Decay estimate for solutions to a semilinear plate equation with memory

Abstract

In this paper we consider the initial value problem of a semilinear plate equation with memory term in $R^n (n \geq 1)$. We study the decay estimates for solutions to the equation in the spirit of reference[1, 2], and obtain the global existence and decay estimates of solutions to the semilinear problem, which generalize the results in reference[1].

Keywords: Decay estimates; Plate equation; Semilinear; Memory term.

1 Introduction

In this paper we consider the initial value problem of the following semilinear plate equation with memory term in R^n ($n \geq 1$) :

$$u_{tt} + \Delta^2 u + u - g * u = f(\partial_x^2 u, u_t) \tag{1.1}$$

with the initial data

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \tag{1.2}$$

Here $u = u(x,t)$ is the unknown function of $x = (x_1, \cdots, x_n) \in R^n$ and $t > 0$, which represents the transversal displacement of the plate at the point x and the time t. The subscript t in u_t and u_{tt} denotes the time derivative $(i.e., u_t = \partial_t u$ and $u_{tt} = \partial_t^2 u)$. The memory term $g * u$ is defined by

$$(g * u)(x,t) := \int_0^t g(t-\tau) u(x,\tau) d\tau ,$$

which means the stress at an instant depends on the whole history of the strains the material has suffered. The memory kernel g is a given function which satisfies the following assumption.

Assumption $[\mathrm{A}]$ **:**

(i) $\quad g \in C^2([0,\infty))$,

(ii) $\quad g(t) > 0, -C_1 g(t) \leq g'(t) \leq -C_2 g(t), |g''(t)| \leq C_3 g(t)$ for t ≥ 0,

(iii) $\quad \int_0^t g(\tau) d\tau \leq 1$ for t ≥ 0,

where, C_j $(j = 1,2,3)$ are positive constants.

The nonlinear term $f(\partial_x^2 u, u_t)$ satisfies the following

Assumption$[\mathrm{B}]$**:** $f(\lambda\partial_x^2 u, \lambda u_t) = \lambda^\alpha f(\partial_x^2 u, u_t), \forall \lambda > 0$,

here α is an integer satisfying $\alpha > \alpha_n$ with $\alpha_n := \begin{cases} 5-n, & n \le 3, \\ 1+\dfrac{2}{n}, & n \ge 4 \end{cases}$.

Denote

$$\sigma(k,n) = 2k + \left[\frac{n+1}{2}\right], n \ge 1, \tag{1.3}$$

then our main theorem can be stated as follows.

Theorem1.1 (Existence and Decay Estimates). Let s be an integer and $s \ge \max\{n+1, 3\}$. Also let $u_0 \in H^{s+2}(R^n) \bigcap L^1(R^n)$ and $u_1 \in H^s(R^n) \bigcap L^1(R^n)$, and put

$$I_1 := \|u_0\|_{H^{s+2}} + \|u_1\|_{H^s} + \|(u_0, u_1)\|_{L^1}.$$

Then there exists a unique solution $u(x,t)$ of (1.1)--(1.2) with

$$u(x,t) \in C^0([0,\infty); H^{s+2}(R^n)) \bigcap C^1([0,\infty); H^s(R^n))$$

and satisfying the following decay estimates:

$$\left\|\partial_x^{k+2} u(t)\right\|_{H^{s-\sigma(k,n)}} \le CI_1(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{1.4}$$

$$\left\|\partial_x^k u_t(t)\right\|_{H^{s-\sigma(k,n)}} \le CI_1(1+t)^{-\frac{n}{8}-\frac{k}{4}}, \tag{1.5}$$

for $k \ge 0$ satisfying $\sigma(k,n) \le s$, here $\sigma(k,n)$ is defined in (1.3).

For the study of the plate equations, many results have been obtained in the literatures. Da Luz and Charão[3] studied the following semilinear dissipative plate equation

$$u_{tt} - \Delta u_{tt} + \Delta^2 u + u_t = f(u). \tag{1.6}$$

They obtained the global existence of solutions and a polynomial decay of the energy by applying an energy method. However the result was confined to the lower spatial dimension $1 \le n \le 5$. This limitation on the spatial dimension was eliminated by Sugitani and Kawashima[4] by using the fundamental method of energy estimates in the Fourier (or frequency) space and some sharp decay estimates. Subsequently, Liu and Kawashima[5,6] studied a more complex inertial model for quasilinear dissipative plate equation whose linear part is presented by

$$u_{tt} - \Delta u_{tt} + \sum_{i,j=1}^n b^{ij}(\partial_x^2 u)_{x_i x_j} + u_t = 0. \tag{1.7}$$

For the case of plate equations with memory term, Liu-Kawashima[7] studied the following semilinear plate equation

$$u_{tt} + \Delta^2 u + u + g * \Delta u = f(u). \tag{1.8}$$

and obtained the global existence and decay estimates of solutions by exploiting the energy method in the Fourier space. Liu[2] studied the following initial value problem of rotational plate equations with memory,

$$u_{tt}-\Delta u_{tt}+\Delta^2 u+u+g*\Delta u=f(u,u_t,\nabla u)\,. \qquad (1.9)$$

Due to the rotation term, Liu obtained the global existence and decay estimates of solutions with more general semilinear term. The results in these papers and the general dissipative plate equation (see [2,5,8,9])show that they are of regularity-loss property.

A similar decay structure of the regularity-loss type was also observed for the dissipative Timoshenko system[10,11] and a hyperbolic-elliptic system related to a radiating gas[12]. For more studies on various aspects of dissipation of plate equations, we refer to reference[13,14,15,16].

The main purpose of this paper is to study the global existence and decay estimates for solutions to the initial value problem (1.1)--(1.2) with semilinear term in the spirit of reference[1,2]. We extend the result in [1] to the semilinear case.

Before closing this section, we give some notations to be used below. Let $\mathcal{F}\left[f\right]$ denote the Fourier transform of f defined by

$$\hat{f}(\xi)=\mathcal{F}\left[f\right](\xi)=)\frac{1}{(2\pi)^{\frac{n}{2}}}\int_{R^n}e^{-ix\cdot\xi}f(x)dx\,,$$

and we denote its inverse transform as $\mathcal{F}^{-1}$.

$L^p=L^p(R^n)$ for $1\leq p\leq\infty$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$. $H^s=H^s(R^n)$ for $s\in N$ denotes the Sobolev space with its norm

$$\|f\|_{H^s}:=\left\|\left(1+|\xi|^2\right)^{\frac{s}{2}}\hat{f}\right\|_{L^2}\,.$$

Also, $C^k(I;H^s(R^n))$ denotes the space of k -times continuously differentiable functions on the interval I with values in the Sobolev space $H^s=H^s(R^n)$.

Finally, we denote every positive constant by the same symbol C or c without confusion, and $[\cdot]$ is Gauss' symbol.

2 Proof of the Main Theorem

In this section, by virtue of the properties of solution operators, we prove the global existence and decay estimates of solutions to the semilinear problem (1.1)--(1.2) by employing the contraction mapping theorem. Now we recall the fundamental solution formula of the linear problems in reference[1], which is given by

$$u(t)=u_s(t):=\mathcal{G}(t)*u_0+\mathcal{H}(t)*u_1\,. \qquad (2.1)$$

Also, the solution $u(x,t)$ to the problem (1.1)--(1.2) can be formally expressed as

$$u(t):=\mathcal{G}(t)*u_0+\mathcal{H}(t)*u_1+\int_0^t\mathcal{H}(t-\tau)*f(\partial_x^2 u,u_t)d\tau\,. \qquad (2.2)$$

To prove theorem 1.1, we need some lemma.

Lemma 1 (see [1]**).** Assume that $\varphi \in H^{s+2}\left(R^n\right) \bigcap L^q\left(R^n\right)$ and $\psi \in H^s\left(R^n\right) \bigcap L^q\left(R^n\right)$ for $s \geq 0$ and $1 \leq q \leq 2$. Let k and l be non-negative integers satisfying $k+l \leq s$, then the following estimate holds:

$$(1)\ \left\|\partial_x^{k+2}\mathcal{G}(t)*\varphi\right\|_{L^2} \leq \sqrt{2}C_{q,k}\left(1+t\right)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}}\left\|\varphi\right\|_{L^q} + \sqrt{2}C_l\left(1+t\right)^{-\frac{l}{4}}\left\|\partial_x^{k+l+2}\varphi\right\|_{L^2},$$

$$(2)\ \left\|\partial_t\partial_x^{k}\mathcal{G}(t)*\varphi\right\|_{L^2} \leq \sqrt{2}C_{q,k}\left(1+t\right)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}}\left\|\varphi\right\|_{L^q} + \sqrt{2}C_l\left(1+t\right)^{-\frac{l}{4}}\left\|\partial_x^{k+l+2}\varphi\right\|_{L^2},$$

$$(3)\ \left\|\partial_x^{k+2}\mathcal{H}(t)*\psi\right\|_{L^2} \leq C_{q,k}\left(1+t\right)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}}\left\|\psi\right\|_{L^q} + C_l\left(1+t\right)^{-\frac{l}{4}}\left\|\partial_x^{k+l}\psi\right\|_{L^2},$$

$$(4)\ \left\|\partial_t\partial_x^{k}\mathcal{H}(t)*\psi\right\|_{L^2} \leq C_{q,k}\left(1+t\right)^{-\frac{n}{4}\left(\frac{1}{q}-\frac{1}{2}\right)-\frac{k}{4}}\left\|\psi\right\|_{L^q} + C_l\left(1+t\right)^{-\frac{l}{4}}\left\|\partial_x^{k+l}\psi\right\|_{L^2},$$

where $C_{q,k}$ is a positive constant depending only on q and k, C_l is a positive constant depending only on l.

By some modification of the theorem 2.7 in Liu-Ueda[1], we have the following

Lemma 2 (see [1]**).** Let $s \geq 0$ be an integer. Suppose that $u_0 \in H^{s+2}\left(R^n\right),\ u_1 \in H^s\left(R^n\right)$. Furthermore, assume $u_0 \in L^1\left(R^n\right),\ u_1 \in L^1\left(R^n\right)$, and put

$$I_1 := \left\|u_0\right\|_{H^{s+2}} + \left\|u_1\right\|_{H^s} + \left\|(u_0,u_1)\right\|_{L^1}.$$

then the following decay estimate holds:

If $s \geq \left[\frac{n+1}{2}\right]$, then

$$\left\|\partial_x^{k+2}u(t)\right\|_{H^{s-\sigma(k,n)}} \leq C\left(1+t\right)^{-\frac{n}{8}-\frac{k}{4}}\left(\left\|u_0\right\|_{H^{s+2}} + \left\|u_1\right\|_{H^s} + \left\|(u_0,u_1)\right\|_{L^1}\right),$$

$$\left\|\partial_x^{k}u_t(t)\right\|_{H^{s-\sigma(k,n)}} \leq C\left(1+t\right)^{-\frac{n}{8}-\frac{k}{4}}\left(\left\|u_0\right\|_{H^{s+2}} + \left\|u_1\right\|_{H^s} + \left\|(u_0,u_1)\right\|_{L^1}\right).$$

Where $k \geq 0$, $\sigma(k,n) \leq s$.

Proof. Let k and m are non-negative integers.

By virtue of (2.1) and Lemma 1 (1) (3)with $q=1$ and $C = \max\left(\sqrt{2}C_{q,k}, \sqrt{2}C_l\right)$, we have

$$\begin{aligned}
\left\|\partial_x^{k+m+2}u(t)\right\|_{L^2} &\leq \left\|\partial_x^{k+m+2}\mathcal{G}(t)*u_0(t)\right\|_{L^2} + \left\|\partial_x^{k+m+2}\mathcal{H}(t)*u_1(t)\right\|_{L^2} \\
&\leq C\left(1+t\right)^{-\frac{n}{8}-\frac{k}{4}}\left\|u_0\right\|_{L^1} + C\left(1+t\right)^{-\frac{l_1}{4}}\left\|\partial_x^{k+m+l_1+2}u_0\right\|_{L^2} \\
&\quad + C\left(1+t\right)^{-\frac{n}{8}-\frac{k}{4}}\left\|u_1\right\|_{L^1} + C\left(1+t\right)^{-\frac{l_2}{4}}\left\|\partial_x^{k+m+l_2}u_1\right\|_{L^2} \\
&\leq C\left(1+t\right)^{-\frac{n}{8}-\frac{k}{4}}\left\|(u_0,u_1)\right\|_{L^1} + C\left(1+t\right)^{-\frac{l_1}{4}}\left\|\partial_x^{k+m+l_1+2}u_0\right\|_{L^2} \\
&\quad + C\left(1+t\right)^{-\frac{l_2}{4}}\left\|\partial_x^{k+m+l_2}u_1\right\|_{L^2}.
\end{aligned}$$

here $l_1 \geq 0$, $l_2 \geq 0$, $k+m+l_1 \leq s$, $k+m+l_2 \leq s$.

Choose the smallest integers l_1 and l_2 satisfying

$$\frac{l_1}{4} \geq \frac{n}{8} + \frac{k}{4}, \qquad \frac{l_2}{4} \geq \frac{n}{8} + \frac{k}{4}.$$

It yields that

$$l_1 \geq \left[\frac{n+1}{2}\right] + k, \qquad l_2 \geq \left[\frac{n+1}{2}\right] + k.$$

Take $l_1 = l_2 = \sigma(k,n) - k$, where $\sigma(k,n)$ is defined by (1.3). Hence that holds m satisfying $0 \le m \le s - \sigma(k,n)$.Taking sum with m, we get that

$$\left\|\partial_x^{k+2} u(t)\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\left(\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s} + \|(u_0,u_1)\|_{L^1}\right).$$

Similarly, we can prove that

$$\left\|\partial_x^k u_t(t)\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\left(\|u_0\|_{H^{s+2}} + \|u_1\|_{H^s} + \|(u_0,u_1)\|_{L^1}\right).$$

So far we complete the proof of Lemma 2.

Lemma 3 (see [2]**).** Assume that p, q, r, k, α and β are integers, $1 \le p, q, r \le \infty$, $\frac{1}{p} = \frac{1}{q} + \frac{1}{r}$ and $k \ge 0$, $\alpha \ge 1$ and $\beta \ge 1$, then

$$\left\|\partial_x^k\left(u^\alpha v^\beta\right)\right\|_{L^p} \le C\|u\|_{L^\infty}^{\alpha-1}\|v\|_{L^\infty}^{\beta-1}\left(\|u\|_{L^q}\left\|\partial_x^k v\right\|_{L^r} + \|v\|_{L^q}\left\|\partial_x^k u\right\|_{L^r}\right).$$

Proposition 1. Let $a \ge 0$ and $b \ge 0$ be real numbers. If $a + b \ge 1$, then there exists $C > 0$ (independent of $t > 0$) such that the following estimate holds,

$$\int_0^t (1+t-\tau)^{-a}(1+\tau)^{-b}\,d\tau \le C.$$

Proof. Directly computation!

Now in order to prove theorem 1.1, we define

$$X := \left\{u \in C^0([0,\infty); H^{s+2}((R^n))) \cap C^1([0,\infty); H^s((R^n))), \|u\|_X < \infty\right\},$$

here

$$\begin{aligned}\|u\|_X := &\sum_{\{k;\sigma(k,n)\le s\}} \sup_{t\ge 0}(1+t)^{\frac{n}{8}+\frac{k}{4}}\left\|\partial_x^{k+2}u(t)\right\|_{H^{s-\sigma(k,n)}} \\ &+ \sum_{\{k;\sigma(k,n)\le s\}} \sup_{t\ge 0}(1+t)^{\frac{n}{8}+\frac{k}{4}}\left\|\partial_x^k u_t(t)\right\|_{H^{s-\sigma(k,n)}}.\end{aligned}$$

Denote

$$\begin{aligned}&U := (\partial_x^2 u, u_t),\\ &S_R := \left\{u \in X; \|u\|_X \le R\right\}, \forall R > 0,\\ &\phi[u](t) := \mathcal{G}(t) * u_0 + \mathcal{H}(t) * u_1 + \int_0^t \mathcal{H}(t-\tau) * f(U)(\tau)\,d\tau,\\ &\phi_0(t) := \mathcal{G}(t) * u_0 + \mathcal{H}(t) * u_1.\end{aligned}$$

Noticing that $f(v) = O(|v|^\alpha)$ and applying lemma 3, we have the following inequalities for $k \ge 0$:

$$\begin{aligned}\left\|\partial_x^k\left(f(V) - f(W)\right)(\tau)\right\|_{L^1} \le C&\left\|(V,W)(\tau)\right\|_{L^\infty}^{\alpha-2}\Big(\left\|(V,W)(\tau)\right\|_{L^2}\left\|\partial_x^k(V-W)(\tau)\right\|_{L^2} \\ &+ \left\|\partial_x^k(V,W)(\tau)\right\|_{L^2}\left\|(V-W)(\tau)\right\|_{L^2}\Big),\end{aligned} \tag{2.3}$$

$$\begin{aligned}\left\|\partial_x^k\left(f(V) - f(W)\right)(\tau)\right\|_{L^2} \le C&\left\|(V,W)(\tau)\right\|_{L^\infty}^{\alpha-2}\Big(\left\|(V,W)(\tau)\right\|_{L^\infty}\left\|\partial_x^k(V-W)(\tau)\right\|_{L^2} \\ &+ \left\|\partial_x^k(V,W)(\tau)\right\|_{L^2}\left\|(V-W)(\tau)\right\|_{L^\infty}\Big).\end{aligned} \tag{2.4}$$

Now we will prove that $u \to \phi[u]$ is a contraction mapping on S_R for some $R > 0$. We divide the proof into the following four steps.

Step 1. First we give an estimate on the L^∞ – norm by using the Gagliardo-Nirenberg inequality which will be regularly used in the succeeding computation.

Set $s_0 = \left[\frac{n}{2}\right] + 1$, $\theta_n = \frac{n}{2s_0}$. Take $u \in X$, by using the Gagliardo-Nirenberg inequality, we have

$$\|U(t)\|_{L^\infty} \le C \|U(t)\|_{L^2}^{1-\theta_n} \|\partial_x^{s_0} U(t)\|_{L^2}^{\theta_n}.$$

When $n=1$, since $s \ge 3$, i.e. $s - \sigma(1,1) \ge 0$, hence we get $\|U(t)\|_{L^2} \le C(1+t)^{-\frac{1}{8}} \|u\|_X$, $\|\partial_x^{s_0} U(t)\|_{L^2} \le C(1+t)^{-\frac{3}{8}} \|u\|_X$ by the definition of $\|u\|_X$. It yields $\|U(t)\|_{L^\infty} \le C(1+t)^{-\frac{1}{4}} \|u\|_X$.

When $n \ge 2$, since $s \ge n+1$ and $\frac{[n]}{2} + \frac{[n+1]}{2} = n$, i.e. $s - \sigma(0,n) \ge s_0$, thus we obtain $\|U(t)\|_{L^2} \le C(1+t)^{-\frac{n}{8}} \|u\|_X$, $\|\partial_x^{s_0} U(t)\|_{L^2} \le C(1+t)^{-\frac{n}{8}} \|u\|_X$ by the definition of $\|u\|_X$. Then we have $\|U(t)\|_{L^\infty} \le C(1+t)^{-\frac{n}{8}} \|u\|_X$.

Denote

$$d_n = \begin{cases} \frac{1}{4}, & n=1, \\ \frac{n}{8}, & n \ge 2, \end{cases}$$

Therefore we have

$$\|U(t)\|_{L^\infty} \le C \|u\|_X (1+t)^{-d_n}. \tag{2.5}$$

Step 2. Take any $v, w \in X$, denote $V := (\partial_x^2 v, v_t)$, $W := (\partial_x^2 w, w_t)$, then we have

$$\phi[v](t) - \phi[w](t) := \int_0^t \mathcal{H}(t-\tau) * \left(f(V) - f(W) \right)(\tau) d\tau.$$

Assume that k, m and l are non-negative integers and $s \ge \sigma(k,n)$, then we have that

$$\begin{aligned} \left\|\partial_x^{k+m+2} (\phi[v](t) - \phi[w](t))\right\|_{L^2} &\le C \left(\int_0^{\frac{t}{2}} + \int_{\frac{t}{2}}^t \right) \left\|\partial_x^{k+m+2} \mathcal{H}(t-\tau) * \left(f(V) - f(W) \right)(\tau)\right\|_{L^2} d\tau \\ &=: I_1 + I_2. \end{aligned} \tag{2.6}$$

By applying Lemma 1 (3) with $q=1$ and $C = \max(C_{q,k}, C_l)$, we have that

$$\begin{aligned} I_1 &\le C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{n}{8} - \frac{k+m}{4}} \left\| \left(f(V) - f(W) \right)(\tau) \right\|_{L^1} d\tau \\ &\quad + C \int_0^{\frac{t}{2}} (1+t-\tau)^{-\frac{l}{4}} \left\| \partial_x^{k+m+l} \left(f(V) - f(W) \right)(\tau) \right\|_{L^2} d\tau \\ &=: I_{11} + I_{12}. \end{aligned} \tag{2.7}$$

By applying (2.3) with $k=0$, we obtain that

$$\left\| \left(f(V) - f(W) \right)(\tau) \right\|_{L^1} \le C \|(V,W)(\tau)\|_{L^\infty}^{\alpha-2} \|(V,W)(\tau)\|_{L^2} \|(V-W)(\tau)\|_{L^2}.$$

By virtue of (2.5), we get that

$$\begin{aligned}\left\|\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\,(1+\tau)^{-d_n(\alpha-2)-\frac{n}{4}}\\ &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\begin{cases}(1+\tau)^{-\frac{\alpha-1}{4}}, & n=1,\\ (1+\tau)^{-\frac{n\alpha}{8}}, & n\ge 2.\end{cases}\end{aligned}\tag{2.8}$$

In view of Assumption [B], we have that $\frac{\alpha-1}{4}>1$ and $\frac{n\alpha}{8}>1$. Therefore, by Proposition 1, we obtain that

$$I_{11}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\tag{2.9}$$

for $m\ge 0$.

Similarly, if $k+m+l\le s$, by using (2.4) with k replaced by $k+m+l$, we obtain that

$$\begin{aligned}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} \le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(&\|(V,W)(\tau)\|_{L^\infty}\left\|\partial_x^{k+m+l}(V-W)(\tau)\right\|_{L^2}\\ &+\left\|\partial_x^{k+m+l}(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^\infty}\Big).\end{aligned}$$

By virtue of (2.5), it holds that

$$\begin{aligned}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\,(1+\tau)^{-d_n(\alpha-1)-\frac{n}{8}}\\ &\le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\begin{cases}(1+\tau)^{-\frac{\alpha-1}{4}}, & n=1,\\ (1+\tau)^{-\frac{n\alpha}{8}}, & n\ge 2.\end{cases}\end{aligned}\tag{2.10}$$

Take $l=\sigma(k,n)-k$, then $\frac{l}{4}\ge\frac{n}{8}+\frac{k}{4}$. By virtue of Assumption [B] and Proposition 1, we have that

$$I_{12}\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\tag{2.11}$$

with $0\le m\le s-\sigma(k,n)$.

Put the estimates for I_{11} and I_{12} in (2.7), then we obtain

$$I_1\le C(1+t)^{-\frac{n}{8}-\frac{k}{4}}\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X\tag{2.12}$$

with $0\le m\le s-\sigma(k,n)$.

Also, by using Lemma 1 (3) with $q=1$ and $C=\max(C_{q,k},C_l)$, we have that

$$\begin{aligned}I_2 &\le C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\partial_x^k\left(f(V)-f(W)\right)(\tau)\right\|_{L^1}d\tau\\ &\quad+C\int_{\frac{t}{2}}^{t}(1+t-\tau)^{-\frac{l}{4}}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2}d\tau\\ &=:I_{21}+I_{22}.\end{aligned}\tag{2.13}$$

By using (2.3) , we have

$$\begin{aligned}\left\|\partial_x^{k}\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} \le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(&\|(V,W)(\tau)\|_{L^2}\left\|\partial_x^{k}(V-W)(\tau)\right\|_{L^2}\\ &+\left\|\partial_x^{k}(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^2}\Big),\end{aligned}$$

hence we obtain that

$$\left\|\partial_x^{k}\left(f(V)-f(W)\right)(\tau)\right\|_{L^1}\le C\|(V,W)(\tau)\|_X^{\alpha-1}\|(V-W)(\tau)\|_X\,(1+t)^{-d_n(\alpha-2)-\frac{n}{4}-\frac{k}{4}}.$$

It holds that

$$I_{21} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

with $0 \le m \le s-\sigma(k,n)$.

Similarly, by using (2.4) with k replaced by $k+m+l$, we obtain that

$$\begin{aligned}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} \le C\|(V,W)(\tau)\|_{L^\infty}^{\alpha-2}\Big(&\|(V,W)(\tau)\|_{L^\infty}\left\|\partial_x^{k+m+l}(V-W)(\tau)\right\|_{L^2} \\ &+\left\|\partial_x^{k+m+l}(V,W)(\tau)\right\|_{L^2}\|(V-W)(\tau)\|_{L^\infty}\Big).\end{aligned}$$

By applying (2.5), it holds that

$$\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} \le C\|(v,w)\|_X^{\alpha-1}\|(v-w)\|_X(1+\tau)^{-d_n(\alpha-2)-\frac{n}{4}-\frac{k}{4}}$$

with $0 \le m \le s-\sigma(k,n)$.

It yields that

$$I_{22} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

for $0 \le m \le s-\sigma(k,n)$.

Put the estimates for I_{21} and I_{22} in (2.13), then we obtain

$$I_2 \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X \tag{2.14}$$

for $0 \le m \le s-\sigma(k,n)$.

Combining the estimates (2.6), (2.12) and (2.14), we obtain that

$$\left\|\partial_x^{k+m+2}\left(\phi[v]-\phi[w]\right)(t)\right\|_{L^2} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

Taking sum with $0 \le m \le s-\sigma(k,n)$, we have that

$$\left\|\partial_x^{k+2}(\phi[v](t)-\phi[w](t))\right\|_{H^{s-\sigma(k,n)}} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

It holds that

$$\sup_{t\ge 0}(1+\tau)^{\frac{n}{8}+\frac{k}{4}} \left\|\partial_x^{k+2}(\phi[v](t)-\phi[w](t))\right\|_{H^{s-\sigma(k,n)}} \le C\|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X . \tag{2.15}$$

Step 3. Assume that k, m and l are non-negative integers and $\sigma(k,n) \le s$, then we get that

$$\begin{aligned}\left\|\partial_x^{k+m}\partial_t(\phi[v](t)-\phi[w](t))\right\|_{L^2} &\le C\left(\int_0^{\frac{t}{2}}+\int_{\frac{t}{2}}^t\right)\left\|\partial_x^{k+m}\mathcal{H}_t(t-\tau)*\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} d\tau \\ &=: I_3+I_4.\end{aligned} \tag{2.16}$$

By applying Lemma 1 (4) with $q=1$ and $C=\max(C_{q,k}, C_l)$, we have that

$$\begin{aligned}I_3 &\le C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}}\left\|\left(f(V)-f(W)\right)(\tau)\right\|_{L^1} d\tau \\ &\quad + C\int_0^{\frac{t}{2}}(1+t-\tau)^{-\frac{l}{4}}\left\|\partial_x^{k+m+l}\left(f(V)-f(W)\right)(\tau)\right\|_{L^2} d\tau \\ &=: I_{31}+I_{32}.\end{aligned} \tag{2.17}$$

Similar to the argument of (2.9), we obtain

$$I_{31} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

At the same time, similar to the estimate of I_{12}, we have

$$I_{32} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

Put the estimates for I_{31} and I_{32} in (2.17), then it holds that

$$I_3 \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X \tag{2.18}$$

Similarly, by applying Lemma 1 (4) with $q=1$ and $C=\max(C_{q,k},C_l)$, we have that

$$\begin{aligned} I_4 &\le C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{n}{8}-\frac{k+m}{4}} \left\| \partial_x^k \left(f(V)-f(W) \right)(\tau) \right\|_{L^1} d\tau \\ &\quad + C\int_{\frac{t}{2}}^{t} (1+t-\tau)^{-\frac{l}{4}} \left\| \partial_x^{k+m+l} \left(f(V)-f(W) \right)(\tau) \right\|_{L^2} d\tau \\ &=: I_{41}+I_{42}. \end{aligned} \tag{2.19}$$

Since

$$\begin{aligned} \left\| \partial_x^k \left(f(V)-f(W) \right)(\tau) \right\|_{L^1} \le C \left\| (V,W)(\tau) \right\|_{L^\infty}^{\alpha-2} \Big(&\left\| (V,W)(\tau) \right\|_{L^2} \left\| \partial_x^k (V-W)(\tau) \right\|_{L^2} \\ &+ \left\| \partial_x^k (V,W)(\tau) \right\|_{L^2} \left\| (V-W)(\tau) \right\|_{L^2} \Big), \end{aligned}$$

thus

$$\left\| \partial_x^k \left(f(V)-f(W) \right)(\tau) \right\|_{L^1} \le C \left\| (V,W)(\tau) \right\|_X^{\alpha-1} \left\| (V-W)(\tau) \right\|_X (1+t)^{-d_n(\alpha-2)-\frac{n}{4}-\frac{k}{4}}.$$

It yields that

$$I_{41} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

Similar to the estimate of I_{22}, we obtain

$$I_{42} \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

Put the estimates for I_{41} and I_{42} in (2.19), then it holds that

$$I_4 \le C(1+t)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X \tag{2.20}$$

Combining the estimates (2.16), (2.18) and (2.20), we get that

$$\left\| \partial_x^{k+m} \partial_t \left(\phi[v]-\phi[w] \right)(t) \right\|_{L^2} \le C(1+\tau)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

Taking sum with $0 \le m \le s-\sigma(k,n)$, we have that

$$\left\| \partial_x^k \partial_t (\phi[v](t)-\phi[w](t)) \right\|_{H^{s-\sigma(k,n)}} \le C(1+\tau)^{-\frac{n}{8}-\frac{k}{4}} \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X .$$

It yields that

$$\sup_{t\ge 0}(1+\tau)^{\frac{n}{8}+\frac{k}{4}} \left\| \partial_x^k \partial_t (\phi[v](t)-\phi[w](t)) \right\|_{H^{s-\sigma(k,n)}} \le C \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X . \tag{2.21}$$

Step 6. Combining the estimates (2.15) and (2.21), we get that

$$\left\| \left(\phi[v]-\phi[w] \right) \right\|_X \le C \|(v,w)\|_X^{\alpha-1} \|(v-w)\|_X$$

So far we prove that $\left\| \left(\phi[v]-\phi[w] \right) \right\|_X \le C_1 R^{\alpha-1} \|(v-w)\|_X$, if $v,w \in S_R$. On the other hand, $\phi[0](t)=\phi_0(t)=u_s(t)$, and by lemma 2 we know that $\|\phi_0\|_X \le C_2 I_1$, if I_1 is suitably small. Take $R=2C_2 I_1$, if I_1 is suitably small such that $R<1, C_1 R \le \frac{1}{2}$, then we obtain that

$$\left\| \left(\phi[v]-\phi[w] \right) \right\|_X \le \frac{1}{2} \|(v-w)\|_X .$$

It yields that

$$\|\phi[v]\|_X \leq \|\phi_0\|_X + \frac{1}{2}\|v\|_X \leq C_2 I_1 + \frac{1}{2}R \leq R .$$

Hence $v \to \phi[v]$ is a contraction mapping on S_R , and by the fixed point principle there exists a unique $u \in S_R$ satisfying $\phi[u] = u$, and it is the solution to the semilinear problem (1.1)(1.2) satisfying the decay estimates (1.4) and (1.5). Thus we complete the proof of Theorem 1.1.

References

[1] Y. Liu, Y. Ueda. Decay estimate and asymptotic profile for a plate equation with memory[J]. Journal of Differential Equations 268(2020)2435–2463.

[2] Liu Y. Decay of solutions to an inertial model for a semilinear plate equation with memory[J]. Math. Anal. Appl. 394(2012)616-632.

[3] C.R. da Luz, R.C. Charão. Asymptotic properties for a semi-linear plate equation in unbounded domains [J]. Hyper-bolic Differ. Equ. 6(2009)269–294.

[4] Y. Sugitani, S. Kawashima. Decay estimates of solutions to a semilinear dissipative plate equation [J]. Hyperbolic Differ. Equ. 7(2010)471–501.

[5] Liu Y, Kawashima S. Global existence and asymptotic behavior of solutions for quasi-linear dissipative plate equation. Discrete Contin. Dyn. Syst. 29(2011)1113-1139.

[6] Liu Y, Kawashima S. Global existence and decay of solutions for a quasi-linear dissipative plate equation. J. Hyperbolic Difffferential Equations. 8(2011)591-614.

[7] Liu Y, Kawashima S. Decay property for a plate equation with memory-type dissipation. Kinet. Relat. Mod. 4(2011)531-547.

[8] S. Mao and Y. Liu, Decay of solutions to generalized plate type equations with memory,Kinet. Relat. Mod., 7(2014)121-131.

[9] Y. Sugitani and S. Kawashima, Decay estimates of solutions to a semi-linear dissipative plate equation, J. Hyperbolic Differential Equations, 7(2010)471-501.

[10] Y. Liu, S. Kawashima, Decay property for the Timoshenko system with memory-type dissipation, Math. Models Methods Appl. Sci., 22(2012)1–19.

[11] Y. Liu, S. Kawashima, Global existence and asymptotic decay of solutions to the nonlinear Timoshenko system with memory, Nonlinear Anal. TMA 84(2013)1–17.

[12] T. Hosono and S. Kawashima, Decay property of regularity-loss type and application to some nonlinear hyperbolic-elliptic system, Math. Models Meth. Appl. Sci., 16(2006)1839-1859.

[13] Bradley ME, Lenhart S. Bilinear spatial control of the velocity term in a Kirchhoffff plate equation. Electronic J. Difffferential Equations. 2001(2001)1-15.

[14] Buriol C. Energy decay rates for the Timoshenko system of thermoelastic plates. Nonlinear Analysis. 64(2006)92-108.

[15] R. C. Charão, E. Bisognin, V. Bisognin and A.F. Pazoto, Asymptotic behavior for a Dissipative plate equation in R^N with periodic coefficients, Electronic J. Differential Equations, 2008(2008)1-23.

[16] Enomoto Y. On a thermoelastic plate equation in an exterior domain. Math. Meth. Appl. Sci. 25(2002)443-472.