing different restrictions. Local and global stability of disease free and endemic equilibrium points have been verified with the help of variation matrix and Liapunove functions respectively. The basic reproduction number is drived,and Realistic values are assigned to the parameters. Numerical simulations are obtained using DEDiscover software which approve the analytical results.

Key words: Modeling; Eco-epidemiology; Stability Analysis; Reproduction Number; Simulation Study

1. INTRODUCTION

Prey-predator systems of interaction of species is a remarkable work of Lotka-Volterra in 1920s[1,3,5,6,16,17], and SIR model of Compartmentalization of population is well known area of research of Kermack and Mckendrick[1-3,5-10,15]. Anderson and May combined these two modeling systems, while Chattopadhyay and Arino were the first who used the term "eco-epidemiology" for such models[3,5,7,16,17]. The dynamics of disease in prey-predator systems now become an interesting area of research due to the fact that prey-predator interaction is rich and complex in nature[4,6,7,11-13]. Several mathematical models have been proposed and studied on prey-predator systems[1-7,9-12]. Many studies focused on the study of disease in a prey only[1-5,7,12], other researchers were interested in the study of disease within the predator population only[14], and there are also some studies on diseases in both prey and predators[6,9,11,16,17] In this paper, we proposed and studied infectious disease only in prey.

2. MATHEMATICAL MODEL FORMULATION AND ASSUMPTIONS

We Consider a prey-predator population with three compartments consisting of Susceptible prey $x(t)$, infected prey $w(t)$,and predator $y(t)$ populations.

1. In the absence of infectious disease, the susceptible prey population grows according to logistic function $g(x)$ with intrinsic growth rate r and environmental carrying capacity k and only susceptible prey can reproduce to reach its carrying capacity. On the other hand infected prey does not grow, reproduce and recover from the disease once infected.
2. Infectious disease transmission from infected prey $w(t)$ to susceptible prey $x(t)$ is assumed to follow non-linear incidence rate of function as $\mathrm{I}(x,\ w) = \frac{\beta x w}{1+w}$, which was constructed by Gumel and Moghadas and used by different scholars where, the parameter β is infection rate, the simple mass action law $\beta x w$ measures disease force of infection, and $\frac{1}{1+w}$ measures the inhibition effect from the crowding effect of infected individuals.
3. The predation functional response of predator towards the Susceptible prey $x(t)$ and infected prey $w(t)$ are assumed to follow a Modified holling type II functional response $p_1 f_1(x,\ w,\ y) = \frac{p_1 x y}{s+x+pw}$ and $\mathrm{p}_2 \mathrm{f}_2(x,\ w,\ y) = \frac{\mathrm{p}_2 w y}{s+x+\mathrm{p}w}$,where $p_1,\ p_2$ are predation coefficients of $x(t)$, $w(t)$ due to predator $y(t)$ with predation preference rate p
4. We also supposed that the Consumed susceptible prey and consumed infected prey converted into predator at Conversion rate q_1 and q_2 respectively with half saturated constant s
5. Only infected prey suffers from infectious disease with death rate d_1 and the remaining population predator and susceptible prey suffer with natural death rate d_2, d_3 respectively.
6. Assume that all variables and parameters used in the model are non-negative.

Table 1. Notation and Description of Variables

Variables	Descriptions
$\mathbf{X(t)}$	Population size of susceptible preys at time t
$\mathbf{W(t)}$	Population size of infected preys at time t
$\mathbf{Y(t)}$	Population size of predators at time t

Table 2. Notations and Description of parameters

Parameters	Description of Parameters
r, k	Intrinsic growth rate, Carrying capacity of susceptible prey respectively.
q_1, q_2	Conversion rate of susceptible prey , infected prey respectively
p_1, p_2	Predation coefficient of susceptible prey , infected prey respectively
p , s	Predation preference rate , Half saturated Constant respectively
d_1 , d_2	Death rate of infected prey , predator respectively.
$g(x)$	Logistic growth function of susceptible prey
$I(x,w)$	Nonlinear incidence rate of functions
$f_1(x, w, y)$	Predation functional response of predator towards the susceptible prey
$f_2(x, w, y)$	Predation functional response of predators towards the infected prey

According to the above assumptions, the description of variables, and parameters the present paper will have the following mode diagram given in Fig. 1

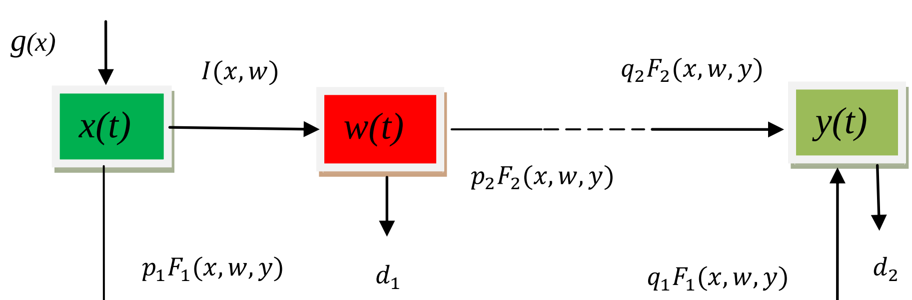

Figure 1. Model Diagram

$$\frac{dx}{dt} = g(x) - I(x, \quad y) - p_1 F_1(x, \quad w, \quad y) \qquad (1)$$

$$\frac{\mathrm{dw}}{\mathrm{dt}} = I(x, \quad y) - p_2 F_2(x, \quad w, \quad y) - \mathrm{d}_1 w \qquad (2)$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = q_1 F_1(x, w, y) + q_2 F_2(x, \quad w, \quad y) - \mathrm{d}_2 y \qquad (3)$$

According to the flow diagram, the Model written as follows:

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} - \frac{p_1 xy}{s+x+pw} \qquad (4)$$

$$\frac{\mathrm{dw}}{\mathrm{dt}} = \frac{\beta xw}{1+w} - \frac{\mathrm{p_2 wy}}{\mathrm{s}+x+pw} - \mathrm{d}_1 w \qquad (5)$$

$$\frac{\mathrm{dy}}{\mathrm{dt}} = \frac{\mathrm{q_1} xy}{\mathrm{s}+x+\mathrm{p}w} + \frac{\mathrm{q_2} wy}{\mathrm{s}+x+\mathrm{p}w} - \mathrm{d}_2 y \qquad (6)$$

with initial conditions $x(0) \geq 0, w(0) \geq 0, y(0) \geq 0$

3. MATHEMATICAL ANALYSIS OF THE MODEL

In this section, positivity, boundedness, and existence of the solution of the model is checked. This mathematical analysis of the model could be considered as primarly results.

Theorem 3.1 [Boundedness] All solutions of Model (4)-(6) are bounded in feasible region $\mathbb{R}_+^3$

Proof: each solutions $x(t), w(t), y(t)$ of the model is bounded if and only if total population N is bounded. Let total population of prey-predator $N = x + w + y$

For $\Lambda > 0$ be constant, $\frac{dN}{dt} + \Lambda N = \frac{dx}{dt} + \frac{dw}{dt} + \frac{dy}{dt} + \Lambda N$ (7)

By substitute all model Equations (4)-(6) into (7) and removing all negative terms, we have the following

$\frac{dN}{dt} + \Lambda N \leq rx\left(1 - \frac{x}{K}\right) - (p_1 - q_1)\frac{p_1 xy}{s+x+pw} - (p_2 - q_2)\frac{p_2 wy}{s+x+pw} - d_1 w - d_2 y = \mu$. Then Solving the differential inequality $\frac{dN}{dt} + \Lambda N \leq \mu$ yields $N(t) \leq \frac{\mu}{\Lambda}\left(1 - e^{-\Lambda t}\right) + N(0)e^{-\Lambda t}$ for $t \to \infty, N \to \frac{\mu}{\Lambda}$. We know that total prey-predator population is non-negative and hence $0 \leq N(t) \leq \frac{\mu}{\Lambda}$. So we have invariant feasible region $\Omega = \left\{(x, \ w, \ y) \in \mathbb{R}_+^3 : 0 \leq N(t) \leq \frac{\mu}{\Lambda}\right\}$. This proves the theorem and the model is Mathematically well posed.

Theorem 3.2 [Positivity] All solutions of Model (4)-(6) are positive.

Proof: To prove theorem 3.2, We have to show that variables $x(t), w(t), y(t)$ of the Model (4)-(6) are all non-negative $\forall t \geq 0$.

i. **Positivity of x(t):** From the Susceptible prey Model in (4), $\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} - \frac{p_1 xy}{s+x+pw}$ Without loss of generality, After removing all the positive terms from the right hand side of the differential equation, we have the following differential inequality; $\frac{dx}{dt} \geq -\left(\frac{rx^2}{k} + \frac{\beta xw}{1+w} + \frac{P_1 xy}{s+x+pw}\right)$ divide both sides by negative yields $-\frac{dx}{dt} \leq \frac{rx^2}{k} + \frac{\beta xw}{1+w} + \frac{p_1 xy}{s+x+pw}$,But It is also clear that the following inequality holds $\frac{rx^2}{k} + \frac{\beta xw}{1+w} + \frac{p_1 xy}{s+x+pw} \leq rx^2 + \beta xw + p_1 xy = x(rx + \beta w + p_1 y)$ Assume that $rx + \beta w + p_1 y = C$,Then the differential inequality reduced to $-\frac{dx}{dt} \leq x(rx + C)$.This inequality can be arranged for integration by partial fraction as $\int \frac{1}{x(rx+C)} dx \geq \int -d\,t$, integrating the integral inequality $\int \left(\frac{1/C}{x} + \frac{-r/C}{rx+C}\right) dx \geq -\int dt$ will give us $\frac{1}{C}\ln|x| - \frac{1}{C}\ln|rx + C| \geq -t + Q$,where Q is integration constant. Using rules of logarithm the inequality can be written as $\ln\left|\frac{x}{rx+C}\right| \geq -Ct + CQ$. Finally solving for x will give as $x(t) \geq \frac{ACe^{-Ct}}{1-rAe^{-Ct}}$, for $A = e^{CQ}$ .Therefore $x(t) > 0$ for $1 - rAe^{-Ct} > 0$. That is $x(t)$ is non-negative for $t > \frac{1}{C}\ln(rA)$

ii. **Positivity of w(t):** From infected prey Model in (5),$\frac{dw}{dt} = \frac{\beta xw}{1+w} - \frac{p_2 wy}{s+x+pw} - d_1 w$,Without loss of original generality, after removing the positive term $\left(\frac{\beta xw}{1+w}\right)$. we obtain the following differential inequality

$\frac{dw}{dt} \geq -\left(\frac{p_2 wy}{s+x+pw} + d_1 w\right) \Leftrightarrow -\frac{dw}{dt} \leq \left(\frac{p_2 wy}{s+x+pw} + d_1 w\right)$, But it is clear that the inequality $\frac{p_2 wy}{s+x+pw} + d_1 w \leq p_2 wy + d_1 w = (p_2 y + d_1)w$ holds true. Now Assume that $p_2 y + d_1 = C$ .Then we have

$-\frac{dw}{dt} \leq Cw$, Now applying integration yield $\ln|w| \geq -Ct + Q$,where Q is integration constant, Then solving for the variable $w(t)$ gives the equation $w(t) \geq e^{-Ct+Q}$ which is exponential function and positive at all time ,Hence $w(t)$ is positive.

iii. **Positivity of y(t):** From the Susceptible predator Model in (6),$\frac{dy}{dt} = \frac{q_1xy}{s+x+pw} + \frac{q_2wy}{s+x+pw} - d_2y$, without loss of original generality, after removing all positive terms$\left(\frac{q_1xy}{s+x+pw} + \frac{q_2wy}{s+x+pw}\right)$, we obtain differential equation $\frac{dy}{dt} \geq -(d_2)y$ Then applying integration by separable of variable method results, $ln|y| \geq -(d_2)t + Q$,where Q integration constant. solving for variable $Y(t)$,we obtain the solution $|y| \geq e^{-(d_2)t+Q}$. Therefore $y(t) \geq e^{-(d_2)t+Q}$ is a positive exponential function. Hence y(t) is positive.

Theorem3.3.[Existence]All Solutions of the model(4)-(6) together with the initial conditions $x(0) > 0, w(0) \geq 0, y(0) \geq 0$ exist in $\mathbb{R}^3_+$ i.e., the model variables $x(t)$, $w(t)$, and $y(t)$ exist for all t and remain in $\mathbb{R}^3_+$.

Proof: Let model(1) represented as given as

$$f_1(x, \; w, \; y) = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} - \frac{p_1xy}{s+x+pw},$$

$$f_2(x, \; w, \; y) = \frac{\beta xw}{1+w} - \frac{p_2wy}{s+x+pw} - d_1w,$$

$$f_3(x, \; w, \; y) = \frac{q_1xy}{s+x+pw} - \frac{q_2wy}{s+x+pw} - d_2y,$$

According to Derrick and Groosman theorem, let Ω denote the region $\Omega = \left\{(x, \; w, \; y) \in \mathbb{R}^3_+; N \leq \frac{\mu}{\Lambda}\right\}$. Then model (4)-(6) have a unique solution if $(\partial f_i)/(\partial x_j)$, $i,j = 1,2,3$ are continuous and bounded in Ω. Here, $x_1 = x$, $x_2 = w$, $x_3 = y$, The continuity and the Boundedness can be shown as follows:

Table 3. Partial Derivatives of Functions

For f_1:	For f_3:
$\left\|\frac{\partial f_1}{\partial x}\right\| = \left\|r - \frac{2rx}{k} - \frac{\beta w}{1+w} - \frac{p_1 y(s+pw)}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_1}{\partial w}\right\| = \left\|-\frac{\beta x}{(1+w)^2} + \frac{p_1 pxy}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_1}{\partial y}\right\| = \left\|-\frac{p_1 x}{s+x+pw}\right\| < \infty$	$\left\|\frac{\partial f_3}{\partial x}\right\| = \left\|\frac{q_1 y(s+pw) - q_2 wy}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_3}{\partial w}\right\| = \left\|\frac{(q_2 - q_1)pxy}{(s+x+pw)^2}\right\| < \infty$ $\left\|\frac{\partial f_3}{\partial y}\right\| = \left\|\frac{q_1 x + q_2 w}{s+x+pw} - d_2\right\| < \infty$
For f_2: $\left\|\frac{\partial f_2}{\partial x}\right\| = \left\|\frac{\beta w}{1+w}\right\| < \infty$ $\left\|\frac{\partial f_2}{\partial w}\right\| = \left\|\frac{\beta x}{(1+w)^2} - \frac{p_2 y(s+x)}{(s+x+pw)^2} - d_1\right\| < \infty$ $\left\|\frac{\partial f_2}{\partial y}\right\| = \left\|-\frac{p_2 w}{s+x+pw}\right\| < \infty$	

Thus, all the partial derivatives $(\partial f_i)/(\partial x_j)$, $i, j = 1,2,3$ exist, continuous, and bounded in a region Ω for all positive values of model variable and model parameter. Hence, by Derrick and Groosman theorem, a solution for the model (4)-(6) exists and unique.

4. STABILITY ANALYSIS

4.1. Stability Analysis of Sub Models in the Absence of Predators

In the absence of predators *(y=0)*, model (4)-(6) can be written as

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} = f(x, w) \qquad (8)$$

$$\frac{dw}{dt} = \frac{\beta xw}{1+w} - d_1 w = g(x, w) \qquad (9)$$

This subsystem has at most three non-negative equilibrium points. The equilibrium points are: trivial equilibrium point $E_o(0,\ 0)$, axial equilibrium point $E_A(k,\ 0)$, and positive equilibrium point $E^*(\mathrm{x},\ \mathrm{w})$ where

$$x = \frac{2kr + kd_1 - k\sqrt{d_1}\sqrt{4\beta + d_1}}{2r}, \quad w = \frac{-d_1 + \sqrt{d_1}\sqrt{4\beta + d_1}}{2r}$$

It is noted that the last W is negative. hence we reject the negative equilibrium point one and take only the non-negative equilibrium point.

The Variation Matrix of (8) and (9) is given by

$$V(x,w) = \begin{pmatrix} r - \frac{2rx}{k} - \frac{\beta w}{1+w} & -\frac{\beta x}{(1+w)^2} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2} - d_1 \end{pmatrix}$$

Theorem 4.1.1. The trivial Equilibrium point $E_o(0, 0)$ of model (8) and (9) always exists and E_o is a saddle point with locally unstable manifold in x-direction and locally stable manifold in the w-direction .

Proof: Consider the variation matrix of sub model(3) at trivial equilibrium point $V(E_o) = \begin{pmatrix} r & 0 \\ 0 & -d_1 \end{pmatrix}$ .Thus Eigen values of $V(E_o)$ are $\lambda_1 = r > 0$ and $\lambda_2 = -d_1 < 0$, Hence E_o is a saddle point with locally unstable manifold in x-direction and local stable manifold in the w-direction

Theorem 4.1.2.The axial Equilibrium point $E_A(k, 0)$ of sub model (8) and (9) always exists and if $\beta k - d_1 < 0$, E_A is locally asymptotically stable point and if $\beta k - d_1 > 0$, E_o is a saddle point with locally stable manifold in x-direction and locally unstable manifold in w-direction.

Proof: Consider the variation matrix of model (8) and (9) at Axial equilibrium point , $V(E_A) = \begin{pmatrix} -r & -\beta k \\ 0 & \beta k - d_1 \end{pmatrix}$.Thus Eigen values are $\lambda_1 = -r < 0$ and $\lambda_2 = \beta k - d_1$ Thus if $\beta k - d_1 < 0$, E_A is locally asymptotically stable point, and if $\beta k - d_1 > 0$, E_A is a saddle point with locally stable manifold in X-direction and locally unstable manifold in W-direction.

Theorem 4.1.3. The positive Equilibrium point $E^*(X^*, W^*)$ of sub model (8) and (9) exists and stable if the inequalities $\left\{r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*}\right\} + \left\{\frac{\beta x^*}{(1+w^*)^2} - d_1\right\} > 0, \& \left\{r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*}\right\}\left\{\frac{\beta x^*}{(1+w^*)^2} - d_1\right\} + \frac{\beta^2 x^* w^*}{(1+w^*)^3} > 0$ holds true. Otherwise unstable

Proof: Consider the variation matrix of sub model (8) and (9)at positive equilibrium point

$$V^*(x^*,w^*) = \begin{pmatrix} r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*} & -\frac{\beta x^*}{(1+w^*)^2} \\ \frac{\beta w^*}{1+w^*} & \frac{\beta x^*}{(1+w^*)^2} - d_1 \end{pmatrix}$$

To find the characteristic polynomial of the variation matrix first compute the $\det(V(E^*) - \lambda I_2) = 0$ That is $\begin{vmatrix} r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*} - \lambda & -\frac{\beta x^*}{(1+w^*)^2} \\ \frac{\beta w^*}{1+w^*} & \frac{\beta x^*}{(1+w^*)^2} - d_1 - \lambda \end{vmatrix} = 0$

Thus Using Routh Hourwith criterion, the characteristic polynomial $\left(r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*} - \lambda\right)\left(\frac{\beta x^*}{(1+w^*)^2} - d_1 - \lambda\right) + \frac{\beta^2 x^* w^*}{(1+w^*)^3} = 0$ is stable, if $\left\{r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*}\right\} + \left\{\frac{\beta x^*}{(1+w^*)^2} - d_1\right\} > 0$ and $\left\{r - \frac{2rx^*}{k} - \frac{\beta w^*}{1+w^*}\right\}\left\{\frac{\beta x^*}{(1+w^*)^2} - d_1\right\} + \frac{\beta^2 x^* w^*}{(1+w^*)^3} > 0$

Otherwise unstable

4.2. Stability Analysis of Sub Model (4)-(6) in the Absence of Infected Prey

In the absence of infected prey $(w = 0)$, model (4)-(6) can be written as

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{p_1 xy}{s+x} = f(x,\ y) \qquad (10)$$

$$\frac{dy}{dt} = \frac{q_1 xy}{s+x} - d_2 y = h(x,\ y) \qquad (11)$$

This subsystem has at most three non-negative equilibrium points. The equilibrium points are: trivial equilibrium point $E_o(0,\ 0)$, axial equilibrium point $E_A(k,\ 0)$, and positive equilibrium point $E^*(x,\ y)$ *where*

$$x = \frac{sd_2}{q_1 - d_2}, y = \frac{rsq_1(kq_1 - kd_2 - sd_2)}{(q_1 - d_2)^2 p_1 k}$$

The Variation matrix of (10) and (11) is given by

$$V(x, y) = \begin{pmatrix} r - \frac{2rx}{k} - \frac{sp_1 y}{(s+x)^2} & -\frac{p_1 x}{s+x} \\ \frac{sq_1 y}{(s+x)^2} & \frac{q_1 x}{s+x} - d_2 \end{pmatrix}$$

Theorem 4.2.1. The trivial Equilibrium point $E_o(0,0)$ of (10) and (11) always exists and E_o is a saddle point with locally unstable manifold in x-direction and locally stable manifold in the y-direction .

Proof: The variation matrix of sub model (10) and (11) at trivial equilibrium point E_o is given by

$$V(E_o) = \begin{pmatrix} r & 0 \\ 0 & -d_2 \end{pmatrix}$$

Thus the eigen values of $V(E_o)$ are: $\lambda_1 = r > 0$ and $\lambda_2 = -d_2 < 0$ Hence E_o is a saddle point with locally unstable manifold in x-direction and local stable manifold in the y-direction.

Theorem 4.2.2. The axial Equilibrium point $E_A(k,0)$ of sub model (10) and (11) always exists and if $\frac{q_1 k}{s+k} - d_2 < 0$, E_A is locally asymptotically stable point, and if $\frac{q_1 k}{s+k} - d_2 > 0$, E_A is a saddle point with locally stable manifold in x-direction and locally unstable manifold in y-direction.

Proof: The variation matrix of sub model (10) and (11) at axial equilibrium point E_A is given by

$$V(E_A) = \begin{pmatrix} -r & \frac{p_1 k}{s+k} \\ 0 & \frac{q_1 k}{s+k} - d_2 \end{pmatrix}$$

Thus the eigen values are: $\lambda_1 = -r < 0$ and $\lambda_2 = \frac{q_1 k}{s+k} - d_2$. Thus if $\frac{q_1 k}{s+k} - d_2 < 0$, E_A is locally asymptotically stable point, and if $\frac{q_1 k}{s+k} - d_2 > 0$, E_A is a saddle point with locally stable manifold in x-direction and locally unstable manifold in y-direction.

Theorem 4.2.3. The positive Equilibrium point $E^*(x^*, y^*)$ of sub model (10) and (11) exists and stable if the following inequalities holds true. $\left\{r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2}\right\} + \left\{\frac{q_1 x^*}{s+x^*} - d_2\right\} > 0, \&$ $\left\{r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2}\right\}\left\{\frac{q_1 x^*}{s+x^*} - d_2\right\} + \frac{sp_1 q_1 x^* y^*}{(s+x^*)^3} > 0$, otherwise unstable.

Proof: Consider the variation matrix of sub model (10) and (11) at positive equilibrium point

$$V(E^*) = \begin{pmatrix} r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2} & -\frac{p_1 x^*}{s+x^*} \\ \frac{sq_1 y^*}{(s+x^*)^2} & \frac{q_1 x^*}{s+x^*} - d_2 \end{pmatrix}$$

To find the characteristic polynomial of the variation matrix first compute $\det(V(E^*) - \lambda I_2) = 0$

That is $$\begin{vmatrix} r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2} - \lambda & -\frac{p_1 x^*}{s+x^*} \\ \frac{sq_1 y^*}{(s+x^*)^2} & \frac{q_1 x^*}{s+x^*} - d_2 - \lambda \end{vmatrix} = 0$$

using Routh Hourwith criterion, the characteristic polynomial

$\left(r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2} - \lambda\right)\left(\frac{q_1 x^*}{s+x^*} - d_2 - \lambda\right) + \frac{sp_1 q_1 x^* y^*}{(s+x^*)^3} = 0$ is stable if

$\left\{r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2}\right\} + \left\{\frac{q_1 x^*}{s+x^*} - d_2\right\} > 0, \& \left\{r - \frac{2r\,x^*}{k} - \frac{sp_1 y^*}{(s+x^*)^2}\right\}\left\{\frac{q_1 x^*}{s+x^*} - d_2\right\} + \frac{sp_1 q_1 x^* y^*}{(s+x^*)^3} > 0$,otherwise unstable.

4.3. Stability Analysis of Model (4)-(6) with no Restriction Imposed on the Prey and Predator

In this section, we are going to determine the stability analysis of equilibrium points of model (4)-(6) and re written as follows

$$\frac{dx}{dt} = rx\left(1 - \frac{x}{k}\right) - \frac{\beta xw}{1+w} - \frac{p_1 xy}{s+x+pw} = f(x, \quad w, \quad y) \qquad (12)$$

$$\frac{dw}{dt} = \frac{\beta xw}{1+w} - \frac{p_2 wy}{s+x+pw} - d_1 w = g(x, \quad w, \quad y) \qquad (13)$$

$$\frac{dy}{dt} = \frac{q_1 xy}{s+x+pw} + \frac{q_2 wy}{s+x+pw} - d_2 y = h(x, \quad w, \quad y) \qquad (14)$$

This model has at most five non negative equilibrium points. The equilibrium points are: i) trivial equilibrium point $E_o(0, \ 0, \ 0)$, ii)axial equilibrium point $E_A(k, \ 0, \ 0)$.iii) disease-free equilibrium point(DFEP) $\bar{E}(\bar{x} \quad 0, \quad \bar{y})$ where $\bar{x} = \frac{sd_2}{q_1 - d_2}$, $\bar{w} = 0$, $\bar{y} = \frac{rsq_1(kq_1 - kd_2 - sd_2)}{(q_1 - d_2)^2 p_1 k}$ iv) Susceptible prey-free equilibrium points $E(0, w, y)$ is not applicable where $x = 0$, $w = -\frac{sd_2}{pd_2 - q_2}$, $y = \frac{sd_1 q_2}{(pd_2 - q_2)p_2}$ iv) predator-free equilibrium point $E(x, \ w, \ 0)$, the values of x and y taken from sub model (8) & (9) . v) Positive equilibrium point $E^*(x^*, \quad w^*, \quad y^*)$.

The variation matrix of model (12)-(14) is given by

$$V(x, \quad w, \quad y) = \begin{pmatrix} f_x & f_w & f_y \\ g_x & g_w & g_y \\ h_x & h_w & h_y \end{pmatrix}$$

That is the variation matrix model (12)-(14) is given by

$$V(x,\quad w,\quad y)=\begin{pmatrix} r-\frac{2rx}{k}-\frac{\beta w}{1+w}-\frac{p_1y(s+pw)}{(s+x+pw)^2} & \frac{-\beta x}{(1+w)^2}+\frac{p_1pxy}{(s+x+pw)^2} & \frac{-p_1x}{s+x+pw} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2}-\frac{p_2y(s+x)}{(s+x+pw)^2}-d_1 & \frac{-p_2w}{s+x+pw} \\ \frac{q_1y(s+pw)-q_2wy}{(s+x+pw)^2} & \frac{(q_2-q_1)xyp}{(s+x+pw)^2} & \frac{q_1x+q_2w}{s+x+pw}-d_2 \end{pmatrix}$$

Theorem 4.3.1. The trivial equilibrium point $E_o(0,\quad 0,\quad 0)$ is a saddle point with locally unstable manifold in x-direction and locally stable manifold in wy-plane.

Proof: The variation matrix at E_o is given by $J(E_o)=\begin{pmatrix} r & 0 & 0 \\ 0 & -d_1 & 0 \\ 0 & 0 & -d_2 \end{pmatrix}$

Then the eigen values are:$\lambda_1=r>0$, $\lambda_2=-d_1<0$, $\lambda_3=-d_2<0$ which is saddle point with locally unstable manifold in x-direction and locally stable manifold in wy-plane.

Theorem 4.3.2. The axial equilibrium point $E_A(k,\quad 0,\quad 0)$ is stable if $\beta k-d_1<0$ & $\frac{q_1k}{s+k}-d_2<0$ and unstable if $\beta k-d_1<0$ & $\frac{q_1k}{s+k}-d_2<0$

Proof: The variation matrix at E_A is given by

$$J(E_A)=\begin{pmatrix} -r & -\beta k & \frac{-p_1k}{s+k} \\ 0 & \beta k-d_1 & 0 \\ 0 & 0 & \frac{q_1k}{s+k}-d_2 \end{pmatrix}$$

Thus the eigen values are:$\lambda_1=-r<0$, $\lambda_2=\beta k-d_1$, $\lambda_3=\frac{q_1k}{s+k}-d_2$

The axial equilibrium point E_A is stable if $\beta k-d_1<0$ & $\frac{q_1k}{s+k}-d_2<0$ and unstable if $\beta k-d_1<0$ & $\frac{q_1k}{s+k}-d_2<0$

Theorem 4.3.3. The disease-free equilibrium point $\bar{E}(\bar{x}\quad 0,\quad \bar{y})$ is stable if $\beta\bar{x}-\frac{p_2\bar{y}}{s+\bar{x}}<0$ and the quadratic polynomial $\left(r-\frac{2r\bar{x}}{k}-\frac{sp_1\bar{y}}{(s+x)^2}-\lambda\right)*\left(\frac{q_1\bar{x}}{s+\bar{x}}-d_2-\lambda\right)+\frac{sp_1q_1\bar{x}\bar{y}}{(s+\bar{x})^3}=0$ is stable

Proof: Consider the variation matrix at $\bar{E}$ is given by

$$V(\bar{E})=\begin{pmatrix} r-\frac{2rx}{k}-\frac{sp_1y}{(s+x)^2} & -\beta x+\frac{p_1pxy}{(s+x)^2} & \frac{-p_1x}{s+x} \\ 0 & \beta x-\frac{p_2y(s+x)}{(s+x)^2}-d_1 & 0 \\ \frac{sq_1y}{(s+x)^2} & \frac{(q_2-q_1)xyp}{(s+x)^2} & \frac{q_1x}{s+x}-d_2 \end{pmatrix}$$

To find eigen values first compute $\det(V(\bar{E})-\lambda I_3)=0$

$$\begin{vmatrix} r-\frac{2rx}{k}-\frac{sp_1y}{(s+x)^2}-\lambda & -\beta x+\frac{p_1pxy}{(s+x)^2} & \frac{-p_1x}{s+x} \\ 0 & \beta x-\frac{p_2y(s+x)}{(s+x)^2}-d_1-\lambda & 0 \\ \frac{sq_1y}{(s+x)^2} & \frac{(q_2-q_1)xyp}{(s+x)^2} & \frac{q_1x}{s+x}-d_2-\lambda \end{vmatrix}=0$$

$$\left(\beta x-\frac{p_2y(s+x)}{(s+x)^2}-d_1-\lambda\right)*\begin{vmatrix} r-\frac{2rx}{k}-\frac{sp_1y}{(s+x)^2}-\lambda & \frac{-p_1x}{s+x} \\ \frac{sq_1y}{(s+x)^2} & \frac{q_1x}{s+x}-d_2-\lambda \end{vmatrix}=0$$

$$\left(\beta x-\frac{p_2y(s+x)}{(s+x)^2}-d_1-\lambda\right)*\left\{\left(r-\frac{2rx}{k}-\frac{sp_1y}{(s+x)^2}-\lambda\right)*\left(\frac{q_1x}{s+x}-d_2-\lambda\right)+\frac{sp_1q_1\bar{x}\bar{y}}{(s+\bar{x})^3}\right\}=0$$

is the characteristic polynomial which implies that eigen value are:

$\lambda_1=\beta x-\frac{p_2y(s+x)}{(s+x)^2}-d_1$ and the remaining eigen values are determined from the quadratic equation $\left(\underbrace{r-\frac{2rx}{k}-\frac{sp_1y}{(s+x)^2}}_{a}-\lambda\right)*\left(\underbrace{\frac{q_1x}{s+x}-d_2}_{b}-\lambda\right)+\underbrace{\frac{sp_1q_1\bar{x}\bar{y}}{(s+\bar{x})^3}}_{c}=0$

the disease free equilibrium point is stable if $\beta x-\frac{p_2y(s+x)}{(s+x)^2}-d_1<0$ and $a+b>0\,, ab+c>0$ otherwise unstable, where $x=\frac{sd_2}{q_1-d_2}$ and $y=\frac{rsq_1(kq_1-kd_2-sd_2)}{(q_1-d_2)^2p_1k}$

Theorem 4.3.4 The predator-free equilibrium point $\tilde{E}(\tilde{x},\ \tilde{w},\ 0)$ stable if $+b>0\,, ab+c>0$, where $a=r-\frac{2rx}{k}-\frac{\beta w}{1+w}$, $b=\frac{\beta x}{(1+w)^2}-d_1$, $c=\frac{\beta^2xw}{(1+w)^3}$

Proof: Consider the variation matrix at predator -free equilibrium point $\tilde{E}$ is given by

$$V(\tilde{E})=\begin{pmatrix} r-\frac{2rx}{k}-\frac{\beta w}{1+w} & \frac{-\beta x}{(1+w)^2} & \frac{-p_1x}{s+x+pw} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2}-d_1 & \frac{-p_2w}{s+x+pw} \\ 0 & 0 & \frac{q_1x+q_2w}{s+x+pw}-d_2 \end{pmatrix}$$

To find eigen values first compute $\det\left(V(\tilde{E})-\lambda I_3\right)=$

$$\begin{vmatrix} r-\frac{2rx}{k}-\frac{\beta w}{1+w}-\lambda & \frac{-\beta x}{(1+w)^2} & \frac{-p_1 x}{s+x+pw} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2}-d_1-\lambda & \frac{-p_2 w}{s+x+pw} \\ 0 & 0 & \frac{q_1 x+q_2 w}{s+x+pw}-d_2-\lambda \end{vmatrix} = 0$$

$$\left(\frac{q_1x+q_2w}{s+x+pw}-d_2-\lambda\right)*\begin{vmatrix} r-\frac{2rx}{k}-\frac{\beta w}{1+w}-\lambda & \frac{-\beta x}{(1+w)^2} \\ \frac{\beta w}{1+w} & \frac{\beta x}{(1+w)^2}-d_1-\lambda \end{vmatrix}=0$$

$\left(\frac{q_1x+q_2w}{s+x+pw}-d_2-\lambda\right)*\left\{\left(r-\frac{2rx}{k}-\frac{\beta w}{1+w}-\lambda\right)*\left(\frac{\beta x}{(1+w)^2}-d_1-\lambda\right)+\frac{\beta^2xw}{(1+w)^3}\right\}=0$ is characteristic polynomial. eigen values are: $\lambda_1=\frac{q_1x+q_2w}{s+x+pw}-d_2$ and the remaining eigen values are obtained from the roots of the quadratic equation

$$\left(\underbrace{r-\frac{2rx}{k}-\frac{\beta w}{1+w}}_{a}-\lambda\right)*\left(\underbrace{\frac{\beta x}{(1+w)^2}-d_1}_{b}-\lambda\right)+\underbrace{\frac{\beta^2xw}{(1+w)^3}}_{c}=0$$

Thus the characteristic polynomial is stable if $\frac{q_1x+q_2w}{s+x+pw}-d_2<0$ and

$\left(\underbrace{r-\frac{2rx}{k}-\frac{\beta w}{1+w}}_{a}-\lambda\right)*\left(\underbrace{\frac{\beta x}{(1+w)^2}-d_1}_{b}-\lambda\right)+\underbrace{\frac{\beta^2xw}{(1+w)^3}}_{c}=0$is stable iff $a+b>0\,,ab+c>0$ by Routh Hourwith criterion

Theorem 4.3.5 The positive equilibrium point $E^*(x^*,\quad w^*,\quad y^*)$ is globally asymptotically stable

Proof: Take appropriate liapunove function $L(x,\quad w,\quad y)=\frac{\alpha_1}{2}(x-x^*)^2+\frac{\alpha_2}{2}(w-w^*)^2+\frac{\alpha_3}{2}(y-y^*)^2$ (15)

The derivative of the Liapunove function (15) L with respect to time t

$$\frac{dL}{dt}=\alpha_1(x-x^*)\frac{dx}{dt}+\alpha_2(w-w^*)\frac{dw}{dt}+\alpha_3(y-y^*)\frac{dy}{dt}$$

Substitute Equations (12)-(14) into (15),we have the following equations

$$\frac{dL}{dt}=\alpha_1(x-x^*)\left[rx\left(1-\frac{x}{k}\right)-\frac{\beta xw}{1+w}-\frac{p_1xy}{s+x+pw}\right]+\alpha_2(w-w^*)\left[\frac{\beta xw}{1+w}-\frac{p_2wy}{s+x+pw}-d_1w\right]$$
$$+\alpha_3(y-y^*)\left[\frac{q_1xy}{s+x+pw}+\frac{q_2wy}{s+x+pw}-d_2y\right]$$

take out $x, w, \&\ y$ from each bracket and put as change,

$$\frac{dL}{dt} = \alpha_1(x-x^*)^2\left[r\left(1-\frac{x}{k}\right)-\frac{\beta w}{1+w}-\frac{p_1 y}{s+x+pw}\right]+\alpha_2(w-w^*)^2\left[\frac{\beta x}{1+w}-\frac{\mathrm{p}_2\mathrm{y}}{\mathrm{s}+x+pw}-\mathrm{d}_1\right]$$
$$+\alpha_3(y-y^*)^2\left[\frac{\mathrm{q}_1 x}{\mathrm{s}+x+\mathrm{p}w}+\frac{\mathrm{q}_2 w}{\mathrm{s}+x+\mathrm{p}w}-\mathrm{d}_2\right]$$

take out negative sign from each square bracket

$$\frac{dL}{dt} = -\alpha_1(x-x^*)^2\left[-r\left(1-\frac{x}{k}\right)+\frac{\beta w}{1+w}+\frac{p_1 y}{s+x+pw}\right]$$
$$-\alpha_2(w-w^*)^2\left[-\frac{\beta x}{1+w}+\frac{\mathrm{p}_2\mathrm{y}}{\mathrm{s}+x+pw}+\mathrm{d}_1\right]$$
$$-\alpha_3(y-y^*)^2\left[-\frac{\mathrm{q}_1 x}{\mathrm{s}+x+\mathrm{p}w}-\frac{\mathrm{q}_2 w}{\mathrm{s}+x+\mathrm{p}w}+\mathrm{d}_2\right]$$

we could properly choose the value of $\alpha_1, \alpha_2, \alpha_3$ such that $\frac{dL}{dt} < 0$ .The endemic equilibrium point is globally stable

Theorem 4.3.6 The Basic reproduction number is given by

$$R_0 = \frac{\beta s d_2 p_1 k}{k(q_1-d_2)(rp_2+d_1p_1)-rsd_2p_2}$$

Proof: Consider the infected predator model (5)

$$\frac{\mathrm{dw}}{\mathrm{dt}} = \frac{\beta x w}{1+w}-\frac{\mathrm{p}_2\mathrm{wy}}{\mathrm{s}+x+pw}-\mathrm{d}_1 w = \left[\underbrace{\frac{\beta x}{1+w}}_{\mathrm{F}}-\left(\underbrace{\frac{\mathrm{p}_2\mathrm{y}}{\mathrm{s}+x+pw}+\mathrm{d}_1}_{\mathrm{V}}\right)\right]w$$

Let us Define functions $F = \frac{\beta x}{1+w}$,$\mathrm{V} = \frac{\mathrm{p}_2\mathrm{y}}{\mathrm{s}+x+pw}+\mathrm{d}_1$ Then Evaluate the function at disease free equilibrium point (DFEP) $\bar{E} = (\bar{x} \quad 0, \quad \bar{y})$ where

$$\bar{x} = \frac{sd_2}{q_1-d_2}, \qquad \bar{w} = 0, \qquad \bar{y} = \frac{rsq_1(kq_1-kd_2-sd_2)}{(q_1-d_2)^2 p_1 k}$$

$F(\bar{E}) = \beta\bar{x} = \frac{\beta s d_2}{q_1-d_2}$

$$V(\bar{E}) = \frac{\mathrm{p}_2\bar{\mathrm{y}}}{\mathrm{s}+\bar{x}} + \mathrm{d}_1 = \frac{\mathrm{p}_2\left[\frac{rsq_1(kq_1-kd_2-sd_2)}{(q_1-d_2)^2 p_1 k}\right]}{\mathrm{s}+\left[\frac{sd_2}{q_1-d_2}\right]} + \mathrm{d}_1 = \frac{rsq_1p_2(kq_1-kd_2-sd_2)[q_1-d_2]}{(q_1-d_2)^2p_1k[sq_1-sd_2+sd_2]} + d_1$$

$$= \frac{rsq_1p_2(kq_1-kd_2-sd_2)}{(q_1-d_2)p_1k[sq_1]} + d_1 = \frac{rp_2(kq_1-kd_2-sd_2)}{(q_1-d_2)p_1k} + d_1$$

$$V(\bar{E}) = \frac{rp_2(kq_1-kd_2-sd_2)+(q_1-d_2)d_1p_1k}{(q_1-d_2)p_1k}$$

Then Basic reproduction number is

$$R_0 = [F(\bar{E})][V(\bar{E})]^{-1} = \left[\frac{\beta sd_2}{q_1-d_2}\right]\left[\frac{(q_1-d_2)p_1k}{rp_2(kq_1-kd_2-sd_2)+(q_1-d_2)d_1p_1k}\right]$$

$$R_0 = \frac{\beta sd_2p_1k}{rp_2(kq_1-kd_2-sd_2)+(q_1-d_2)d_1p_1k}$$

$$R_0 = \frac{\beta sd_2p_1k}{k(q_1-d_2)(rp_2+d_1p_1)-rsd_2p_2}$$

5. SIMULATION STUDY

In the absence of predators our model becomes a 2 by 2 dynamical systems and the parametric values r=0.067,k=0.425,Beta=0.0800,d_1=0.6000, and initial conditions for the system variable x_o =10.000, w_o=15.0000 are used for simulation purposes.

dx/dt=r*x*(1-x/k)-(Beta*x*w)/(1+w)

dw/dt=Beta*x*w/(1+w)-d_1*w

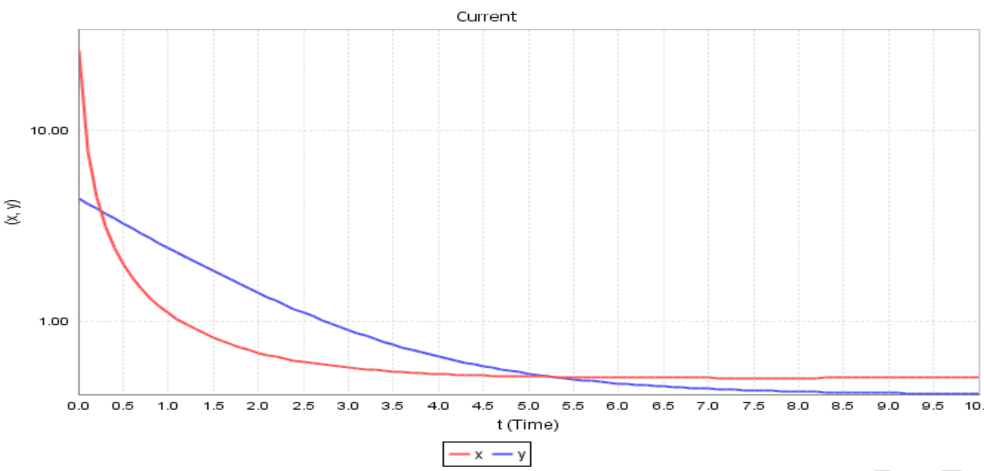

Figure 2. Time Series Plot for Prey-Predator System with no Infected Prey

There is no infected prey populations in the above simulation.The predators has no preference to eat the weaker populaions.Thus the graph is oscilatary with more than two cross and the given data shows graph of a well known prey-predator system of lotika voltera.

In the absence of infected prey the model becomes simpler 2 by 2 dynamical systems and the parametric values r=0.1460,k=0.1990,p_1=0.2120,s=0.2690, q_1=0.2460,d_2=0.7750, and initial conditions model variable x_o=20.0000,y_o=5.0000 are used for simulation purposes

dx/dt=r*x*(1-x/k)-(p_1*x*y)/(s+x)

dy/dt=(q_1*x*y)/(s+x)-d_2*y

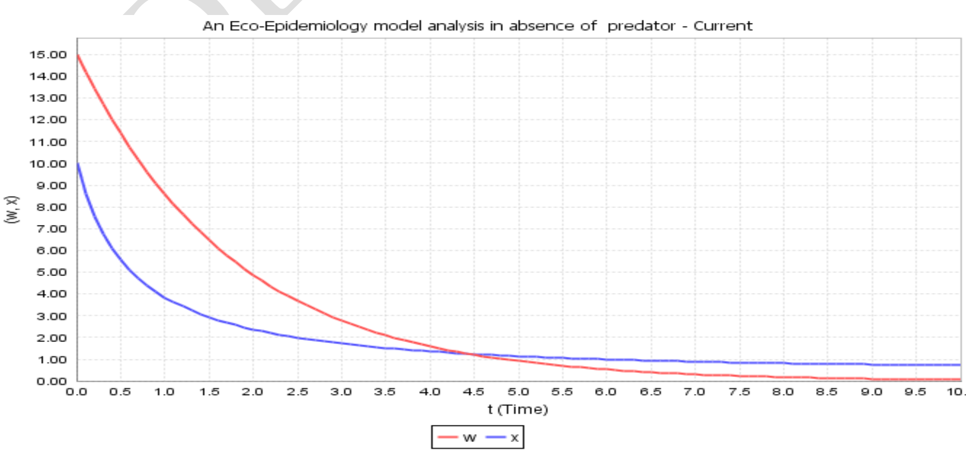

Figure 3. Time Series Plot for Prey-predator System with no Predator

Initially both infected prey and susceptible prey populations decline up to some time and then infected prey Continue declining more than susceptible prey. This prevails that predators prefer to consume the weaker populations. Thus the disease die out due to the predator consume more infected preys and susceptible prey have got time to survive and reproduce.

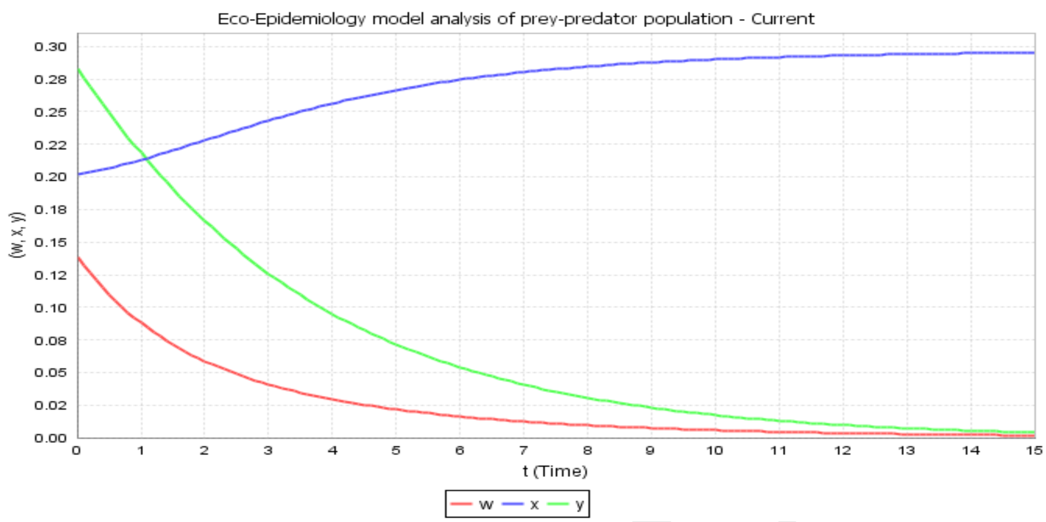

Figure 4. Time Series Plot for Eco-epidemic Prey-predator System

6. CONCLUSSION

In this Paper, an Eco-epidemiological Mathematical Model describing the spread of infectious disease in a prey-predator system has been Proposed and analyzed with assumptions that an infectious disease is spreading among the prey population only and the predators are Consuming both susceptible and infected preys with two different modified functional responses.

It is proved that the solutions of the constructed model exist, positive and bounded. we have also investigated and Computed that different Equilibrium points are exist for the proposed model. Moreover, Local and global stability analysis of equilibrium points of the proposed model is studied Using Variation matrix and liapunove function respectively. The trivial equilibrium point $E_o(0, \quad 0, \quad 0)$ is a saddle point which is locally asymptotically unstable and the positive equilibrium point $E^*(x^*, \quad w^*, \quad y^*)$ is globally asymptotically stable.

We have obtained the basic reproduction number $R_0 = \frac{\beta s d_2 p_1 k}{k(q_1 - d_2)(r p_2 + d_1 p_1) - r s d_2 p_2}$ If the basic reproduction number is less than one, then the infectious disease will be dies out from the system and if the basic reproduction number is greater than one, then the infectious disease continue to spread in the system. Finally, Realistic values are assigned to the model parameters and numerical simulations are obtained using DEDiscover software which approve the analytical results.

Conflicts of Interest: The author(s) declare that there are no conflicts of interest regarding the publication of this paper.

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