Original Research Article
S-INDEX OF DIFFERENT GRAPH OPERATIONS

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ABSTRACT

In this paper, we introduced new index from the Zagreb index family, named as 'S-index'. We derive the S-index of some different graph operations such that Join, Cartesian product, Composition, Corona product, Tensor product, Strong product, Disjunction, Symmetric difference, Corona join product, Subdivision vertex join are obtained.

Keywords: Zagreb Indices, F-index, S-index, Graph operations

1. INTRODUCTION

Topological indices are a numeric value which is associated with a chemical structure of a particular chemical compound. Graph operations play a important role in chemical mathematics, since some chemically interesting graphs can be derived from some simpler graphs by different graph operations.

Let G be a connected graph with vertex set $V(G)$ and edge set $E(G)$, respectively. The degree of a vertex v is the number of edges incident to v and is denoted by $\gamma_G(v)$.

The first general Zagreb index is defined as:

$$M_1^{\alpha+1}(G) = \sum_{v \in V(G)} \gamma_G^{\alpha+1}(v) = \sum_{uv \in E(G)} \gamma_G^{\alpha}(u) + \gamma_G^{\alpha}(v)$$

The first and second Zagreb index is defined as:

$$M_1(G) = \sum_{v \in V(G)} \gamma_G(v)^2 = \sum_{uv \in E(G)} [\gamma_G(u) + \gamma_G(v)]$$

$$M_2(G) = \sum_{uv \in E(G)} \gamma_G(u)\gamma_G(v)$$

The F-index is defined as:

$$F(G)=\sum_{v\in V(G)}\gamma_G(v)^3=\sum_{uv\in E(G)}\left[\gamma_G(u)^2+\gamma_G(v)^2\right]$$

The Y-index is defined as:

$$Y(G)=\sum_{v\in V(G)}\gamma_G(v)^4=\sum_{uv\in E(G)}\left[\gamma_G(u)^3+\gamma_G(v)^3\right]$$

The S-index of a graph G is defined as:

$$S(G)=\sum_{v\in V(G)}\gamma_G(v)^5=\sum_{uv\in E(G)}[\gamma_G(u)^4+\gamma_G(v)^4]$$

2. MAIN RESULTS

In this section, we study about the S-index of some different graph operations.

Join:

The join $G+H$ of graphs G and H with vertex sets $V(G)$ and $V(H)$ and edge sets $E(G)$ and $E(H)$ is the graph union $G\cup H$ together with all the edges between $V(G)$ and $V(H)$. Obviously,

$$|V(G+H)|=|V(G)|+|V(H)|$$

$$|E(G+H)|=|E(G)|+|E(H)|+|V(G)||V(H)|$$

Theorem : 2.1

The S-index of $G+H$ is given by

$$S(G+H)=S(G)+S(H)+j_1\left(2k_2^4+k_1^4\right)+Y(G)k_2+k_2^5k_1+4\left[F(G)k_2^2+F(H)k_1^2\right]+6\left[M_1(G)k_2^3+M_1(H)k_1^3\right]$$
$$+8\left[k_2^4j_1+k_1^4j_2\right]+Y(H)k_1+k_1^5k_2.$$

Proof:

From definition of S-index, we have

$$S(G+H)=\sum_{uv\in E(G+H)}\left[\gamma_{G+H}(u)^4+\gamma_{G+H}(v)^4\right]$$
$$=\sum_{uv\in E(G)}\left[\gamma_{G+H}(u)^4+\gamma_{G+H}(v)^4\right]+\sum_{uv\in E(H)}\left[\gamma_{G+H}(u)^4+\gamma_{G+H}(v)^4\right]$$
$$+\sum_{u\in V(G)}\sum_{v\in V(H)}\left[\gamma_{G+H}(u)^4+\gamma_{G+H}(v)^4\right]$$

$$= \sum_{uv \in E(G)} \left[\gamma_G(u)^4 + \gamma_H(v)^4 + 2k_2^4 \right] + \sum_{uv \in E(H)} \left[\gamma_G(u)^4 + \gamma_H(v)^4 + 2k_1^4 \right] + \sum_{u \in V(G)} \sum_{v \in V(H)} \left[(\gamma_G(u) + k_2)^4 + (\gamma_H(v) + k_1)^4 \right]$$

$$S(G+H) = S(G) + S(H) + j_1(2k_2^4 + k_1^4) + Y(G)k_2 + k_2^5 k_1 + 4\left[F(G)k_2^2 + F(H)k_1^2\right] + 6\left[M_1(G)k_2^3 + M_1(H)k_1^3\right] + 8\left[k_2^4 j_1 + k_1^4 j_2\right] + Y(H)k_1 + k_1^5 k_2.$$

which is complete the proof.

Cartesian Product:

The Cartesian Product $G \times H$ of graphs G and H has the vertex set $V(G \times H) = V(G) \times V(H)$ and $(u, x)(v, y)$ is an edge of $G \times H$ if $uv \in E(G)$ and $x = y$ or $u = v$ and $xy \in E(H)$ Obviously, $|V(G \times H)| = |V(G)||V(H)|$ and $|E(G \times H)| = |E(G)||V(H)| + |V(G)||E(H)|$.

Theorem : 2.2

The S-index of $G \times H$ is given by

$$S(G \times H) = k_2 S(G) + k_1 S(H) + 10Y(G) j_2 + 10F(G)M_1(H) + 10M_1(G)F(H) + 10 j_1 Y(H).$$

Proof:

From definition of S-index, we have

$$S(G \times H) = \sum_{(u,v) \in V(G \times H)} \gamma_{G \times H}(u,v)^5$$

$$= \sum_{u \in V(G)} \sum_{v \in V(H)} \left[\gamma_G(u) + \gamma_H(v) \right]^5$$

$$S(G \times H) = k_2 S(G) + k_1 S(H) + 10Y(G) j_2 + 10F(G)M_1(H) + 10M_1(G)F(H) + 10 j_1 Y(H).$$

which is complete the proof.

Composition

The Composition $G[H]$ of graphs G and H with disjoint vertex sets $V(G)$ and $V(H)$ and edge set $E(G)$ and $E(H)$ is the graph with vertex set $V(G)\times V(H)$ and $u=(u_1,v_1)$ is adjacent to $v=(u_2,v_2)$ whenever u_1 is adjacent to u_2 or $u_1=u_2$ and v_1 is adjacent to v_2

$$\gamma_{G[H]}(a,b)=k_2\gamma_G(a)+\gamma_H(b).$$

Theorem: 2.3.

The S-index of $G[H]$ is given by

$$S(G[H])=k_2^6S(G)+k_1S(H)+10k_2^3F(G)M_1(H)+10k_2^2F(H)M_1(G)+10k_2j_1Y(H)+10k_2^4j_2Y(G).$$

Proof:

From definition of S-index, we have

$$S(G[H])=\sum_{(u,v)\in V(G[H])}\gamma_{G[H]}(u,v)^5$$

$$=\sum_{u\in V(G)}\sum_{v\in V(H)}[k_2\gamma_G(u)+\gamma_H(v)]^5$$

$$S(G[H])=k_2^6S(G)+k_1S(H)+10k_2^3F(G)M_1(H)+10k_2^2F(H)M_1(G)+10k_2j_1Y(H)+10k_2^4j_2Y(G).$$

which completes the proof.

Tensor Product

The Tensor Product $G\otimes H$ of graphs G and H has the vertex set $V(G\otimes H)=V(G)\times V(H)$ and $(u,x)(v,y)$ is an edge of $G\otimes H$ if $uv\in E(G)$ and $xy\in E(H)$. Obviously, $|V(G\otimes H)|=|V(G)||V(H)|, |E(G\otimes H)|=2|E(G)||E(H)|$ and

$$\gamma_{G\otimes H}(u,x)=\gamma_G(u)\gamma_H(x).$$

Theorem: 2.4.

The S-index of $G \otimes H$ is given by

$$S(G \otimes H) = S(G)S(H).$$

Proof:

From definition of S-index, we have

$$S(G \otimes H) = \sum_{(u,v)\in V(G\otimes H)} \gamma_{G\otimes H}(u,v)^5$$

$$= \sum_{u\in V(G)} \sum_{v\in V(H)} (\gamma_G(u)\gamma_H(v))^5$$

$$S(G \otimes H) = S(G)S(H).$$

which complete the proof.

Disjunction :

The Disjunction $G \vee H$ of a graphs G and H is the graph with vertex set $V(G)\times V(H)$ and u_1v_1 is adjacent with u_2v_2 whenever $u_1u_2 \in E(G)$ and $v_1v_2 \in E(H)$

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Theorem: 2.5:

The S-index of $G \vee H$ is given by

$$S(G \vee H) = k_2^5 S(G) + k_1^5 S(H) - S(G)S(H) + 10k_2^4 k_1 Y(G) j_2 - 10k_2^4 S(G) j_2 + 10k_2^3 F(G) k_1^2 M_1(H)$$
$$10k_2^3 S(G) M_1(H) + 10k_2^2 M_1(G) k_1^3 F(H) + 10k_2 j_1 k_1^4 Y(H) - 10k_1^4 S(H) j_1 + 10k_1^3 S(H) M_1(G)$$
$$-10k_2^2 S(G) F(H) - 10k_1^2 S(H) F(G) - 20k_2^3 Y(G) k_1 M_1(H) - 20k_2 k_1^3 M_1(G) Y(H) - 20k_2 k_1 Y(G) Y(H)$$
$$+5k_2 S(G) Y(H) + 5k_1 S(H) Y(G) + 30k_2^2 k_1 Y(G) F(H) + 30k_2 k_1^2 F(G) Y(H) - 30k_1^2 k_2^2 F(G) F(H).$$

Proof:

From definition of S-index, we have

$$S(G \vee H) = \sum_{(u_1,u_2)\in V(G\vee H)} \gamma_{G\vee H}(u_1,u_2)^5$$

$$= \sum_{u_1 \in V(G)} \sum_{u_2 \in V(H)} [k_2 \gamma_G(u_1) + k_1 \gamma_H(u_2) - \gamma_G(u_1)\gamma_G(u_2)]^5$$

$$S(G \vee H) = k_2^5 S(G) + k_1^5 S(H) - S(G)S(H) + 10k_2^4 k_1 Y(G) j_2 - 10k_2^4 S(G) j_2 + 10k_2^3 F(G) k_1^2 M_1(H)$$
$$10k_2^3 S(G) M_1(H) + 10k_2^2 M_1(G) k_1^3 F(H) + 10k_2 j_1 k_1^4 Y(H) - 10k_1^4 S(H) j_1 + 10k_1^3 S(H) M_1(G)$$
$$-10k_2^2 S(G) F(H) - 10k_1^2 S(H) F(G) - 20k_2^3 Y(G) k_1 M_1(H) - 20k_2 k_1^3 M_1(G) Y(H) - 20k_2 k_1 Y(G) Y(H)$$
$$+5k_2 S(G) Y(H) + 5k_1 S(H) Y(G) + 30k_2^2 k_1 Y(G) F(H) + 30k_2 k_1^2 F(G) Y(H) - 30k_1^2 k_2^2 F(G) F(H).$$

which completes the proof.

Symmetric Difference :

The Symmetric Difference $G \oplus H$ of two graphs G and H is a graph with vertex set $V(G) \times V(H)$ and

$$E(G \oplus H) = \{(u_1, u_2)(v_1, v_2) / u_1 v_1 \in E(G) or u_2 v_2 \in E(H) but \ \ not \ \ both\}$$

Theorem :2.6.

The S-index of $G \oplus H$ is given by

$$S(G \oplus H) = k_2^5 S(G) + k_1^5 S(H) - 32S(G)S(H) + 10k_2^4 k_1 Y(G) j_2 - 20k_2^4 S(G) j_2 + 10k_2^3 F(G) k_1^2 M_1(H) +$$
$$40k_2^3 S(G) M_1(H) + 10k_2^2 M_1(G) k_1^3 F(H) + 10k_2 j_1 k_1^4 Y(H) - 20k_1^4 S(H) j_1 + 40k_1^3 S(H) M_1(G)$$
$$-80k_2^2 S(G) F(H) - 80k_1^2 S(H) F(G) - 40k_2^3 Y(G) k_1 M_1(H) - 40k_2 k_1^3 M_1(G) Y(H) - 160k_2 k_1 Y(G) Y(H)$$
$$+80k_2 S(G) Y(H) + 80k_1 S(H) Y(G) + 120k_2^2 k_1 Y(G) F(H) + 120k_2 k_1^2 F(G) Y(H) - 60k_1^2 k_2^2 F(G) F(H).$$

Proof:

From definition of S-index, we have

$$S(G \oplus H) = \sum_{(u_1, u_2) \in V(G \oplus H)} \gamma_{G \oplus H}(u_1, u_2)^5$$

$$= \sum_{u_1 \in V(G)} \sum_{u_2 \in V(H)} [k_2 \gamma_G(u_1) + k_1 \gamma_H(u_2) - \gamma_G(u_1)\gamma_G(u_2)]^5$$

$$S(G\oplus H)=k_2^5S(G)+k_1^5S(H)-32S(G)S(H)+10k_2^4k_1Y(G)j_2-20k_2^4S(G)j_2+10k_2^3F(G)k_1^2M_1(H)+$$
$$40k_2^3S(G)M_1(H)+10k_2^2M_1(G)k_1^3F(H)+10k_2j_1k_1^4Y(H)-20k_1^4S(H)j_1+40k_1^3S(H)M_1(G)$$
$$-80k_2^2S(G)F(H)-80k_1^2S(H)F(G)-40k_2^3Y(G)k_1M_1(H)-40k_2k_1^3M_1(G)Y(H)-160k_2k_1Y(G)Y(H)$$
$$+80k_2S(G)Y(H)+80k_1S(H)Y(G)+120k_2^2k_1Y(G)F(H)+120k_2k_1^2F(G)Y(H)-60k_1^2k_2^2F(G)F(H).$$

which completes the proof.

Strong Product

The Strong Product $G*H$ of a graph G and H is a graph with vertex set $V(G)\times V(H)$ and any two vertices (u_p,v_r) and (u_q,v_s) are adjacent if and only if if $\lfloor u_p=u_q and v_rv_s\in E(H)\rfloor$ or $\lfloor v_r=v_s and u_pu_q\in E(H)\rfloor$ or $\lfloor u_pu_q\in E(G) and v_rv_s\in E(H)\rfloor$.

$$\gamma_{G*H}(a,b)=\gamma_G(a)+\gamma_H(b)+\gamma_G(a)\gamma_H(b)$$

Theorem: 2.7

The S-index of $G*H$ is given by

$$S(G*H)=S(G)k_2+S(H)k_1+S(G)S(H)+10Y(G)j_2+10S(G)j_2+10j_1Y(H)+10S(H)j_1$$
$$+5S(G)Y(H)+5S(H)Y(G)+10F(G)M_1(H)+20Y(G)M_1(H)+10S(G)M_1(H)+10M_1(G)F(H)$$
$$+20M_1(G)Y(H)+10S(H)M_1(G)+10S(G)F(H)+20Y(G)Y(H)+10S(H)F(G)+30F(G)F(H)$$
$$+30Y(G)M_1(H)+30F(G)Y(H).$$

Proof:

From definition of S-index, we have

$$S(G*H)=\sum_{(u,v)\in V(G*H)}\gamma_{G*H}(u,v)^5$$
$$=\sum_{u\in V(G)}\sum_{v\in V(H)}[\gamma_G(u)+\gamma_H(v)+\gamma_G(u)\gamma_H(v)]^5$$
$$S(G*H)=S(G)k_2+S(H)k_1+S(G)S(H)+10Y(G)j_2+10S(G)j_2+10j_1Y(H)+10S(H)j_1$$
$$+5S(G)Y(H)+5S(H)Y(G)+10F(G)M_1(H)+20Y(G)M_1(H)+10S(G)M_1(H)+10M_1(G)F(H)$$
$$+20M_1(G)Y(H)+10S(H)M_1(G)+10S(G)F(H)+20Y(G)Y(H)+10S(H)F(G)+30F(G)F(H)$$
$$+30Y(G)M_1(H)+30F(G)Y(H).$$

which completes the proof.

Corona Product:

The Corona Product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and $|V(G)|$ copies of H and then joining by an edge each vertex of the i^{th} copy of H is named (H, i) with the i^{th} vertices of G.

$$\gamma_{G \circ H}(v) = \begin{Bmatrix} \gamma_G(v) + k_2, v \in V(G) \\ \gamma_H(v) + 1, v \in V(H) \end{Bmatrix}$$

Theorem : 2.8

The S-index of $G \circ H$ is given by

$$S(G \circ H) = S(G) + k_1 S(H) + 5k_2 Y(G) + 5k_1 Y(H) + 10k_2^2 F(G) + 10k_2^3 M_1(G) + 10k_2^4 j_1 + k_2^4 k_1 + 10k_1 F(H) + 10k_1 M_1(H) + 10k_1 j_2 + k_1 k_2.$$

Proof:

From definition of S-index, we have

$$S(G \circ H) = \sum_{v \in V(G)} [\gamma_G(v) + k_2]^5 + k_1 \sum_{v \in V(H)} [\gamma_H(v) + 1]^5$$

$$S(G \circ H) = S(G) + k_1 S(H) + 5k_2 Y(G) + 5k_1 Y(H) + 10k_2^2 F(G) + 10k_2^3 M_1(G) + 10k_2^4 j_1 + k_2^4 k_1 + 10k_1 F(H) + 10k_1 M_1(H) + 10k_1 j_2 + k_1 k_2.$$

which completes the proof.

Corona join product :

Let $G(k_1, j_1)$ and $H(k_2, j_2)$ be simple connected graphs, and the Corona join graph of G and H is obtained by taking one copy of G , k_1 copies of H , and joining each vertex of the i^{th} copy of H with all vertices of G. The Corona join product of G and H is denoted by $\gamma_{G\oplus H}(v)=\begin{Bmatrix}\gamma_G(v)+k_1k_2, v\in V(G)\\ \gamma_H(v)+k_1, v\in V(H)\end{Bmatrix}$

Theorem : 2.9

The $\mathcal{S}$-index of $G\oplus H$ is given by

$$S(G\oplus H)=S(G)+5Y(G)k_1k_2+10F(G)k_1^2k_2^2+10M_1(G)k_1^3k_2^3+10j_1k_1^4k_2^4+k_1^6k_2^5+k_1S(H)$$
$$+5k_1^2Y(H)+10F(H)k_1^3+10M_1(H)k_1^4+10k_1^5j_2+k_1^6k_2.$$

Proof:

From definition of $\mathcal{S}$-index, we have

$$S(G\oplus H)=\sum_{v\in V(G\oplus H)}\gamma_{G\oplus H}(v)^5$$

$$=\sum_{v\in V(G)}(\gamma_G(v)+k_1k_2)^5+\sum_{v\in V(G)}\sum_{v\in V(H)}(\gamma_H(v)+k_1)^5$$

$$S(G\oplus H)=S(G)+5Y(G)k_1k_2+10F(G)k_1^2k_2^2+10M_1(G)k_1^3k_2^3+10j_1k_1^4k_2^4+k_1^6k_2^5+k_1S(H)$$
$$+5k_1^2Y(H)+10F(H)k_1^3+10M_1(H)k_1^4+10k_1^5j_2+k_1^6k_2.$$

which completes the proof.

Subdivision vertex join :

For $G(k_1, j_1)$ and $H(k_2, j_2)$, the subdivision vertex join is denoted by $G+H$ and is obtained by joining the each new vertex of $S(G)$ to all vertices of H.

$$\gamma_{G+H}(v)=\begin{Bmatrix}\gamma_G(v), v\in V(G)\\ 2+k_2, v\in V_s(G)\\ \gamma_H(v)+j_1, v\in V(H)\end{Bmatrix}$$

Theorem: 2.10

The $\mathcal{S}$-index of $G+H$ is given by

$$S(G+H)=S(G)+(2+k_2)^5 j_1+S(H)+j_1^5 k_2+5Y(H)j_1+10F(H)j_1^2+10M_1(H)j_1^3+10j_2 j_1^4.$$

Proof:

From definition of S-index, we have

$$S(G+H)=\sum_{v\in V(G+H)}\gamma_{G+H}(v)^5$$

$$=\sum_{v\in V(G)}\gamma_G(v)^5+\sum_{v\in V_s(G)}(2+k_2)^5+\sum_{v\in V(H)}(\gamma_H(v)+j_1)^5$$

$$S(G+H)=S(G)+(2+k_2)^5 j_1+S(H)+j_1^5 k_2+5Y(H)j_1+10F(H)j_1^2+10M_1(H)j_1^3+10j_2 j_1^4.$$

which completes the proof.

Example: 2.11

In this part, the S-index of some special graphs are calculated.

A. $S(K_n)=n(n-1)^5, n\geq 3$

B. $S(C_n)=32n \;, n\geq 3$

C. $S(P_n)=32n-62 \;, n\geq 3$

D. $S(S_n)=(n-1)^5+(n-1) \;, n\geq 3$

E. $S(W_n)=243n+n^5 \;, n\geq 3$

F. $S(L_n)=128+(2n-4)243 \;, n\geq 2$

Example: 2.12

$$S(G) = \sum_{v \in V(G)} \gamma_G(v)^5 = 1{,}61{,}314$$

3. CONCLUSION

In this paper, we compute some exact expressions for the S-index of some graph operations such as Join, Cartesian product, Composition, Corona product, Tensor product, Strong product, Disjunction, Symmetric difference, Corona join product, Subdivision vertex join.

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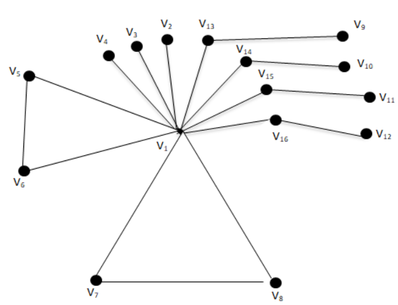

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