***Original Research Article***

# Some Common Fixed Point Results for OWC Mappings in Cone Metric Spaces



**Abstract**

In this paper, we obtained some common fixed point results for OWC (Occasionally Weakly Compatible) mappings in cone metric spaces. Recently, Amari and Moutawakil have proved some new common fixed point theorems under strict contractive conditions in metric space with property (E.A). Our results are generalized; improved and extends the results existing in the literature.

**Keywords:** Cone metric space, fixed point, occasionally weakly compatible (owc).

## 1. Introduction and Preliminaries

The concept of cone metric space was introduced in 2007 by Haung and Zhang [5] and they have generalized the concept of a metric space, and they replacing the real numbers by on ordered Banach space and obtained some fixed point theorems for mappings satisfying different contractive conditions in cone metric space. Later on many authors were, inspired in these results and generalized, extended and improved these results in many ways (for e.g. see [ 1] ,[4] ). Recently Amari and Moutawakil [2] have proved some fixed point theorems results under strict contractive conditions in metric space and they were using the property (E.A) . We have extended these results into cone metric space. In this paper we obtained some results for OWC in cone metric spaces, which are generalized and improved the results of [2].

The following definitions are useful in our main results which are due to in [5].

**Definition 1.4** Let X be a non-empty set of E. Suppose that the map d: $X \times X \to E$
Satisfies the following conditions:

(d1). $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2). $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3). $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is clear that the concept of a cone metric space is more general than that of a metric space.

**Example 1.5** [5] Let $E = \mathbb{R}^2$, $P = \{(x, y) \in E$ such that : $x, y \geq 0\} \subseteq \mathbb{R}^2$, $X = \mathbb{R}$ and

$d: X \times X \rightarrow E$ such that $d(x, y) = (|x - y|, \alpha|x - y|)$, where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

**Definition 1.2.** Let P be a cone in a Banach space E, define partial ordering '$\leq$' with respect to P by $x \leq y$ if and only if $y-x \in P$. We shall write $x<y$ to indicate $x \leq y$ but $x \neq y$ while $x<<y$ will stand for $y-x \in \text{int } P$, where int P denotes the interior of the set P. This cone P is called an order cone.

**Definition 1,3.** Let E be a Banach space and $P \subset E$ be an order cone. The order cone P is called normal if there exists K>0 such that for all $x, y \in E$,

$$0 \leq x \leq y \text{ implies } \|x\| \leq K\|y\|.$$

The least positive number K satisfying the above inequality is called the normal constant of P.

**Definition 1.6** Let (X, d) be a cone metric space .We say that $\{x_n\}$ is

(i) a convergent sequence if for any $c>>0$, there is a natural number N such that for all $n>N$, $d(x_n, x) <<c$, for some fixed x in X .We denote this $x_n \rightarrow x$ (as $n \rightarrow \infty$) .

(ii) a Cauchy sequence if for every c in E with $c>>0$, there is a natural number N such that for all $n, m>N$, $d(x_n, x_m)<<c$ .

A cone metric space (X, d) is said to be complete if every Cauchy sequence is convergent.

**Definition 1.7[1].** Let f and g be self-mappings of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called a point of coincidence of f and g.

**Proposition 1.8 [3].** Let f and g be Occasionally Weakly Compatible (OWC ) self-mappings of a set X if and only if there is a point x in X which is coincidence point of f and g at which f and g are commute.

**Lemma 1.1** [3] . Let X be a set f, g are OWC self-mappings of X. If f and g have a unique point of coincidence $w = fx = gx$, then w is the unique common fixed point of f and g.

## 2. Main Results

In this section, we prove fixed point theorems, which are generalized and improved the results of [2].

**Theorem 2.1**: Let (X, d) be a cone metric space and p be a normal cone. Suppose that f and g are OWC self-mappings of X and satisfy the following conditions.

(i) $fX \subset gX$

(ii) $d(fx, fy) < \max\{d(gx, gy), [d(gx, fx) + d(gy, fy) / 2] , [d(gy, fx) + d(gx, fy) / 2] \}$,

for all $x \neq y \in X$.

Then f and g have a unique common fixed point.

**Proof :** By the given assumption that there exists a point $x \in X$ such that $fx = gx$. And suppose that there exists a point $y \in X$ for which $fy = gy$. Then from (ii)

$$d(fx, fy) < \max\{d(fx, fy), [d(fx, fx) + d(fy, fy) / 2 ], [d(fy, fx) + d(fx, fy) / 2] \},$$

$$< \max\{d(fx, fy), 0 , d(fx, fy) \},$$

$$= d(fx,fy) < d(fx,fy),$$

which is a contradiction.

Therefore , $fx = fy$ , and $fx$ is unique. From Lemma 1.1, we get that f and g have a unique common fixed point.

**Theorem 2.2** : Let (X, d) be a cone metric space and p be a normal cone. Suppose that f ,g , S and T are self-mappings of X and that the pairs {f, S} and {g, T}are OWC. If

( i) $d(fx, gy) < \max\{d(Sx, Sy), [d(Sx, Sy) + d(Ty, fy) / 2] , [d(Sx, gy) + d(Ty, fx) / 2 ]\}$ for each $x, y \in X$ for which $fx \neq gy$, then f,g,S and T have a unique common fixed point.

**Proof:** Given {f, S} and {g, T}are OWC, then there exists $x,y \in X$ such that $fx = Sx$ and $gy = Ty$.

We claim that $fx = gy$. For otherwise by ( i )

$$d(fx, gy) < \max\{d(Sx, Ty), [d(Sx, fx) + d(Ty, gy) / 2] , [d(Sx, gy) + d(Ty, fx) / 2 ]\}$$

since, $fx = gx = w$ and $gy = Ty = z$ are points of coincidence of {f, S} and {g, T} respectively.

$$d(fx, gy) < \max\{d(fx, gy), [d(fx, fx) + d(gy, gy) / 2] , [d(fx, gy) + d(gy, fx) / 2 ]\}$$

$$< \max\{d(fx, gy), 0, d(fx, gy) \} < d(f, gy),$$

which is a contradiction.

Therefore, $fx = gy$ , that is $fx = Sx = gy = Ty$.

Moreover, if there is another point z such that $fz = gz$, then by (i), we get that

$fz = Sz = gy = Ty$ (or) $fx = fz$.

Therefore, $fx = Sx = w$ is the unique point of coincidence of f and S.

By the Lemma 1.1. , w is the unique common fixed point of f and S. Similarly, there is a unique point $z \in X$ such that $gz = Tz = z$.

Now we prove uniqueness: suppose $w \neq z$. Then by ( i ) we get that

$$d(w, z) = d(fw, fz) < \max\{d(Sw, Tz), [d(Sw, fw) + d(Tz, gz) / 2] , [d(Sw, gz) + d(Tz, fw) / 2 ]\}$$

$$= \max\{d(w, z), [d(w,fw) + d(z, z) / 2] , [d(w,z) + d(z, w) / 2 ]\}$$

$$= \max\{d(w, z), d(w, z) \} = d(w, z) < d(w, z),$$ which is a contradiction.

Therefore, $w = z$ and w is the unique common fixed point of f, g, S, and T.

Hence proved.

**Remark:** Our results are more general than the results of [2] and we have improved and extended the results of [2].

**References:**


[1] M. Abbas and G. Jungck, Common fixed point results for non commuting mappings without continuity in cone metric spaces, J. Math. Anal. Appl. 341(2008) 416-420.

[2] Amari and Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Appl., 270(2002), 181-188.

[3] Arvind Bhatt and Harish Chandra, Occasionally weakly compatible mappings in cone metric space , Applied Mathematical Sciences, Vol. 6, 2012, no. 55, 2711 – 2717.

[4] Guangxing Song, Xiaoyan Sun, Yian Zhao, Guotao Wang, New common fixed point theorems for maps on cone metric spaces, Appl. Math. Lett. 32(2010)1033-1037.

[5] G. Junck, Compatible mappings and common fixed points, Internat . J. Math. Sci.9(1986) 771-779.

[6] G. Junck and B.E.Rhoads, Fixed point theorems for occasionally weakly compatible mappings, Fixed point theory, 7(2006), 286- 296.

[7] G. Junck and B.E.Rhoads, Fixed point theorems for occasionally weakly compatible mappings, Erratum Fixed point theory, 9(2008), 383- 384.

[8] L.G. Huang, X. Zhang, Cone metric spaces and fixed point theorems of contractive mappings J. Math. Anal. Appl. 332(2)(2007) 1468-1476.