

Original Research Article

Bound Estimation of Some Special Functions

ABSTRACT

This article estimates bounds of several special functions. Mathematical proof and the graph of the corresponding functions are given. The results are can also be Applicable.in aspect of inequalities.

Keywords: Estimation, bound, special function, inequalities

1. INTRODUCTION

T_3 tree, which was introduced in article [1], brought some interesting inequalities and bound estimation for special functions, as shown in [2,3,4,5]. During the study of the tree several new special functions are found. They are

$$f(a) = \frac{\sqrt{a}}{2-\sqrt{a}} - \left(\frac{a+1}{2}\right)^2, f(\sigma) = \sqrt{\frac{2^\sigma \sqrt{2}}{2^\sigma - 1}}, f(\sigma) = \sqrt{\frac{2^\sigma}{(2^\sigma - 1)\sqrt{2}}}, f(\sigma) = \sqrt{\frac{2^{\sigma+1}}{2^\sigma + \sqrt{2}}}, f(\sigma) = \sqrt{\frac{2^{\sigma+1}}{2^\sigma + 1}} \text{ and } f(\sigma) = \sqrt{\frac{2^\sigma \sqrt{2}}{(2^\sigma + 1)}}.$$

It also needs to estimate the bound of them. This paper presents the required answers. Results of this paper are helpful for further study of T_3 tree.

2. MAIN RESULT AND PROOF

Theorem 1. Let α be a real number with $\alpha \in (1,4) \cup (4,\infty)$; then

$$f(a) = \frac{\sqrt{a}}{2-\sqrt{a}} - \left(\frac{a+1}{2}\right)^2$$

thus $0 < f(\alpha) < \infty, \alpha \in (1,4)$

and $-\infty < f(\alpha) < (-17.345), \alpha \in (4,6)$

Proof. Direct calculation shows

$$\begin{aligned} f'(\alpha) &= \frac{\frac{1}{2}\alpha^{-\frac{1}{2}}(2-\sqrt{\alpha}) - \sqrt{\alpha}\left(-\frac{1}{2}\alpha^{-\frac{1}{2}}\right)}{(2-\sqrt{\alpha})^2} - \frac{1}{2}\alpha - \frac{1}{2} \\ &= \frac{1}{\sqrt{\alpha}(2-\sqrt{\alpha})^2} - \frac{\alpha}{2} - \frac{1}{2}. \end{aligned}$$

The original function can be changed to

$$\frac{8\sqrt{\alpha} + 4\alpha - (4-\alpha)(\alpha+1)^2}{4(4-\alpha)} > 0 \quad (1)$$

Since $1 < \alpha < 4$, $4(4-\alpha) > 0$. Next is to show

$$8\sqrt{\alpha} + 4\alpha - (4-\alpha)(\alpha+1)^2 > 0$$

Let $h(\alpha) = 8\sqrt{\alpha} + 4\alpha - (4-\alpha)(\alpha+1)^2$; then

$$\begin{aligned} h(\alpha) &= 8\sqrt{\alpha} + 4\alpha - 4(\alpha+1)^2 + \alpha(\alpha+1)^2 \\ &= 8\sqrt{\alpha+1-1} + 4(\alpha+1-1) - 4(\alpha+1)^2 + (\alpha+1-1)(\alpha+1)^2. \end{aligned}$$

Assume $\alpha+1=t$; then $2 < t < 5$ and

$$\begin{aligned} f(t) &= 8\sqrt{t-1} + 4(t-1) - 4t^2 + (t-1)t^2 \\ &= t^3 - 5t^2 + 4t + 8\sqrt{t-1} - 4. \end{aligned}$$

Direct calculation yields

$$f'(t) = 3t^2 - 10t + 4 + \frac{4}{\sqrt{t-1}}.$$

$$\begin{aligned} f''(t) &= 6t - \frac{2}{\sqrt{(t-1)^3}} - 10 \\ &= 6t - \frac{2}{(t-1)\sqrt{t-1}} - 10. \end{aligned}$$

Note that, $\frac{1}{4} < \frac{2}{(t-1)\sqrt{t-1}} < 2$ and when $f''(t) > 0$. This means $f'(t)$ is monotonically increasing. Since $f'(t) > 0$ when $t > 2$, $f(t)$ is monotonically increasing. Considering $f(2) = 0$, it is obtained

$$f(\alpha) > 0$$

Since $f(4-0) = +\infty$ and $f(\alpha) > 0$, it holds

$$0 < f(\alpha) < +\infty \quad \alpha \in (1, 4)$$

Since $\alpha_0 = \left(\frac{2 + (35 + 3\sqrt{129})^{\frac{1}{3}} + 4(35 + 3\sqrt{129})^{-\frac{1}{3}}}{3} \right)^2 \approx 5.566$, it holds and consequently

$$f''(\alpha_0) = \frac{-\left(\frac{2}{\sqrt{\alpha}} + \frac{3\sqrt{\alpha}}{2} - 4\right)}{\alpha(2-\sqrt{\alpha})^4} - \frac{1}{2} \approx -25.98 < 0$$

It is obvious that $f(\alpha_0) = -17.34544147$ when $4 < \alpha < 6$.

Since $f(4+0) = +\infty$ and $f(\alpha_0) = -17.34544147$, it holds

$$-\infty < f(\alpha) < (-17.345), \alpha \in (4,6)$$

□

Figure 1 shows the graph of the function $f(x) = \frac{\sqrt{x}}{2-\sqrt{x}} - (\frac{x+1}{2})^2$ with $a \in (1,4) \cup (4,\infty)$, which is just what Theorem 1 states.

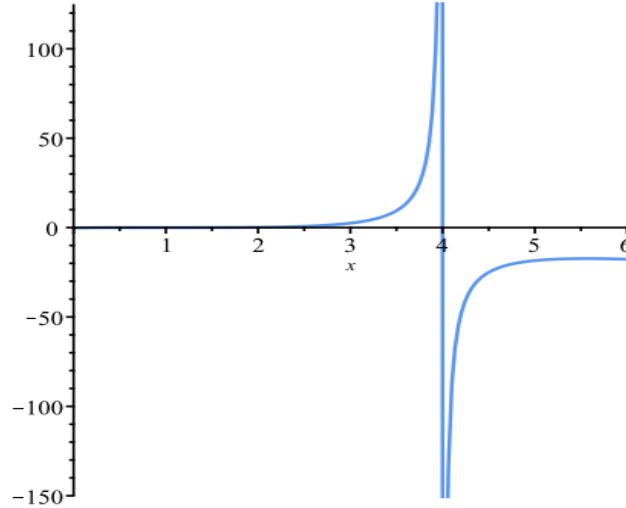


Fig. 1. The graph of the function

Theorem 2. Let $f(\sigma) = \sqrt{\frac{a2^\sigma}{2^\sigma - 1}}$ with $\sigma > 1$ and $a > 0$; then $f'(\sigma) < 0$ and $\sqrt{a} \leq f(\sigma) < \sqrt{2a}$.

Proof. Let $g(\sigma) = \frac{a2^\sigma}{2^\sigma - 1}$

Direct calculation shows

$$\begin{aligned} g'(\sigma) &= \frac{a2^\sigma \ln 2(2^\sigma - 1) - a2^\sigma (2^\sigma \ln 2)}{(2^\sigma - 1)^2} \\ &= \frac{-a2^\sigma \ln 2}{(2^\sigma - 1)^2}. \end{aligned} \quad (2)$$

The conditions $a > 0$ and $\sigma > 1$ mean $-a2^\sigma \ln 2 < 0$; thus

$$g'(\sigma) < 0 \quad (3)$$

Obviously, (2) and (3) result in $f'(\sigma) < 0$. Since $\sigma = 1$, it can be $f(1) = \sqrt{2a}$. When σ is to infinity, the value of $f(\sigma)$ is equal to \sqrt{a} . It leads to $\sqrt{a} \leq f(\sigma) < \sqrt{2a}$.

□

Corollary 1. Let $f(\sigma) = \sqrt{\frac{2^\sigma \sqrt{2}}{2^\sigma - 1}}$ with $\sigma > 1$; then $1.189 \leq f(\sigma) < 1.682$.

Proof. According to Theorem 2, since $a = \sqrt{2}$, it can obtain $1.189 \leq f(\sigma) < 1.682$ when $\sigma > 1$.

Figure 2 displays the graph of the function $f(x) = \sqrt{\frac{2^x \sqrt{2}}{2^x - 1}}$ when $\sigma > 1$, which is just what Corollary 1 states.

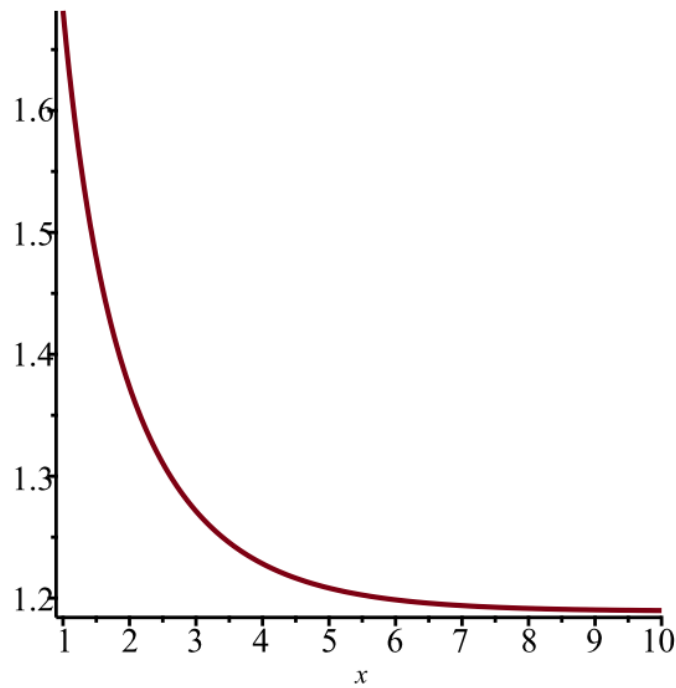


Figure.2 The graph of the function

Corollary 2. Let $h(\sigma) = \sqrt{\frac{2^\sigma}{(2^\sigma - 1)\sqrt{2}}}$ with $\sigma > 1$; then $0.841 \leq h(\sigma) < 1.189$

Proof. By Theorem 2, Since $\sigma > 1$ and $a > 0$, it holds

$$\sqrt{a} \leq \sqrt{\frac{a2^\sigma}{2^\sigma - 1}} < \sqrt{2a}$$

If $a = \frac{\sqrt{2}}{2}$, it can get $0.841 \leq h(\sigma) < 1.189$.

□

As shown in Figure 3 illustrates this.

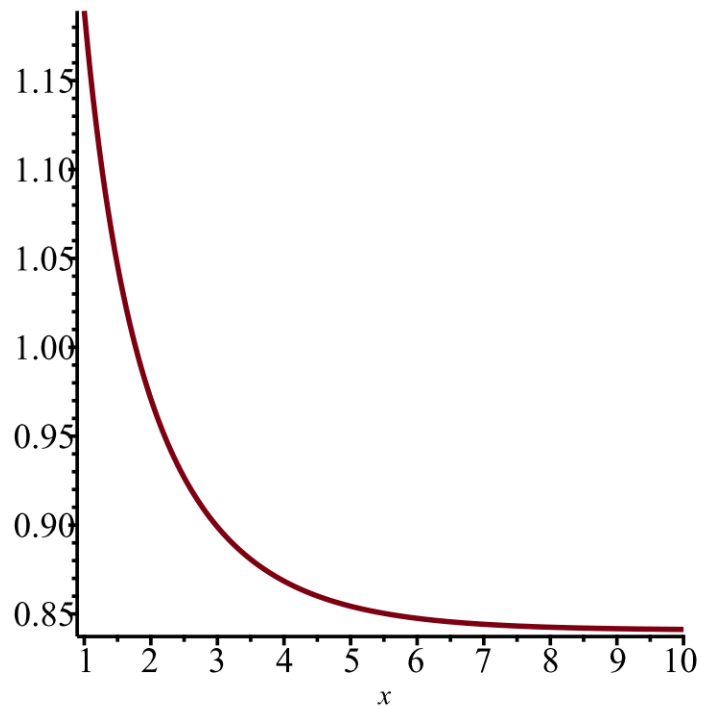


Figure.3 The graph of the function

Theorem 3. Let $f(\sigma) = \sqrt{\frac{2^\sigma}{2^\sigma + a}}$ with $\sigma > 1$ and $a > 0$; then $f'(\sigma) > 0$ and $\sqrt{\frac{2}{2+a}} < f(\sigma) < 1$.

Proof. Let $g(\sigma) = \frac{2^\sigma}{2^\sigma + a}$;

Direct calculation yields

$$g'(\sigma) = \frac{2^\sigma a \ln 2}{(2^\sigma + a)^2}$$

When $\sigma > 1$ and $a > 0$, it is obviously $g(\sigma)' > 0$. So

$$f'(\sigma) > 0.$$

Then by $f(1) = \sqrt{\frac{2}{2+a}}$ and $\lim_{\sigma \rightarrow \infty} f(\sigma) = \sqrt{\frac{2^\sigma}{2^\sigma + a}} = 1$ thus $\sqrt{\frac{2}{2+a}} \leq f(\sigma) < 1$.

□

Corollary 3. Let $f(\sigma) = \sqrt{\frac{2^{\sigma+1}}{2^\sigma + \sqrt{2}}}$ with $\sigma > 1$; then $1.082 < \sqrt{\frac{2^{\sigma+1}}{2^\sigma + \sqrt{2}}} \leq 1.414$.

Proof. According to theorem 3, Since $f(1) = \sqrt{\frac{2^2}{2 + \sqrt{2}}} = 1.082$ and $\lim_{\sigma \rightarrow \infty} f(\sigma) = \sqrt{2} = 1.414$ hence $1.082 \leq f(\sigma) < 1.414$ when $\sigma > 1$.

□

Figure 4 shows the graph of the function $f(x) = \sqrt{\frac{2^{x+1}}{2^x + \sqrt{2}}}$. It is seen $1.082 < f(x) \leq 1.414$ when $x > 1$, which is just what Corollary 3 states.

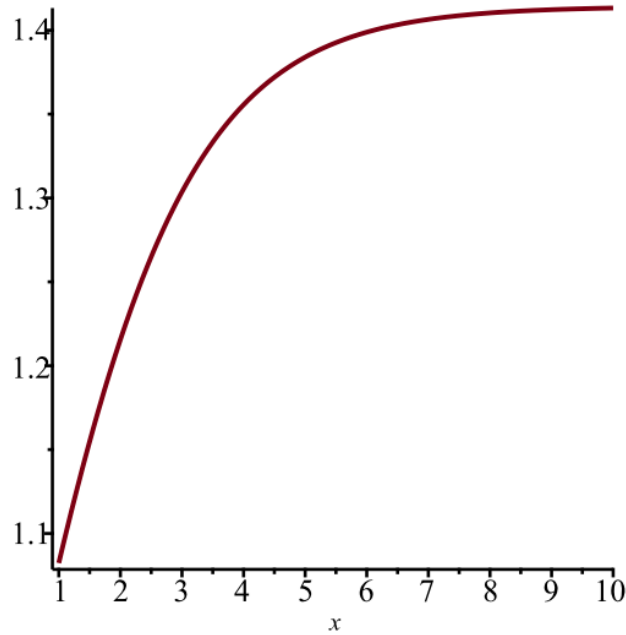


Figure.4 The graph of the function

Corollary 4. Let $f(\sigma) = \sqrt{\frac{2^{\sigma+1}}{2^\sigma + 1}}$ with $\sigma > 1$; then $1.155 < \sqrt{\frac{2^{\sigma+1}}{2^\sigma + 1}} \leq 1.414$.

Proof. According to the analysis of Theorem 3, since $f(1) = \sqrt{\frac{2^2}{2+1}} = 1.155$ and $\lim_{\sigma \rightarrow \infty} \sqrt{\frac{2^{\sigma+1}}{2^\sigma + 1}} = 1.414$, So it is directly obtained $1.155 \leq f(\sigma) < 1.414$ when $\sigma > 1$.

□

As shown in Figure 5.

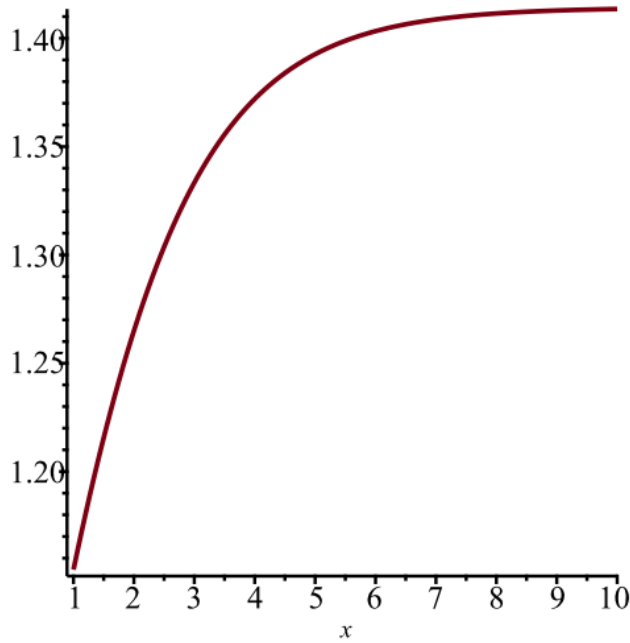


Figure.5 The graph of the function

Theorem 4. Suppose $f(\sigma) = \sqrt{\frac{a2^\sigma}{2^\sigma + 1}}$ with $\sigma > 1$ and $a > 0$; then $f'(\sigma) > 0$ and $\sqrt{\frac{2a}{3}} < \sqrt{\frac{2^\sigma}{2^\sigma + a}} < \sqrt{a}$.

Proof. Let $g(\sigma) = \frac{a2^\sigma}{2^\sigma + 1}$; then

$$g'(\sigma) = \frac{2^\sigma a \ln 2}{(2^\sigma + 1)^2}.$$

When $\sigma > 1$ and $a > 0$, this is obviously $g(\sigma)' > 0$ thus

$$f'(\sigma) > 0.$$

Since $\sigma = 1$, it can be $f(1) = \sqrt{\frac{2a}{3}}$. When σ is to infinity, the value of $f(\sigma)$ is equal to \sqrt{a} . Consequently

$$\sqrt{\frac{2a}{3}} \leq f(\sigma) < \sqrt{a}.$$

□

Corollary 5. Assume $f(\sigma) = \sqrt{\frac{2^\sigma \sqrt{2}}{2^\sigma + 1}}$ with $\sigma > 1$; then $f'(\sigma) > 0$ and $0.971 < \sqrt{\frac{2^\sigma \sqrt{2}}{2^\sigma + 1}} \leq 1.189$.

Proof. According to theorem 4, It yields $0.971 \leq f(\sigma) < 1.189$ when $\sigma > 1$.

□

Figure 6 shows the graph of the function $f(x) = \sqrt{\frac{2^x \sqrt{2}}{2^x + 1}}$. It is seen $0.971 < f(x) \leq 1.189$ when

$x > 1$. This is exactly what Corollary 5 says.

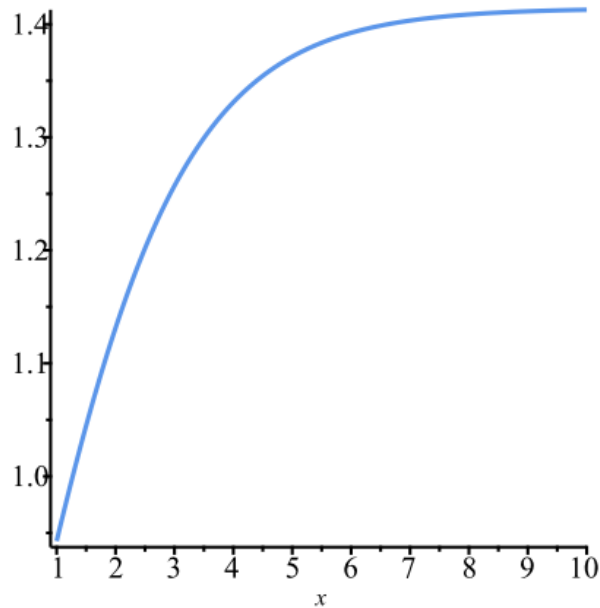


Figure 6: The graph of the function

3. CONCLUSION

This paper proves the estimation of several special function boundaries encountered in the research process, provides a mathematical foundation and solution ideas for estimating function boundaries, so that it can be applied to engineering practice, and it is hoped that it will be helpful to researchers.

COMPETING INTERESTS DISCLAIMER:

Authors have declared that no competing interests exist. The products used for this research are commonly and predominantly use products in our area of research and country. There is absolutely no conflict of interest between the authors and producers of the products because we do not intend to use these products as an avenue for any litigation but for the advancement of knowledge. Also, the research was not funded by the producing company rather it was funded by personal efforts of the authors

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