Short Research Article

Average Clustering Coefficient of the Products of Complete Graphs

Abstract

The average clustering coefficient $Cc(G)$ of a connected graph G of order at least 3 is a metric that somehow measures how close G to being a complete graph. Its value ranges from 0 to 1. In this paper, we will show that for the tensor product $K_m \times K_m$ and cartesian product $K_m \square K_m$, $Cc(K_m \times K_m)$ and $Cc(K_m \square K_m)$ approach to 1 and $\frac{1}{2}$, respectively, as $m \to \infty$.

Keywords: clustering coefficient; tensor product; cartesian product; regular graphs

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1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ be the open neighborhood of a vertex $v \in V(G)$, $\deg_G v$ the degree of v, and $t_G(v) = |E(\langle N_G(v)\rangle)|$ the number of triangles in G which are incident to v. The *local clustering coefficient* of vertex v in G, denoted by $Cc_v(G)$, is a measure that evaluates the local triangle density of G at the level of vertex v. This number $Cc_v(G)$ can be defined as

$$Cc_v(G) = \begin{cases} 0, & \text{if } \deg_G v \leq 1, \\ \frac{t_G(v)}{\binom{\deg_G v}{2}}, & \text{if } \deg_G v \geq 2. \end{cases} \quad (1.1)$$

This formula is a unifying version between its treatment in [13] and [10]. On the other hand, the *average clustering coefficient* $Cc(G)$ of a graph G with order n is a measure that indicates the overall clustering of G, obtained by averaging the local clustering coefficients of all the vertices in G. That is, $Cc(G) = 0$ if $\deg_G v \leq 1$ for each $v \in V(G)$; otherwise,

$$Cc(G) = \frac{1}{n} \sum_{\substack{v \in V(G) \\ \deg_G v \geq 2}} Cc_v(G) = \frac{1}{n} \sum_{\substack{v \in V(G) \\ \deg_G v \geq 2}} \frac{2t_G(v)}{\deg_G v(\deg_G v - 1)}. \tag{1.2}$$

This measure was introduced in the field of social network analysis by Duncan J. Watts and Steven Strogatz [14] in 1998, where one of its goals was to determine whether a graph was a "small-world network". Since then, several studies from various perspectives have also emerged. In [3], the authors gave some expressions and bounds for the average clustering coefficient of the tensor product of graphs, although a related study on finding the number of distinct triangles in the tensor product $G \times H$ was done in [4], while a triangle-counting algorithm for large networks appeared in [12].

In this paper, we investigate the average clustering coefficients of the tensor and cartesian product of complete graphs using some properties that the tensor and cartesian product hold and some inherent characteristics possessed by the complete graphs K_m such as in the observation given below. This work falls within the general motivation of investigating graphs under some binary operations and expressing some of their parameterized values in terms of some relevant invariants of the constituent graphs such as the ones done in [1, 3, 5, 6, 7, 4, 8, 11]. For basic graph theory terminologies not specifically described nor defined in this paper, please refer to either [2] or [9].

Lemma 1.1 *For the complete graph K_m,*

$$Cc(K_m) = \begin{cases} 1 & \text{if } m \geq 3, \\ 0 & \text{if } m = 1, \ 2. \end{cases}$$

Proof: This is immediate from Equation (1.2). ■

2 Tensor Product of Complete Graphs

The *tensor product* $G \times H$ of two graphs G and H is the graph with vertex set $V(G \times H) = V(G) \times V(H)$ and edge set $E(G \times H)$ satisfying the following adjacency condition: $(u, v)(u', v') \in E(G \times H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$. A *regular graph* is a graph that has uniform degree in its vertices. If G is a regular graph with degree d in all its vertices, then we call G a d-regular graph. In [3], Damalerio and Eballe gave a formula for the average clustering coefficient of the tensor product of regular graphs in terms of the average clustering coefficient of each factor.

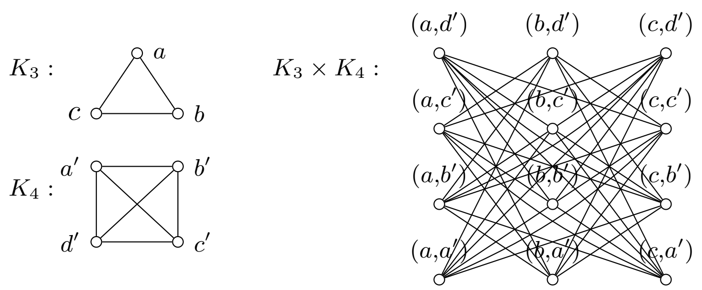

Figure 1: The complete graphs K_3, K_4, and their tensor product $K_3 \times K_4$.

Lemma 2.1 *[3] Let G and H be graphs with orders n_1 and n_2, respectively. If G and H are regular graphs with regularities $d_G \geq 2$ and $d_H \geq 2$, respectively, then*

$$Cc(G \times H) = f \cdot Cc(G) \cdot Cc(H),$$

where $f = (d_G - 1)(d_H - 1)/(d_G \cdot d_H - 1)$.

Proof: See Theorem 3.1 of [3]. ■

Theorem 2.2 *For the tensor product $K_m \times K_n$, where $m,n \geq 3$,*

$$Cc(K_m \times K_n) = \frac{mn - 2m - 2n + 4}{mn - m - n}.$$

Proof: Note that K_m and K_n are regular graphs with regularities $m-1$ and $n-1$, respectively. Using Lemma 1.1 and Lemma 2.1, we have

$$\begin{aligned} Cc(K_m \times K_n) =& \frac{(m-2)(n-2)}{(m-1)(n-1)-1} \cdot Cc(K_m) \cdot Cc(K_n) \\ =& \frac{mn - 2m - 2n + 4}{mn - m - n} \cdot 1 \cdot 1. \end{aligned}$$ ■

In the next result, we give an asymptotic value to $Cc(K_m \times K_n)$, where $n \geq 3$ is held constant and $m \to \infty$.

Corollary 2.3 *If $n \geq 3$ is considered constant, then $Cc(K_m \times K_n) \to \frac{n-2}{n-1}$ as $m \to \infty$.*

Proof: Using Theorem 2.2, we have

$$\lim_{m \to \infty} Cc(K_m \times K_n) = \lim_{m \to \infty} \frac{n - 2 - \frac{2n}{m} + \frac{4}{m}}{n - 1 - \frac{n}{m}} = \frac{n-2}{n-1}.$$ ■

Corollary 2.4 *For the tensor product $K_m \times K_m$, $Cc(K_m \times K_m) \to 1$ as $m \to \infty$.*

Proof: Using Theorem 2.2, we have

$$\lim_{m \to \infty} Cc(K_m \times K_m) = \lim_{m \to \infty} \frac{m^2 - 4m + 4}{m^2 - 2m} = \lim_{m \to \infty} \frac{1 - \frac{4}{m} + \frac{4}{m^2}}{1 - \frac{2}{m}} = 1.$$ ■

Actually, the values of $Cc(K_m \times K_m) = \frac{m^2-4m+4}{m^2-2m}$ can be shown to be strictly increasing. The next result gives an asymptotic value to $Cc(K_m \times K_n)$, wherein both orders of the complete graphs, m and n, approach positive infinity.

Corollary 2.5 *For tensor product $K_m \times K_n$, $Cc(K_m \times K_n) \to 1$ as both m, $n \to \infty$.*

Proof: Using Theorem 2.2, we have

$$\lim_{m,n \to \infty} Cc(K_m \times K_n) = \lim_{m,n \to \infty} \frac{mn - 2m - 2n + 4}{mn - m - n} = \lim_{m,n \to \infty} \frac{1 - \frac{2}{n} - \frac{2}{m} + \frac{4}{mn}}{1 - \frac{1}{n} - \frac{1}{m}} = 1.$$ ■

3 Cartesian Product of Complete Graphs

Recall that the *cartesian product* $G\mathsf{X}H$ of two graphs G and H is the graph with vertex set $V(G\mathsf{X}H) = V(G) \times V(H)$ and edge-set $E(G\mathsf{X}H)$ satisfying the following conditions: $(u,v)(u',v') \in E(G\mathsf{X}H)$ if and only if either $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$.

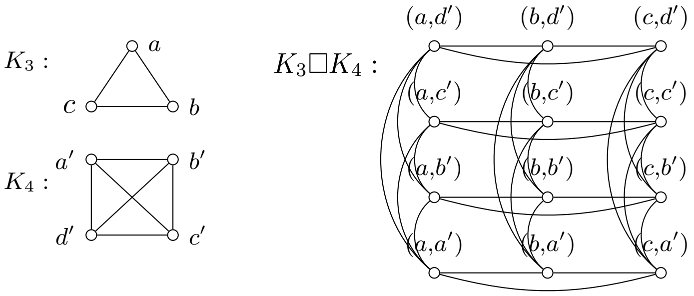

Figure 2: The complete graphs K_3, K_4, and their cartesian product $K_3 \Box K_4$.

Lemma 3.1 *Let G and H be graphs with orders n_1 and n_2, respectively, and sizes m_1 and m_2, respectively. If $u \in V(G)$ and $v \in V(H)$, then the following properties hold.*

1. $\deg_{G\mathsf{X}H}(u,v) = \deg_G u + \deg_H v$,
2. $|E(G\mathsf{X}H)| = n_1 m_2 + n_2 m_1$,
3. $t_{G\mathsf{X}H}(u,v) = t_G(u) + t_H(v)$.

Proof: The proofs of (1) and (2) above follow directly from the definition of the cartesian product of graphs. As for (3), let $(u,v) \in V(G\mathsf{X}H)$ such that $(u',v')(u'',v'') \in E(\langle N_{G\mathsf{X}H}(u,v)\rangle)$. Then vertices (u,v), (u',v'), $(u'',v'') \in V(G\mathsf{X}H)$ are pairwise adjacent in $G\mathsf{X}H$.

Case 1: Suppose that $uu' \in E(G)$. This means that $v = v'$ is a must. Since (u,v) and (u'',v'') are adjacent in $G\mathsf{X}H$, we must have either $u = u''$ and $vv'' \in E(H)$, or $uu'' \in E(G)$ and $v = v''$. But the option $u = u''$ and $vv'' \in E(H)$, together with the observation that $(u',v) = (u',v')$ is adjacent to $(u'',v'') = (u,v'')$ in $G\mathsf{X}H$, leads to $u' = u$, which is imposible since we already have $uu' \in E(G)$. As a consequence, we only have $uu'' \in E(G)$ and $v = v''$. Since $v = v'$, we now have $v = v' = v''$ and, hence, $u'u'' \in E(G)$, so that $\langle\{u, u', u''\}\rangle$ is a triangle incident with vertex u in G. Thus, in this case, a particular triangle in $G\mathsf{X}H$ incident with $(u,v) \in V(G\mathsf{X}H)$ clearly shows a unique triangle in G incident with $u \in V(G)$.

Case 2: In this case we have $uu' \notin E(G)$ and hence we have $u = u'$ and $vv' \in E(H)$. Since $G \Box H \simeq H\mathsf{X}G$, we can apply the argument used in Case 1 above to a starting premise that $vv' \in E(H)$, producing a similar conclusion that in this case, a particular triangle in $G\mathsf{X}H$ incident with $(u,v) \in V(G\mathsf{X}H)$ would show a unique triangle in H incident with $v \in V(H)$.

The two cases together imply that every triangle in $G\mathsf{X}H$ incident with the vertex $(u,v) \in V(G\mathsf{X}H)$ is attributable either to a unique triangle in G incident with $u \in V(G)$ or to a unique triangle in H incident with $v \in V(H)$.

Conversely, every triangle in G(or in H) incident with $u \in V(G)$ (or with $v \in V(H)$) produces a corresponding unique triangle in $G\mathsf{X}H$ incident with (u,y) for every $y \in V(H)$ (or with (x,v) for every $x \in V(G)$). Finally, we can now conclude that $t_{G\mathsf{X}H}(u,v) = t_G(u) + t_H(v)$. ■

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Theorem 3.2 *If $u \in V(G)$ and $v \in V(H)$ such that $\deg_G u \geq 2$ and $\deg_H v \geq 2$, then the local clustering coefficient of (u,v) in $G \times H$ is given by the formula*

$$Cc_{(u,v)}(G \times H) = p(u,v) \cdot Cc_u(G) + q(u,v) \cdot Cc_v(H),$$

where $p(u,v) = (\deg_G u)(\deg_G u - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$ *and* $q(u,v) = (\deg_H v)(\deg_H v - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$.

Proof: Using Equation (1.1) and Lemma 3.1(3), we have

$$\begin{aligned}
Cc_{(u,v)}(G \Box H) &= \frac{t_{G\Box H}(u,v)}{\binom{\deg_{G\Box H}(u,v)}{2}} = \frac{t_G(u) + t_H(v)}{\binom{\deg_G u + \deg_H v}{2}} \\
&= \frac{Cc_u(G)\binom{\deg_G u}{2} + Cc_v(H)\binom{\deg_H v}{2}}{\binom{\deg_G u + \deg_H v}{2}} \\
&= \frac{Cc_u(G)\deg_G u(\deg_G u - 1) + Cc_v(H)\deg_H v(\deg_H v - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} \\
&= \frac{Cc_u(G)\deg_G u(\deg_G u - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} + \frac{Cc_v(H)\deg_H v(\deg_H v - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)},
\end{aligned}$$

and the claimed formula follows. ■

The next result, which is for $Cc(G \Box H)$, is a consequence of Theorem 3.2.

Corollary 3.3 *Let G and H be graphs of orders n_1 and n_2, respectively. Suppose $\delta(G) \geq 2$ and $\delta(H) \geq 2$. Then the average clustering coefficient of $G \times H$ is given by the expression*

$$Cc(G \Box H) = \frac{1}{n_1 n_2} \sum_{u \in V(G)} \sum_{v \in V(H)} p(u,v) \cdot Cc_u(G) + q(u,v) \cdot Cc_v(H),$$

where $p(u,v) = (\deg_G u)(\deg_G u - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$ *and* $q(u,v) = (\deg_H v)(\deg_H v - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$.

Proof: Using Equation (1.2) and Theorem 3.2, we obtain

$$\begin{aligned}
Cc(G \Box H) =& \frac{1}{n_1 n_2} \sum_{(u,v) \in V(G \times H)} Cc_{(u,v)}(G \Box H) \\
=& \frac{1}{n_1 n_2} \sum_{u \in V(G)} \sum_{v \in V(H)} \left(\frac{Cc_u(G)(\deg_G u)(\deg_G u - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} \right. \\
&\left. + \frac{Cc_v(H)(\deg_H v)(\deg_H v - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} \right).
\end{aligned}$$

The claimed formula follows. ■

Theorem 3.4 *Let G and H be graphs with orders n_1 and n_2, respectively. If G and H are regular graphs with regularities $d_G \geq 2$ and $d_H \geq 2$, respectively, then*

$$Cc(G \Box H) = p \cdot Cc(G) + q \cdot Cc(H),$$

where $p = d_G(d_G - 1)/[(d_G + d_H)(d_G + d_H - 1)]$ *and* $q = d_H(d_H - 1)/[(d_G + d_H)(d_G + d_H - 1)]$.

Proof: Using Equation (1.2), Corollary 3.3, and the fact that $G \times H$ is also a regular graph with regularity $d_{G \times H} = d_G + d_H$ from Lemma 3.1(1), we have

$$\begin{aligned} Cc(G \Box H) =& \frac{1}{n_1 n_2} \sum_{u \in V(G)} \sum_{v \in V(H)} \left(\frac{Cc_u(G) d_G(d_G - 1)}{(d_G + d_H)(d_G + d_H - 1)} + \frac{Cc_v(H) d_H(d_H - 1)}{(d_G + d_H)(d_G + d_H - 1)} \right) \\ =& \frac{1}{n_1 n_2} \left(\left(\frac{n_2 d_G(d_G - 1)}{(d_G + d_H)(d_G + d_H - 1)} \sum_{u \in V(G)} Cc_u(G) \right) \right. \\ & \left. + \left(\frac{n_1 d_H(d_H - 1)}{(d_G + d_H)(d_G + d_H - 1)} \sum_{v \in V(H)} Cc_v(H) \right) \right) \\ =& \frac{d_G(d_G - 1)}{(d_G + d_H)(d_G + d_H - 1)} \cdot Cc(G) + \frac{d_H(d_H - 1)}{(d_G + d_H)(d_G + d_H - 1)} \cdot Cc(H), \end{aligned}$$

and the claimed formula holds. ■

Theorem 3.5 *For the cartesian product* $K_m \times K_n$*, where* $m, n \geq 3$*,*

$$Cc(K_m \Box K_n) = \frac{m^2 - 3m + n^2 - 3n + 4}{m^2 - 5m + 2mn + n^2 - 5n + 6}.$$

Proof: Given that K_m is $(m-1)$-regular, K_n is $(n-1)$-regular, and $Cc(K_m) = Cc(K_n) = 1$, Theorem 3.4 asserts that

$$\begin{aligned} Cc(K_m \times K_n) =& \frac{(m-1)(m-1)}{(m+n-2)(m+n-3)} + \frac{(n-1)(n-2)}{(m+n-2)(m+n-3)} \\ =& \frac{(m-1)(m-1) + (n-1)(n-2)}{(m+n-2)(m+n-3)} \\ =& \frac{m^2 - 3m + n^2 - 3n + 4}{m^2 - 5m + 2mn + n^2 - 5n + 6}, \end{aligned}$$

which completes the proof. ■

The next result gives an asymptotic value to $Cc(K_m \times K_m)$ as $m \to \infty$.

Corollary 3.6 *For the cartesian product* $K_m \times K_m$*,* $Cc(K_m \times K_m) \to \frac{1}{2}$ *as* $m \to \infty$*.*

Proof: Using Corollary 3.5, we have

$$\lim_{m \to \infty} Cc(K_m \times K_m) = \lim_{m \to \infty} \frac{2m^2 - 6m + 4}{4m^2 - 10m + 6} = \lim_{m \to \infty} \frac{2 - \frac{6}{m} + \frac{4}{m^2}}{4 - \frac{10}{m} + \frac{6}{m^2}} = \frac{1}{2}.$$

■

4 Final Remarks

In this paper we were able to generate some useful formulas for the average clustering coefficient of the tensor product $K_m \times K_n$ and the cartesian product $K_m \times K_n$. It was interesting to see that as m increases without bound, $Cc(K_m \times K_m) \to 1$ while $Cc(K_m \times K_m) \to \frac{1}{2}$.

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