Original Research Article

Modified Shanker Distribution and its Applications

Abstract

In this paper, we study a new distribution called the Modified inverse Shanker distribution. The proposed distribution has one special case, inverse Shanker distribution. Besides the basic properties of the distribution, the maximum likelihood technique of estimating the parameters of the distribution and some of the reliability measures are also discussed. We also illustrate the applicability of the proposed distribution using two real datasets.

Keywords: Modified Shanker distribution, Exponentiated distribution, Inverse Shanker distribution, Generalized Shanker distribution, Inverse distributions.

1 Introduction

The modeling and analysis of lifetime data, for instance, datasets from medicine, engineering, insurance, actuarial science and so on, is considered a significant facet of statistical study, specifically, in the area of statistical modelling and distribution theory. Consequently, the construction of new distributions or modification of existing ones turns to be the most vital work if looked at, from a probabilistic point of view. However, recently attention have been given to this area by many researchers, see [1-4], and many more. The essence of developing new or modifying existing distributions is to get distributions that are more robust and flexible, unbiased, efficient and sufficient to capture the situation under study, and many researchers have been working on this. One of the proof of this, is the emergence of the Shanker distribution proposed by [5] for modelling real lifetime data-sets from various fields of knowledge as claimed by the author. One major defect of the shanker distribution is its inability to capture or model datasets with decreasing function in their hazard rate. That is, inverted hazard rate shape. Thus, the major objective of this study is to suggest a better distribution that can model some lifetime datasets that follows inverse Shanker distribution.

1.2 Inverse Shanker distribution

The one parameter inverse Shanker distribution was developed by applying the method of inverse transformation in [6]. Suppose X is a nonnegative continuous random variable from Shanker distribution with probability density function (pdf), $f(x)$ and cumulative density function (cdf), $F(x)$, if

$Y = h_1(x) = \frac{1}{X}$, then the pdf of Y is given by;

$$f_{IR}(y) = f_{IR}\left(h_1^{-1}(y)\right)\left|\frac{dx}{dy}\right| \tag{1.1}$$

With the method in (1.1), the inverse Shanker distribution has been derived. Thus, we obtained equations (1.2) and (1.3)

Definition 1: A random variable X is said to have an Inverse Shanker distribution if the probability density function (pdf) and cumulative density function (cdf) are respectively given as

$$g(x;\theta)=\left\{\frac{\theta^2}{\theta^2+1}\left(\frac{\theta x+1}{x^3}\right)\ell^{-\theta/x},x;\theta>0\right\}$$

(1.2)

$$G(x;\theta)=1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x};x,\theta>0$$

(1.3)

1.3 Modified Inverse Shanker Distribution

In order to increase the flexibility of the new distribution while still maintaining few parameters, the authors decided to modify the inverse Shaker distribution using another method suggested by [7] as follows:

Suppose X is a random variable with a baseline cumulative density function, cdf $F(x)$, the new family of distributions with the form

$$F(x,\beta) = [G(x,\beta)]^{\beta};\ x\in R\ ;\ \beta>0 \quad (1.4)$$

where for $\beta = 1$, (1.4) reduces to the cdf of the baseline distribution.

The corresponding probability density function, pdf is obtained by taking the first derivative of (1.4). Consequently, we have

$$f(x,\beta) = \beta[G(x,\beta)]^{\beta-1}f(x,\beta);\ x\in R\ ;\ \beta>0 \quad (1.5)$$

Substituting equations (1.2) and (1.3) into (1.5) and (1.4) respectively, we obtain the pdf and cdf of modified Shanker distribution given in equation (1.6) and (1.7).
Definition 2: Let $X\sim MIS(\theta)$, the probability density function of X is given as follows

$$g(x;\theta,\beta)=\frac{\beta\theta^2}{\theta^2+1}\left(\frac{\theta x+1}{x^3}\right)\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta-1}\ell^{-\theta/x}\ for\beta,\theta,x>0 \quad (1.6)$$

The corresponding cumulative density function is given by

$$G(x;\theta,\beta)=\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta}\ x;\theta,\beta>0 \quad (1.7)$$

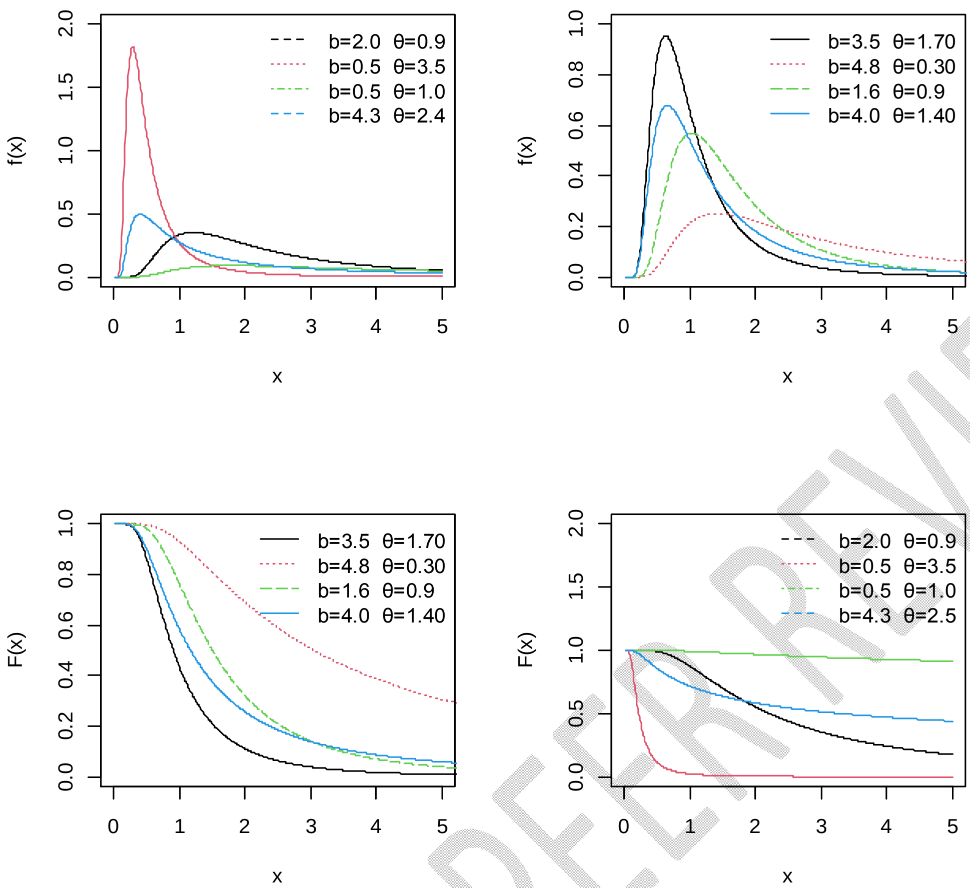

Fig. 1: pdf and cdf plot of the modified inverse Shanker distribution

Fig. 1 shows that the pdf and cdf of the modified inverse Shanker distribution can take varying shapes depending on the value of $\theta \, and \, \beta$

2 Mathematical Characteristics of Modified Inverse Shanker Distribution

In this section, some of the properties of modified inverse Shanker distribution is considered.

2.1 Moments

Moments enable one study vital features of a distribution such as mean, variance, skewness and kurtosis. The most significant aspect of a moment is the rth moment which supports easy derivation of other moments.

Definition 3: Let X be a random variable that follows modified inverse Shanker distribution with parameters (θ, β). Then, the rth moment is given as

$$E(x^j) = \mathrm{A}_{k,l,m}\frac{\Gamma(l-j-m+1)}{(k+1)^{(l-j-m+1)}} + \mathrm{B}_{k,l,m}\frac{\Gamma(l-j-m+2)}{(k+1)^{(l-j-m+2)}}; \quad j = 1,2,3,4$$

Where

$$\mathrm{A}_{k,l,m}=\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta\theta^{k+j-l+2}}{\left(\theta^{2}+1\right)^{k+1}} \text{ and } \mathrm{B}_{k,l,m}=\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta\theta^{k+j-l}}{\left(\theta^{2}+1\right)^{k+1}}$$

The rth moment of a distribution is given by

$$E(x^{j})=\int_{-\infty}^{\infty}x^{j}g(x)dx \tag{1.8}$$

$$=\int_{0}^{\infty}\frac{\beta\theta^{2}x^{j}}{\theta^{2}+1}\left(\frac{\theta x+1}{x^{3}}\right)\left\{1-\left[\frac{\theta x+x+\theta}{(\theta^{2}+1)x}\right]\ell^{\theta/x}\right\}^{\beta-1}\ell^{\theta/x}dx \tag{1.9}$$

A binomial expansion approach can be used to show that:

$$\left\{1-\left[\frac{\theta x+x+\theta}{(\theta^{2}+1)x}\right]\ell^{\theta/x}\right\}^{\beta-1}=\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\frac{x^{-k}}{\left(\theta^{2}+1\right)^{k}}\sum_{l=0}^{\infty}\binom{k}{l}(\theta x)^{k-l}\sum_{m=0}^{l}\binom{l}{m}\theta^{lm}x^{m}\ell^{-k\theta/x}$$

Consequently, equation (1.9) reduces to

$$E(x^{j})=\left\{\begin{array}{l}\displaystyle\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{l=0}^{\infty}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta\theta^{k-m+3}}{\left(\theta^{2}+1\right)^{k+1}}\int_{0}^{\infty}x^{j+m-l-2}\ell^{-\frac{\theta(k+1)}{x}}dx\\ \displaystyle+\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{l=0}^{\infty}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta\theta^{k-m+2}}{\left(\theta^{2}+1\right)^{k+1}}\int_{0}^{\infty}x^{j+m-l-3}\ell^{-\frac{\theta(k+1)}{x}}dx\end{array}\right\}$$

Recall that $\int_{0}^{\infty}x^{-(\alpha+1)}\ell^{-\beta/x}dx=\frac{\Gamma\alpha}{\beta^{\alpha}}$, thus

$$=\left\{\begin{array}{l}\displaystyle\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta\theta^{k+j-l+2}}{\left(\theta^{2}+1\right)^{k+1}}\frac{\Gamma\left(l-j-m+1\right)}{(k+1)^{(l-j-m+1)}}\\ \displaystyle+\sum_{k=0}^{\infty}\binom{\beta-1}{k}(-1)^{k}\sum_{l=0}^{k}\binom{k}{l}\sum_{m=0}^{l}\binom{l}{m}\frac{\beta\theta^{k+j-l}}{\left(\theta^{2}+1\right)^{k+1}}\frac{\Gamma\left(l-j-m+2\right)}{(k+1)^{(l-j-m+2)}}\end{array}\right\}$$

Therefore,

$$E(x^{j})=\mathrm{A}_{k,l,m}\frac{\Gamma\left(l-j-m+1\right)}{(k+1)^{(l-j-m+1)}}+\mathrm{B}_{k,l,m}\frac{\Gamma\left(l-j-m+2\right)}{(k+1)^{(l-j-m+2)}};\qquad j=1,2,3,4$$

(1.10)

The first four crude moments also known as moment about origin, for $j=1,2,3$ and 4 are as follows

$$\mu_1' = \mathrm{A}_{k,l,m} \frac{\Gamma(l-m)}{(k+1)^{(l-m)}} + \mathrm{B}_{k,l,m} \frac{\Gamma(l-m+1)}{(k+1)^{(l-m+1)}} \quad (1.11)$$

$$\mu_2' = \mathrm{A}_{k,l,m} \frac{\Gamma(l-m-1)}{(k+1)^{(l-m-1)}} + \mathrm{B}_{k,l,m} \frac{\Gamma(l-m)}{(k+1)^{(l-m)}} \quad (1.12)$$

$$\mu_3' = \mathrm{A}_{k,l,m} \frac{\Gamma(l-m-2)}{(k+1)^{(l-m-2)}} + \mathrm{B}_{k,l,m} \frac{\Gamma(l-m-1)}{(k+1)^{(l-m-1)}} \quad (1.13)$$

$$\mu_4' = \mathrm{A}_{k,l,m} \frac{\Gamma(l-m-3)}{(k+1)^{(l-m-3)}} + \mathrm{B}_{k,l,m} \frac{\Gamma(l-m-2)}{(k+1)^{(l-m-2)}}$$

(1.14)

2.2 Moment generating function

Apart from moments, some of the remarkable features such as the crude moments, mean, variance and so on, of a statistical distribution can as well be derived from its moment generating function (mgf).

Definition 4: Given a random variable X , such that $X \sim MIS(\theta, \beta)$, the moment generating function is given by

$$M_x(t) = \sum_{j=0}^{\infty} \frac{t^j}{j!} \left[\mathrm{A}_{k,l,m} \frac{\Gamma(l-j-m+1)}{(k+1)^{(l-j-m+1)}} + \mathrm{B}_{k,l,m} \frac{\Gamma(l-j-m+2)}{(k+1)^{(l-j-m+2)}} \right]$$

Let X denote a random variable having the modified inverse Shanker distribution (MIS) with parameters θ and β , then its moment generating function (mgf) is

$$M_x(t) = E(\ell^{tx}) = \int_0^{\infty} \ell^{tx} g(x) dx \quad (1.15)$$

Using Tailor's series

$$\ell^{tx} = 1 + tx + \frac{(tx)^2}{2!} + \ldots = \sum_{j=0}^{\infty} \frac{t^j x^j}{j!} \quad (1.16)$$

Substituting (1.16) into (1.15) gives

$$M_x(t) = \int_0^{\infty} \sum_{j=0}^{\infty} \frac{t^j x^j}{j!} g(x) dx = \sum_{j=0}^{\infty} \frac{t^j}{j!} \int_0^{\infty} x^j g(x) dx$$

Where $\int_0^{\infty} x^j g(x) dx = E(x^j)$

Consequently, $M_x(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}E(x^j)$. Substituting, one obtains

$$M_x(t)=\sum_{j=0}^{\infty}\frac{t^j}{j!}\left[\mathrm{A}_{k,l,m}\frac{\Gamma\left(l-j-m+1\right)}{(k+1)^{(l-j-m+1)}}+\mathrm{B}_{k,l,m}\frac{\Gamma\left(l-j-m+2\right)}{(k+1)^{(l-j-m+2)}}\right] \tag{1.17}$$

2.3 Order Statistics of Modified Inverse Shanker Distribution

Suppose $X_1, X_2 ..., X_n$ are random samples of size n from a continuous distribution with pdf and cdf, $f\left(x\right)$ and $F\left(x\right)$ respectively. If these random variables are arranged in ascending order, they are referred to as order statistics. The smallest of the $X_{\omega}^{'s}$ is denoted by $X_{(1)}$, the second smallest is denoted by $X_{(2)},...,$ and the largest is denoted by $X_{(n)}$. That is, the order statistics is such that $x_1 \le x_2 \le x_3 \le ... \le x_n$, the pdf of the ω^{th} order statistics defined by [8]

$$f_x(x)=\frac{n!}{\left(p-1\right)!\left(n-p\right)!}G_{(x)}^{p-1}\left[1-G\left(x\right)\right]^{n-p}g\left(x\right)$$

$$f_x(x)=\frac{n!}{\left(p-1\right)!\left(n-p\right)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i G\left(x\right)^{p+i-1}g\left(x\right) \tag{1.18}$$

substituting (1.6) and (1.7) into (1.18) gives

$$f_x\left(x\right)=\frac{n!}{\left(p-1\right)!\left(n-p\right)!}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left\{1-\left[\frac{\theta^2x+x+\theta}{\left(\theta^2+1\right)x}\right]\ell^{-\theta/x}\right\}^{\beta(p+i)-1}.$$

$$\times\frac{\beta\theta^2}{\theta^2+1}\left(\frac{\theta x+1}{x^3}\right)\left\{1-\left[\frac{\theta^2x+x+\theta}{\left(\theta^2+1\right)x}\right]\ell^{-\theta/x}\right\}^{\beta-1}\ell^{-\theta/x}$$

$$=\frac{\beta\theta^2n!\left(\theta x+1\right)x^{-3}}{\left(p-1\right)!\left(n-p\right)!\left(\theta^2+1\right)}\ell^{-\theta/x}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\left\{1-\left[\frac{\left(\theta^2x+x+\theta\right)x^{-1}}{\left(\theta^2+1\right)}\right]\ell^{-\theta/x}\right\}^{\beta(p+i)-1}$$

(1.19)

Using Binomial series expansion on equation (1.19) gives the pdf of the order statistics of the modified inverse Shanker distribution

$$f_x\left(x\right)=\frac{n!\beta\left(\theta x+1\right)}{\left(p-1\right)!\left(n-p\right)!}\ell^{-\frac{\theta(j+1)}{x}}\sum_{i=0}^{n-p}\binom{n-p}{i}(-1)^i\sum_{j=0}^{\infty}\binom{\beta\left(p+i\right)-1}{j}(-1)^j$$
$$\times\sum_{k=0}^{j}\binom{j}{k}\sum_{l=0}^{k}\binom{k}{l}\frac{\theta^{2j-k-l+2}}{\left(\theta^2+1\right)^{j+1}}x^{k+l-3} \tag{1.20}$$

However, the cdf of pth order statistics of the modified inverse Shanker distribution is given by

$$F_x(x)=\sum_{j=p}^{n}\binom{n}{j}G^{j}(x)\left[1-G(x)\right]^{n-j}$$

$$=\sum_{j=p}^{n}\binom{n}{j}\sum_{\varphi=0}^{n-j}\binom{n-j}{\varphi}(-1)^{\varphi}G^{j+\varphi}(x) \tag{1.21}$$

Substituting (1.7) in (1.21), complex algebra shows that the cdf of the pth

$$F_x(x)=\sum_{j=p}^{n}\binom{n}{j}\sum_{\varphi=0}^{n-j}\binom{n-j}{\varphi}(-1)^{\varphi}\sum_{q=0}^{\infty}\binom{\beta(j+\varphi)}{q}(-1)^{q}\sum_{r=o}^{q}\binom{q}{r}\sum_{t=0}^{r}\binom{r}{t}\frac{\theta^{2q-r-t}}{(\theta^2+1)^{q}}x^{t-r} \tag{1.22}$$

2.4 Renyi Entropy

The entropy of a random variable, say X is a measure of its variation of uncertainty. One of the common entropy measure is introduced by [9]. If X is a continuous random variable having probability density function $g(\bullet)$, the Rényi entropy of a random variable X from a continuous distribution is given by

$$T_r(\alpha)=\frac{1}{1-\alpha}\log\left[\int g_r^{\alpha}(x)\,dx\right];\alpha>0,\alpha\neq 1 \tag{1.23}$$

$$=\frac{1}{1-\alpha}\log\left\{\frac{\beta\theta^2}{\theta^2+1}\left(\frac{\theta x+1}{x^3}\right)\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)}\right]\ell^{-\theta/x}\right\}^{\beta-1}\ell^{-\theta/x}\right\}^{\alpha}dx$$

$$=\frac{1}{1-\alpha}\log\left\{\begin{array}{l}\sum_{i=0}^{\infty}\binom{\alpha(\beta-1)}{i}(-1)^{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\beta^{\alpha}\theta^{3\alpha+2i-j-k}}{(\theta^2+1)^{\alpha+i}}\int_0^{\infty}x^{-(j-k+2\alpha+1)}\ell^{-\frac{\theta(\alpha+i)}{x}}dx\\+\sum_{i=0}^{\infty}\binom{\alpha(\beta-1)}{i}(-1)^{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\beta^{\alpha}\theta^{2\alpha+2i-j-k}}{(\theta^2+1)^{\alpha+i}}\int_0^{\infty}x^{-(j-k+3\alpha+1)}\ell^{-\frac{\theta(\alpha+i)}{x}}dx\end{array}\right\}$$

Recall that $\int_0^{\infty}x^{-(\alpha+1)}\ell^{-\beta/x}=\frac{\Gamma\alpha}{\beta^{\alpha}}$, substituting, one obtains

$$=\frac{1}{1-\alpha}\log\left\{\begin{array}{l}\sum_{i=0}^{\infty}\binom{\alpha(\beta-1)}{i}(-1)^{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\beta^{\alpha}\theta^{-2j+\alpha+2i}}{(\theta^2+1)^{\alpha+i}}.\frac{\Gamma(j-k+2\alpha)}{(\alpha+i)^{(j-k+2\alpha)}}\\+\sum_{i=0}^{\infty}\binom{\alpha(\beta-1)}{i}(-1)^{i}\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\beta^{\alpha}\theta^{-2j-\alpha+2i}}{(\theta^2+1)^{\alpha+i}}.\frac{\Gamma(j-k+3\alpha)}{(\alpha+i)^{(j-k+3\alpha)}}\end{array}\right\}$$

$$T_r(\alpha)=\frac{1}{1-\alpha}\log\left\{\lambda_{ijk}\frac{\Gamma(j-k+2\alpha)}{(\alpha+i)^{(j-k+2\alpha)}}+\gamma_{ijk}\frac{\Gamma(j-k+3\alpha)}{(\alpha+i)^{(j-k+3\alpha)}}\right\} \tag{1.24}$$

where

$$\lambda_{ijk}=\sum_{i=0}^{\infty}\binom{\alpha(\beta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\beta^{\alpha}\theta^{-2j+\alpha+2i}}{(\theta^2+1)^{\alpha+i}}$$

$$\gamma_{ijk}=\sum_{i=0}^{\infty}\binom{\alpha(\beta-1)}{i}(-1)^i\sum_{j=0}^{i}\binom{i}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{\beta^{\alpha}\theta^{-2j-\alpha+2i}}{(\theta^2+1)^{\alpha+i}}$$

3 Measures of Reliability

In this study, the authors only consider three measures of reliability, that is the survival function, hazard rate and reverse hazard rate.

Definition 5: Let X be a positive random variable from modified inverse Shanker distribution with probability density function $f(x)$ and cumulative distribution function $F(x)$. Thus, the survival function and hazard rate are respectively given by

$$S_{MIS}(x)=1-\left\{1-\left[\frac{(\theta^2x+x+\theta)}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta} \tag{1.25}$$

$$h_{MIS}(x)=\frac{\frac{\beta\theta^2}{\theta^2+1}\left(\frac{\theta x+1}{x^3}\right)\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta-1}\ell^{-\theta/x}}{1-\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta}} \tag{1.26}$$

Survival function $s(x)$ is monotone decreasing over the interval $[0,\infty)$, $\lim_{x\to 0} s(x)=1$, implies a proper functioning system, while $\lim_{x\to\infty} s(\infty)=0$, means that the no system remains working forever.

Also, the reversed hazard rate $\Omega_\tau(x)$ is given by

$$\Omega_\tau(x)=\frac{f(x)}{F(x)} \tag{1.27}$$

Substituting for $f(x)$ and $F(x)$, we obtain the reverse hazard function of modified inverse Shanker distribution. Thus,

$$\Omega_\tau(x)=\frac{\frac{\beta\theta^2}{\theta^2+1}\left(\frac{\theta x+1}{x^3}\right)\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta-1}\ell^{-\theta/x}}{\left\{1-\left[\frac{\theta^2x+x+\theta}{(\theta^2+1)x}\right]\ell^{-\theta/x}\right\}^{\beta}} \tag{1.28}$$

Figures 2 and 3 show the plot of the survival function and hazard rate of the modified inverse Shanker distribution using different values of β and θ

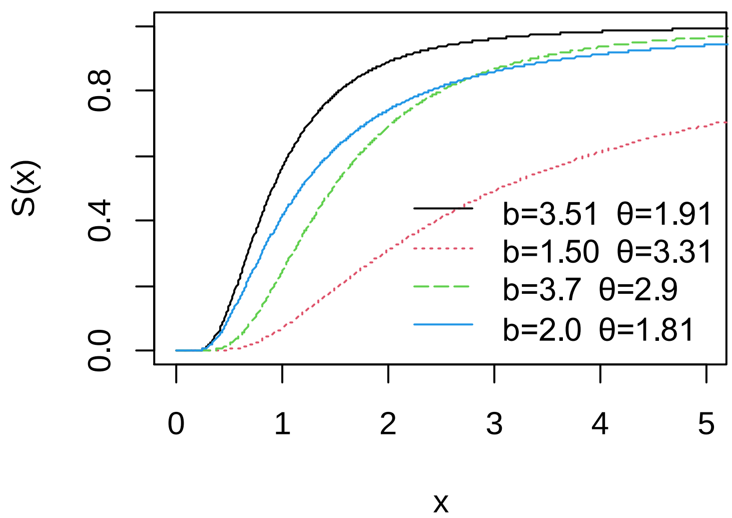

Fig. 2: Survival function of MIS distribution

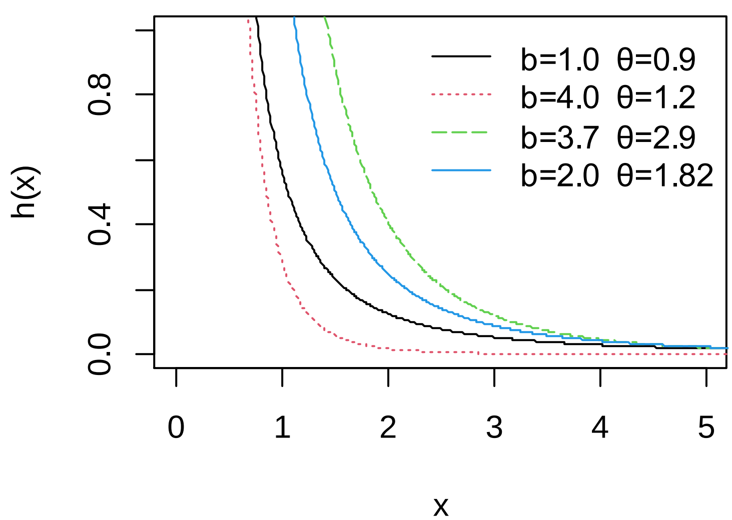

Fig. 3: Hazard rate function of MIS distribution

4 Maximum Likelihood Estimation of parameters of MIS distribution

Let $(x_1, x_2, x_3, ..., x_n)$ be an independent identically distributed random samples of size n with probability density function given in (1.6). Then, the likelihood function L of modified inverse Shanker distribution is given by

$$L=\prod_{i=1}^{n}\frac{\beta\theta^{2}}{\theta^{2}+1}\left(\frac{\theta x+1}{x^{3}}\right)\left\{1-\left[\frac{\theta^{2}x+x+\theta}{\left(\theta^{2}+1\right)x}\right]\ell^{-\theta/x}\right\}^{\beta-1}\ell^{-\theta/x} \tag{1.29}$$

The log – likelihood function is given by ℓ is

$$\ln L=\left\{\begin{array}{l} n\ln\beta+2n\ln\theta-n\ln\left(\theta^{2}+1\right)+\sum_{i=1}^{n}\ln\left[\frac{\theta}{x_i^{2}}\right]+\sum_{i=1}^{n}\ln\left[\frac{i}{x_i^{3}}\right] \\ +\left(\beta-1\right)\sum_{i=1}^{n}\ln\left\{1-\left[\frac{\theta^{2}x+x+\theta}{\left(\theta^{2}+1\right)x}\right]\ell^{-\theta/x}\right\}-\theta\sum_{i=1}^{n}\frac{1}{x_i} \end{array}\right\} \tag{1.30}$$

Differentiating (1.30) once with respect to $\theta \text{ and } \beta$, we arrived at the following

$$\frac{\partial\ln L}{\partial\beta}=\frac{n}{\beta}-\frac{2\theta n}{\theta}+\sum_{i=1}^{n}\ln\left\{1-\left(\frac{\theta^{2}x+x+\theta}{\left(\theta^{2}+1\right)x}\right)\ell^{-\frac{\theta}{x}}\right\} \tag{1.31}$$

$$\frac{\partial\ln L}{\partial\theta}=\frac{2n}{\theta}-\frac{2\theta n}{\left(\theta^{2}+1\right)}+\frac{n}{\theta}+\left(\beta-1\right)\sum_{i=1}^{n}\left\{\frac{\left(\theta^{2}+1\right)x\left(2\theta x-\theta^{2}x-\frac{\theta}{x}-2\theta x\left(\theta^{2}x+x+\theta\right)\right)}{\theta^{2}x+x-\theta}\right\}$$

(1.32)

The maximum likelihood estimates of the parameters $\theta \text{ and } \beta$ are obtained by solving the nonlinear equations (1.31) and (1.32) numerically, and evaluating at

$$\frac{\partial\ln L}{\partial\theta}=0 \text{ and } \frac{\partial\ln L}{\partial\beta}=0$$

The interval estimation of any of the parameters of the MIS distribution is possible when the necessary standard error estimate is known. As $n\to\infty$, the maximum likelihood $\hat{\varepsilon}=\left(\hat{\beta},\hat{\theta}\right)$ of $\varepsilon=(\beta,\theta)$ is asymptotically normally distributed with mean Θ and variance – covariance matrix.

$$v=\begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}=\begin{pmatrix} \frac{\partial^{2}l}{\partial\beta^{2}} & \frac{-\partial^{2}l}{\partial\beta\partial\gamma_1} & \frac{\partial^{2}l}{\partial\beta\partial\theta} \\ \frac{-\partial^{2}l}{\partial\beta\partial\gamma_1} & \frac{-\partial^{2}l}{\partial\beta^{2}} & \frac{-\partial^{2}l}{\partial\alpha\partial\theta} \\ \frac{\partial^{2}l}{\partial\beta\partial\theta} & \frac{-\partial^{2}l}{\partial\alpha\partial\theta} & \frac{-\partial^{2}l}{\partial\theta^{2}} \end{pmatrix}^{-1}$$

Therefore, an appropriate 100 (1 – α)% confidence intervals for $\beta \; and \; \theta$ are defined as follows:

$$\hat{\beta}\pm z_{\frac{\alpha}{2}}\sqrt{\hat{v}_{11}}\,,\; \hat{\alpha}\pm z_{\frac{\alpha}{2}}\sqrt{\hat{v}_{22}} \text{ and } \hat{\theta}\pm z_{\frac{\alpha}{2}}\sqrt{\hat{v}_{33}} \tag{1.33}$$

To obtain the parameter estimates and their corresponding standard error, R package shall be used.

4.1 Applications of Modified Inverse Shanker distribution on real data sets

In this section, the suggested distribution has been applied on two real datasets and compared with two parameters Quasi Shanker distribution (QSD), one parameter Shanker distribution (SD), Power Shanker distribution (PSD) and Length Biased Weighted Shanker Distribution (LBWS). Goodness of fit of the distributions has been carried out by means of Akaike information criteria (AIC), Bayesian Information and criteria (BIC) values respectively. The values are computed for each distribution and also compared. The best goodness of fit of the distribution is judged base on the one with minimum value of AIC and BIC. Comparison of distributions are shown in table 1 and 2 for the first and second datasets respectively. It shows the calculated values of AIC and BIC, log-likelihood, Anderson-Darling statistic and the p-values of the distribution. Also, Table 3 and 4 show calculated 95% confidence intervals for the parameters of the distributions.

Data set 1: This data set given by [10] represents the failure times of the air conditioning system of an airplane.
23, 261 ,87 ,7, 120, 14 ,62, 47 ,225, 71 ,246 ,21, 42 ,20, 5, 12, 120 ,11, 3, 14, 71 ,11, 14, 11, 16, 90, 1, 16 ,52, 95

Data set 2: The data-sets represent the survival times of two groups of patients suffering from Head and Neck cancer disease. The patients were treated using a combined radiotherapy and chemotherapy. These real life data-sets were previously analyzed by [10] and are presented below:

12.20, 23.56, 23.74, 25.87, 31.98, 37, 41.35, 47.38, 55.46, 58.36,63.47, 68.46, 78.26, 74.47, 81.43, 84, 92, 94, 110, 112, 119, 127,130, 133, 140, 146, 155, 159, 173, 179, 194, 195, 209, 249, 281,319, 339, 432, 469, 519, 633, 725, 817, 1776

Table 1: MLEs, S.E, LL, AIC, BIC and AICc (1st dataset)

Model	Parameters	S.E	LL	AIC	BIC	AD	p
MIS	**β = 0.6494092** **θ = 8.2431982**	0.1506 2.2641	-157.19	318.3941	321.1965	1.3512	0.2163
QSD	α= 0.00001446231 θ = 0.03357515	0.0029 0.0052	-162.87	329.7443	332.5467	5.1448	0.002502
SD	θ = 0.03357872	0.0043	-162.87	327.7443	329.1455	∞	2.00E-05
LBWS	θ = 0.05036684	0.0053	-178.23	358.4545	359.8557	13.609	2.00E-05

Table 2: MLEs, S.E, LL, AIC, BIC and AICc (2nd dataset)

Model	Parameters	S.E	LL	AIC	BIC	AD	p
EIS	**β =1.167573** **θ=84.870922**	**0.2431443** **16.503699**	**-279.3058**	**562.6116**	**566.18**	**0.50446**	**0.7413**

PSD	**α=0.6680546**	0.0692831	-279.9994	563.9989	567.5673	0.55803	0.6876
	θ=0.0610248	0.0244364					
SD	θ = 0.00898583	0.0009458	-289.7547	581.5093	583.2935	∞	1.36E-05
LBWS	θ = 0.0134245	0.0011619	-304.9733	611.9466	613.7308	11.079	1.42E-05

Table 3: MLEs of the parameters EIS distribution and their C.I (1st dataset)

Model	parameter	S.E	95% Confidence Interval	
			Lower Limit	Upper Limit
EIS	**β = 0.6494**	0.1506	0.3542	0.9446
	θ = 8.2432	2.2641	3.8056	12.6808
QSD	α=0.00001446	0.002865	-0.0056001	0.005629
	θ = 0.0335752	0.005188305	0.02341	0.04374
SD	θ = 0.0335787	0.00432867	0.02509	0.04206
LBWS	θ = 0.0503668	0.00530472	0.03997	0.06076

Table 4: MLEs of the parameters EIS distribution and their C.I (2nd dataset)

Model	parameter	S.E	95% Confidence Interval	
			Lower Limit	Upper Limit
EIS	**β =1.167573**	0.2431443	0.691	1.6441
	θ=84.870922	16.5036994	52.5237	117.2182
PSD	α=0.6680546	0.0692831	0.5323	0.8038
	θ=0.0610248	0.02443642	0.0131	0.1089
SD	θ = 0.00898583	0.00094582	0.0071	0.01084
LBWS	θ = 0.01342449	0.00116189	0.0111	0.0157

In table 1 and 2, the values of AIC and BIC of MIS distribution are least when compared to the other distributions, thus, considered to provide best fit. However, a look at the 95% confidence intervals show that all the parameter estimates for the distributions lie within the confidence interval. A test of goodness of fit for the proposed distribution using Anderson-darling (AD) showed that modified Inverse Shaker distribution has a good fit.

5 Conclusion

Efforts are made every day by statisticians to provide a model that can be used to fit or describe a situation under study. All these efforts emanate from the fact that the quality of the empirical results obtained by applying many parametric approaches of analysis significantly rest on on how sound a chosen distribution fits the data under consideration. In this study, we have introduced and studied the properties of a new distribution called the modified inverse Shanker distribution. Specifically, we have derived the rth moments, first four raw moments, moment generating function, the pdf and cdf of the order statistics respectively. Using a numerical illustration, the distribution is found to provide a better fit than the competing distributions.

References

[1] Chrisogonus K. Onyekwere, George A. Osuji and Samuel U. Enogwe. Inverted Power Rama Distribution with Applications to Life Time Data; *Asian Journal of Probability and Statistics*. 9(4): 1-21, 2020; Article no.AJPAS.62214, ISSN: 2582-0230.

[2] Mouna Z. The Kumaraswamy-Rani Distribution and Its Applications. *J Biom Biostat*, Volume 11:1, 2020; DOI: 10.37421/jbmbs.2020.11.436

[3] Umeh E.U., Umeokeke E. and Ibenegbu H. A. Two-parameter Rama distribution, *Earthline Journal of Mathematical Sciences*.p365-382.DOI:10.34198/ejms.2219.365382,(2019)

[4] Chrisogonus Kelechi Onyekwere, George Amaeze Osuji, Samuel Ugochukwu Enogwe, Michael Chukwuemeka Okoro, Victor Nduka Ihedioha. Exponentiated Rama Distribution: Properties and Application. *Mathematical Theory and Modeling*; ISSN 2224-5804 (Paper) ISSN 2225-0522 (Online), Vol.11, No.1, 2021

[5] Shanker R. Shanker Distribution and Its Applications, *International Journal of Statistics and Applications* 2015, 5(6): 338-348

[6] Hogg, R.V., J.W. McKean and A.T. Craig. *Introduction to Mathematical Statistics*, 8^{th} edn, Pearson Education, Inc.2019.

[7] Mudholkar, G.S. and Srivastava, D.K. Exponentiated Weibull family for analyzing bathtub failure-rate data, *IEEE Transact. Reliab*, 42(2), 299 – 302, 1993.

[8] Hogg, R.V., J.W. McKean and A.T. Craig. *Introduction to Mathematical Statistics*, 8^{th} edn, Pearson Education, Inc.2019b.

[9] Re'nyi Alfred (1961) On Measure of Entropy and Information. *Mathematical Institute Hungarian Academy of Sciences*; fourth Berkkeley Symposium**.**

[10] Linhart, H. and Zucchini, W.(1986) Model Selection, *John Wiley, New York*.

[11] Vikas, K. S., Sanjay K.S and Umesh S.(2014). A new upside-down bathtub shaped hazard rate model for survival data analysis, *Applied Mathematics and Computation.*2014;239:242-253.DOI:10.1016/j.amc.2014.04.048